Palestine Journal of Mathematics

Vol. 12(1)(2023) , 946–954

ALGEBRAS OF LEFT VARIABLE TERMS

Nuthawud Sungtong, Thodsaporn Kumduang and Sarawut Phuapong

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: 03B15; 08A30.

Keywords and phrases: Left variable term, identity, term operation, hypersubstitution.

This work is dedicated to Professor Sorasak Leeratanavalee on the occasion of his 60th birthday.
This work is supported by Rajamangala University of Technology Lanna, Chiang Mai.

Abstract Terms are fundamental notions in universal algebra for describing algebraic properties and classifying algebras. In this paper, we propose the concept of a specific class of terms which are either variables or composed terms where the first input of terms is an arbitrary variable. In this situation, we call such term that a left variable term. Particularly, we define the partial many-sorted superposition operation on the set of such terms and construct the partial many sorted algebras, called the partial clone of left variable terms, satisfying the superassotive law. Based on the idea of clones, algebraic structures consisting a set of operations with compositions of operations and projection mappings, the partial many-sorted superposition operation is defined on sets of left variable term operations and its properties are given. We also present the notion of term operations on left variable terms and study some important properties. Finally, a hypersubstitution which takes any operation symbol to a left variable term of the corresponding arity is considered. Several properties of such hypersubstitution are examined and it turns out that the collection of them forms a partial monoid under a suitable partial binary associative operation.

1 Introduction

The concept of terms in the classical theory of theoretical computer science is one of the significant tools for defining input or output data. Similarly, in the study of algebra, terms play a key role for describing the fundamental properties of algebras. Let $X := \{x_1, x_2, \ldots\}$ be an infinite set of symbols and whose elements called *variables*. We also refer to the set $X_n := \{x_1, x_2, \ldots, x_n\}$ as an n-element alphabet of variables. By the symbol $(f_i)_{i\in I}$, we denote an indexed set and each f_i is called an n_i*-ary operation symbol*, where $n_i \geq 1$ is its arity. The type $\tau = (n_i)_{i\in I}$ is a sequence of all arities of f_i. Recall from [9, 17] that an n*-ary term of type* τ is inductively defined as follows:

(1) Each variable $x_j \in X_n$ is an n-ary term of type τ.

(2) $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ if $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol.

The set of all n-ary terms of type τ containing $x_1, \ldots, x_n$ and is closed under finite number of applications of (2), is denoted by $W_\tau(X_n)$. The set $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ of all terms of type τ. See [4, 7, 22, 26] for several possibilities to define terms. A connection of terms with tree in the theory of graph was also determined. Furthermore, in the combinatoric aspects, the complexity of terms, for instance, a total number of occurrence variables in a term, was studied by a number of authors. For more details, the reader is referred to [1, 3, 11].

One of important operations on the set of terms is the superposition operation. For each natural numbers $m, n \geq 1$, the superposition operation is a many-sorted mapping

$$S^n_m : W_\tau(X_n) \times W_\tau(X_m)^n \to W_\tau(X_m)$$

defined by

(1) $S^n_m(x_j, t_1, \ldots, t_n) := t_j$ if $x_j \in X_n$,

(2) $S^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) := f_i(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n))$.

Then the many-sorted algebra $clone(\tau)$ can be defined by

$$clone(\tau) := ((W_\tau(X_n))_{n\geq 1}, (S^n_m)_{n,m\geq 1}, (x_i)_{i\leq n}),$$

which is called the *clone of all terms of type* τ. Normally, the many-sorted algebra $clone(\tau)$ can be regarded as one of concrete examples of Menger systems [10, 15, 18, 25].

Actually, terms play a critical role for classifying algebras into subclasses. For this, the corresponding term operations derived from terms are needed. Every term t can be extended to a term operation t on the algebras. Let $\mathcal{A} := (A, (f_i^{\mathcal{A}})_{i\in I})$ be an algebra of type τ and let t be an n-ary term of type τ. Then t induces an n-ary operation $t^{\mathcal{A}}$ on $\mathcal{A}$, by the following steps:

(1) If $t = x_j \in X_n$, then $t^{\mathcal{A}} = x_j^{\mathcal{A}} = e_j^{n,\mathcal{A}}$ where $e_j^{n,\mathcal{A}}$ is a projection mapping defined by $e_j^{n,\mathcal{A}}(a_1, \ldots, a_n) = a_j$ for all $a_1, \ldots, a_n \in A$.

(2) If $t = f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ, and $t_1^{\mathcal{A}}, \ldots, t_{n_i}^{\mathcal{A}}$ are the term operations which are induced by $t_1, \ldots, t_{n_i}$, then $t^{\mathcal{A}} = f_i^{A}(t_1^{\mathcal{A}}, \ldots, t_{n_i}^{\mathcal{A}})$.

Hence, $t^{\mathcal{A}}$ is called the *term operation induced by the term* t *on the algebra* $\mathcal{A}$. The set of all n-ary term operations on $\mathcal{A}$ will be denoted by $W_\tau(X_n)^{\mathcal{A}}$.

Moreover, the many-sorted algebra

$$Clone\mathcal{A} = ((W_\tau(X_n)^{\mathcal{A}})_{n\geq 1}, (S_m^{n,A})_{n,m\geq 1}, (e_i^{n,A})_{i\leq n})$$

is constructed. The set of all identities satisfied in algebra $\mathcal{A}$ is denoted by $Id\mathcal{A}$,. i.e.,

$$Id\mathcal{A} = \{s \approx t \in W_\tau(X) \times W_\tau(X) \mid s^{\mathcal{A}} = t^{\mathcal{A}}\}.$$

For an extensive information, see [8, 13, 14, 23, 24].

The main aim of this paper is to introduce a novel class of term. We further present some interesting examples of our new terms and try to define the partial operations on the set of such terms. Based on the idea of term operations, many properties of them can be used to study the results. Finally, we apply the hypersubstitution theory for describing the structural properties of a mapping which takes the set of all operation symbols to the set of such terms.

2 The Partial Clone of Left Variable Terms

The main goal of this section is to introduce the concept of a left variable term.

Definition 2.1. An *n-ary left variable term of type* τ is defined inductively as follows:

(1) Every variable x_i in X_n is an n-ary left variable term of type τ.

(2) If $t_1, \ldots, t_{n_i}$ are n-ary left variable terms of type τ, and if $t_1 = x_k$ for some $k \in \{1, \ldots, n\}$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary left variable term of type τ.

The set of all n-ary left variable terms of type τ is denoted by $W_\tau^{lv}(X_n)$. By $W_\tau^{lv}(X) := \bigcup_{n\in\mathbb{N}} W_\tau^{lv}(X_n)$, we mean the set of all left variable terms of type τ.

Now we will present some examples of left variable terms of some type τ.

Example 2.2. We consider the type $\tau = (2)$ with one binary operation symbol f and the set of variables X_2. Then some examples of binary left variable terms of type (2) are:

$$x_1, x_2, f(x_1, x_1), f(x_2, x_1), f(x_1, f(x_2, x_1)), f(x_1, f(x_1, f(x_2, x_1))).$$

But the following are not binary left variable terms:

$$f(f(x_1, x_1), x_2), f(f(x_1, x_2), f(x_2, x_1)).$$

Example 2.3. Let $\tau = (2)$, i.e, we have only one binary operation symbol f. Consider the set of variables X_3. Then some examples of ternary left variable terms of type (2) are:

$$x_1, x_2, x_3, f(x_1, x_3), f(x_3, x_2), f(x_1, f(x_3, x_2)), f(x_2, f(x_1, f(x_3, x_2))).$$

However, there are many ternary terms of type (2) which are not ternary left variable terms of type (2) such as

$$f(f(x_1, x_3), x_2), f(x_1, f(f(x_1, x_2), f(x_2, x_3))).$$

In sense of the usual superpostition operation S^n_m of terms, the set of all n-ary left variable terms does not closed under such superposition. For an example, let $\tau = (2)$, means that we consider only one binary operation symbol f. Applying the superposition S^2_2. Then $S^2_2(f(x_1, x_2), f(x_2, x_1), x_2) = f(f(x_2, x_1), x_2)$ is not a binary left variable term of type (2), although $f(x_2, x_1)$ and x_2 are binary left variable terms of type (2).

To define the superposition operation on the set of all left variable terms of type τ, we need the concept of subterms which was introduced in [2]. By $var(t)$, we denote the set of all variables that occur in a term t.

Definition 2.4. ([2]) Let $t \in W_\tau(X)$. A *subterm* t, is defined inductively as follows:

(1) Every variable $x \in var(t)$ is a subterm of t.

(2) If $t = f_i(t_1, \ldots, t_{n_i})$, then t itself, $t_1, \ldots, t_{n_i}$ and all subterms of t_j, $1 \leq j \leq n_i$, are subterms of t.

Let $sub(t)$ be the set of all subterms of t. As an example, if $t = f(x_1, f(x_2, x_3))$, then $sub(t) = \{t, x_1, f(x_2, x_3), x_2, x_3\}$ is the set of all subterms of t.

For any term t, by $leftmost(t)$ we mean the first variable (from the left) occurring in a term t. Next, we define the set of all leftmosts of any subterm which is not a variable of a term t.

$$\overline{leftmost(t)} := \{leftmost(s) \mid s \in sub(t) \setminus X\}.$$

For more understanding of that set, we give a noticeable example. Consider a term $t = f(f(x_3, x_1), f(x_5, f(x_1, x_2)))$, then $sub(t) \setminus X = \{t, f(x_3, x_1), f(x_5, f(x_1, x_2)), f(x_1, x_2)\}$ and so $\overline{leftmost(t)} = \{x_1, x_3, x_5\}$.

The following lemmas are needed for setting the many-sorted operation on the set of left variable terms.

Lemma 2.5. *If* $t = x_i \in W^{lv}_\tau(X_n), s_1, \ldots, s_n \in W^{lv}_\tau(X_m)$, *then* $S^n_m(t, s_1, \ldots, s_n) \in W^{lv}_\tau(X_m)$.

Proof. The proof is clear. □

Lemma 2.6. *Let* $t = f_i(t_1, \ldots, t_{n_i}) \in W^{lv}_\tau(X_n)$ *with* $\overline{leftmost(t)} = \{x_{j_1}, \ldots, x_{j_k}\}$. *If* $s_1, \ldots, s_n \in W^{lv}_\tau(X_m)$ *and* s_{j_p} *is a variable in* X_m *for all* $p = 1, \ldots, k$, *then we have*

$$S^n_m(f_i(t_1, \ldots, t_{n_i}), s_1, \ldots, s_n) \in W^{lv}_\tau(X_m).$$

Proof. By the definition of superposition S^n_m, $n, m \geq 1$, we show that $S^n_m(t_j, s_1, \ldots, s_n)$ is a left variable term for all $1 \leq j \leq n_i$ and if $\overline{leftmost(t)} = \{x_{j_1}, \ldots, x_{j_k}\}$ then $S^n_m(t_{j_p}, s_1, \ldots, s_n) \in X_m$ for all $p = 1, \ldots, k$. Since t_j is a left variable term and $\overline{leftmost(t_j)} \subseteq \overline{leftmost(t)}$ for all $j = 1, \ldots, n_i$, by the assumption we have that $S^n_m(t_j, s_1, \ldots, s_n)$ is also a left variable term. Since $\overline{leftmost(t)} = \{x_{j_1}, \ldots, x_{j_k}\}$, we obtain $S^n_m(t_{j_p}, s_1, \ldots, s_n) \in X_m$. The latter term is a variable from X_m because $s_{j_p} \in X_m$ for all $p = 1, \ldots, k$. The proof is finished. □

As a direct consequence of Lemma 2.5 and 2.6, we can define the partial many-sorted mapping

$$S^{lv}{}^n_m : W^{lv}_\tau(X_n) \times (W^{lv}_\tau(X_m))^n \multimap\!\!\rightarrow W^{lv}_\tau(X_m)$$

by

$$S^{lv}{}^{n}_{m}(t,s_1,\ldots,s_n) = \begin{cases} S^n_m(t,s_1,\ldots,s_n) & \text{if } \overline{leftmost(t)} = \{x_{j_1},\ldots,x_{j_k}\} \text{ and} \\ & s_{j_p} \in X_m \text{ for all } p=1,\ldots,k; \\ \text{not defined} & \text{otherwise.} \end{cases}$$

An interesting example is provided in the following.

Example 2.7. Let $\tau = (2)$ be a type. This means that we consider only one binary operation symbol, say g. We illustrate the calculation of the partial operation $S^{lv}{}^{n}_{m}$ where $n = 3$ and $m = 4$. If we set $(g(x_1, g(x_3, x_2)), x_4, g(x_2, x_3), x_1) \in domS^{lv}{}^{3}_{4}$, then$S^{lv}{}^{3}_{4}(g(x_1, g(x_3, x_2)), x_4, g(x_2, x_3), x_1)$ $= S^3_4(g(x_1, g(x_3, x_2)), x_4, g(x_2, x_3), x_1) = g(x_4, g(x_1, g(x_2, x_3))) \in W^{lv}_{(2)}(X_4)$. On the other hand, if $(g(x_2, x_3), x_2, g(x_4, x_4), x_1) \notin domS^{lv}{}^{3}_{4}$ implies that $S^{lv}{}^{3}_{4}(g(x_2, x_3), x_2, g(x_2, x_3), x_1)$ is not defined.

Based on the concept of clone, for example, in [6, 12], we then form an algebraic structure for left variable terms. Using the sequence of the set of all left variable terms of type τ and the partial many-sorted superposition defined on them, one can construct a many-sorted algebra,

$$clone^{lv}(\tau) = ((W^{lv}_\tau(X_n))_{n\in\mathbb{N}^+}, (S^{lv}{}^{n}_{m})_{n,m\in\mathbb{N}^+}, (x_i)_{i\le n, n\in\mathbb{N}^+}),$$

which is called the *partial clone of left variable terms of type* τ.

Unlike algebra, the concept of identity in partial algebra is different [6]. Let s, t be terms of many-sorted partial algebra A. An equation $s \approx t$ is said to be a *weak identity* in A if one side is defined and the other side is defined and both sides are equal.

We now prove the primary result of the paper showing that the partial many-sorted superposition of left variable terms satisfies the weak identities (C1)-(C3).

Theorem 2.8. *The partial clone of left variable terms of type τ satisfies the following weak identities $(C1), (C2), (C3)$ for every natural numbers $m, n, p \geq 1$:*

(C1) $S^{lv}{}^{n}_{m}(S^{lv}{}^{p}_{n}(t, t_1, \ldots, t_p), s_1, \ldots, s_n)$
$= S^{lv}{}^{p}_{m}(t, S^{lv}{}^{n}_{m}(t_1, s_1, \ldots, s_n), \ldots, S^{lv}{}^{n}_{m}(t_p, s_1, \ldots, s_n))$
whenever $t_1, \ldots, t_p \in W_\tau(X_n)$, $s_1, \ldots, s_n \in W_\tau(X_m)$.

(C2) $S^{lv}{}^{n}_{m}(x_i, t_1, \ldots, t_n) = t_i$ *whenever* $t_1, \ldots, t_n \in W_\tau(X_m)$ *for all* $i \in \{1, \ldots, n\}$.

(C3) $S^{lv}{}^{n}_{n}(t, x_1, \ldots, x_n) = t$.

Proof. First, we prove that (C1) is valid. Let $u \in W^{lv}_\tau(X_p)$, $s_1, \ldots, s_p \in W^{lv}_\tau(X_n)$ and $t_1, \ldots, t_n \in W^{lv}_\tau(X_m)$ for all natural numbers m, n, p. If $\overline{leftmost(u)} = \{x_{j_1}, \ldots, x_{j_k}\}$ and $s_{j_q} \in X_n$ for all $q = 1, \ldots, k$, then we have $S^{lv}{}^{p}_{n}(u, s_1, \ldots, s_p) = S^p_n(u, s_1, \ldots, s_p)$. If $\overline{leftmost(S^p_n(u, s_1, \ldots, s_p))} = \{x_{i_1}, \ldots, x_{i_k}\}$ and $t_{i_q} \in X_m$ for all $q = 1, \ldots, k$, then the left hand side is defined and equals to $S^n_m(S^p_n(u, s_1, \ldots, s_p), t_1, \ldots, t_n)$. Since $s_{j_p} \in X_n$ for all $p = 1, \ldots, k$, then $S^{lv}{}^{n}_{m}(s_{j_p}, t_1, \ldots, t_n)$ is defined and equals to $S^n_m(s_{j_p}, t_1, \ldots, t_n)$. For any $i \in \{1, \ldots, p\} \setminus \{j_1, \ldots, j_k\}$, we have that $S^{lv}{}^{n}_{m}(s_i, t_1, \ldots, t_n)$ is defined and equals to the usual superposition $S^n_m(s_i, t_1, \ldots, t_n)$. Then the right hand side of (C1) is defined and equals to $S^p_m(u, S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_n, t_1, \ldots, t_n))$. It is obviously clear that the proof (C2) follows directly form Lemma 2.5. In order to prove (C3), we replace the variables in such equation by arbitrary left variable terms s and nullary opearation symbols by $x_1, \ldots, x_n$. Then $S^{lv}{}^{n}_{n}(s, x_1, \ldots, x_n) = S^n_n(s, x_1, \ldots, x_n) = s$. The proof is completed.

□

3 Left Variable Term Operations

In this section, one concrete operation can be defined by using a left variable term. Let m, n be positive intergers. For any algebra $\mathcal{A}$ of type τ, we define partial many-sorted superposition operations on sets of left variable term operations

$$S^{lv}{}^{n,A}_{m} : W^{lv}_\tau(X_n)^{\mathcal{A}} \times (W^{lv}_\tau(X_m)^{\mathcal{A}})^n \multimap W^{lv}_\tau(X_m)^{\mathcal{A}}$$

by:

$$S^{lv}{}^{n,A}_{m}(t^{\mathcal{A}}, s_1^{\mathcal{A}}, \ldots, s_n^{\mathcal{A}}) = \begin{cases} S_m^{n,A}(t^{\mathcal{A}}, s_1^{\mathcal{A}}, \ldots, s_n^{\mathcal{A}}) & \text{if } \overline{leftmost(t)} = \{x_{j_1}, \ldots, x_{j_k}\} \text{ and} \\ & s_{j_p} \in X_m \text{ for all } p = 1, \ldots, k; \\ \text{not defined} & \text{otherwise.} \end{cases}$$

In general, the partial operation $S^{lv}{}^{n,A}_{m}$ defined on an arbitrary algebea $\mathcal{A}$ of type τ need not necessaliry be well-defined. For example, let $\mathcal{A}$ be a commutative groupoid, i.e., it is an algebra of type (2) with a binary operation symbol f. Let

$$s = f(x_1, x_2), r = f(x_2, x_1), t_1 = x_1, t_2 = f(x_1, x_2).$$

Clearly, $s, r, t_1, t_2 \in W^{lv}_{(2)}(X_2)$. Since $\overline{leftmost(s)} = \{x_1\}$ and $t_1 = x_1 \in X_2$, we have

$$S^{lv}{}^{2,A}_{2}(s^{\mathcal{A}}, t_1^{\mathcal{A}}, t_2^{\mathcal{A}}) = S_2^{2,A}(s^{\mathcal{A}}, t_1^{\mathcal{A}}, t_2^{\mathcal{A}}) = (f(x_1, f(x_1, x_2)))^{\mathcal{A}}.$$

On the other hand, $S^{lv}{}^{2,A}_{2}(r^{\mathcal{A}}, t_1^{\mathcal{A}}, t_2^{\mathcal{A}})$ is undefined since $\overline{leftmost(r)} = \{x_2\}$ and $t_2 \notin X_n$. But $s^{\mathcal{A}} = r^{\mathcal{A}}$ by the commutivity of $\mathcal{A}$, so we should have $S^{lv}{}^{2,A}_{2}(s^{\mathcal{A}}, t_1^{\mathcal{A}}, t_2^{\mathcal{A}}) = S^{lv}{}^{2,A}_{2}(r^{\mathcal{A}}, t_1^{\mathcal{A}}, t_2^{\mathcal{A}})$, which is a contradiction.

To ensure that $S^{lv}{}^{n,A}_{m}$ are partial operations for all $n, m \geq 1$, some conditions are considered.

Lemma 3.1. *Let m, n be positive intergers and let $\mathcal{A}$ be an algebra of type τ. Then the partial operations $S^{lv}{}^{n,A}_{m}$ are well-defined if any left variable terms $s, r \in W^{lv}_{\tau}(X_n)$ and $t_1, \ldots, t_n$, $u_1, \ldots, u_n \in W^{lv}_{\tau}(X_m)$ satisfy the following conditions:*

(A) $t_{j_p} \in X_m$ *for all* $p = 1, \ldots, k$ *if* $\overline{leftmost(s)} = \{x_{j_1}, \ldots, x_{j_k}\}$,

(B) $u_{j_p} \in X_m$ *for all* $p = 1, \ldots, q$ *if* $\overline{leftmost(r)} = \{x_{j_1}, \ldots, x_{j_q}\}$.

Proof. The aim of this lemma is to show that $S^{lv}{}^{n,A}_{m}$ is well defined for any positive integers m, n. To do this, suppose that $s, r \in W^{lv}_{\tau}(X_n)$ and $t_1, \ldots, t_n, u_1, \ldots, u_n \in W^{lv}_{\tau}(X_m)$ satisfying the conditions (A), (B), and $(s^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}) = (r^{\mathcal{A}}, u_1^{\mathcal{A}}, \ldots, u_n^{\mathcal{A}})$. It follows directly that $s^{\mathcal{A}} = r^{\mathcal{A}}, t_j^{\mathcal{A}} = u_j^{\mathcal{A}}$ for all $j = 1, \ldots, n$. Then $s \approx t, t_j \approx u_j$ are identities in $Id\mathcal{A}$ for every $j = 1, \ldots, n$. Since $Id\mathcal{A}$ is congruence in the many-sorted algebra $clone(\tau)$, we obtain that $S_m^n(s, t_1, \ldots, t_n) \approx S_m^n(r, u_1, \ldots, u_n)$ is an identity in $Id\mathcal{A}$. This implies that

$$(S_m^n(s, t_1, \ldots, t_n))^{\mathcal{A}} = (S_m^n(r, u_1, \ldots, u_n))^{\mathcal{A}}$$

and thus

$$S_m^{n,\mathcal{A}}(s^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}) = S_m^{n,\mathcal{A}}(r^{\mathcal{A}}, u_1^{\mathcal{A}}, \ldots, u_n^{\mathcal{A}}).$$

Hence $S^{lv}{}^{n,A}_{m}(s^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}) = S^{lv}{}^{n,A}_{m}(r^{\mathcal{A}}, u_1^{\mathcal{A}}, \ldots, u_n^{\mathcal{A}})$ by the assumption. □

Now the many-sorted partial algebra of left variable term operations and partial operation $S^{lv}{}^{n,A}_{m}$ is constructed as follows:

$$clone^{lv}\mathcal{A} = ((W^{lv}_{\tau}(X_n)^{\mathcal{A}})_{n \in \mathbb{N}^+}, (S^{lv}{}^{n,A}_{m})_{n,m \in \mathbb{N}^+}, (e_i^{n,A})_{i \leq n, n \in \mathbb{N}^+})$$

The algebraic properties of the partial many-sorted algebra $clone^{lv}\mathcal{A}$ are investigated.

Theorem 3.2. *The many-sorted partial algebra $clone^{lv}\mathcal{A}$ satisfies $(C1), (C2), (C3)$ as weak identities.*

Proof. It can be proved in the same precess of Theorem 2.8. □

If $\mathcal{A}, \mathcal{B}$ are partial algebras of the same type with indexed sets $\{f_i^{\mathcal{A}} \mid i \in I\}$ and $\{f_i^{\mathcal{B}} \mid i \in I\}$ of partial operations on A and B, respectively, then by a weak homomorphism we mean a mapping $\phi : A \to B$ satisfying: if $(a_1, \ldots, a_{n_i}) \in dom f_i^{\mathcal{A}}$, then $(\phi(a_1), \ldots, \phi(a_{n_i})) \in dom f_i^{\mathcal{B}}$ and then, for all $i \in I$,

$$\phi(f_i^{\mathcal{A}}(a_1, \ldots, a_{n_i})) = f_i^{\mathcal{B}}(\phi(a_1), \ldots, \phi(a_{n_i})).$$

The situation for many-sorted partial algebra is the same concept.

Theorem 3.3. *For every algebra $\mathcal{A}$ of type τ, the partial many-sorted algebra clone$^{lv}\mathcal{A}$ is a weak homomorphic image of clone$^{lv}(\tau)$.*

Proof. Let n be a positive integer. We define a maping $\phi_n : W_\tau^{lv}(X_n) \to (W_\tau^{lv}(X_n))^{\mathcal{A}}$ by $\phi_n(x_i) := x_i^{\mathcal{A}} = e_i^{n,A}$ for all $1 \leq i \leq n$ and $\phi_n(f_i(t_1,\ldots,t_{n_i})) := (f_i(t_1,\ldots,t_{n_i}))^{\mathcal{A}} = f_i^{\mathcal{A}}(t_1^{\mathcal{A}},\ldots,t_{n_i}^{\mathcal{A}}) = f_i^{\mathcal{A}}(\phi_n(t_1),\ldots,\phi_n(t_{n_i}))$, assumed that $\phi_n(t_j) := t_j^{\mathcal{A}}, 1 \leq j \leq n_i$, are already known. If is clear that the mappings ϕ_n are well-defined since $s = t$ implies $s \approx t \in Id\mathcal{A}, i.e., s^{\mathcal{A}} = t^{\mathcal{A}}$. Assume that $(s,t_1,\ldots,t_n) \in domS_m^n$. Then $s \in W_\tau^{lv}(X_n), t_1,\ldots,t_n \in W_\tau^{lv}(X_m)$ and $t_{j_p} \in X_m$ for all $p = 1,\ldots,k$ if $\overline{leftmost(s)} = \{x_{j_1},\ldots,x_{j_k}\}$ and $(\phi_n(s),\phi_m(t_1),\ldots,\phi_m(t_n)) = (s^{\mathcal{A}},t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}) \in domS^{lv}{}_m^{n,A}$.

Furthermore, since we known that $(S_m^n(s,t_1,\ldots,t_n))^{\mathcal{A}} = S_m^{n,A}(s^{\mathcal{A}},t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}})$, we have $\phi_m(S_m^n(s,t_1,\ldots,t_n)) = S^{lv}{}_m^{n,A}(\phi_n(s),\phi_m(t_1),\ldots,\phi_m(t_n))$. This shows that $\phi = (\phi_n)_{n\geq 1}$ is a weak homomorphism from $clone^{lv}(\tau)$ to $clone^{lv}\mathcal{A}$. ◻

4 The Partial Monoid of Left Variable Hypersubstitutions

We begin this section with recalling the concepts and some notations of hypersubstitutions. Some backgrounds and current developments of hypersubstitutions can be found in [5, 20, 21, 26]. A mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ preserving the arity of both operation symbols and terms, is called a *hypersubstitution of type* τ. Each hypersubstitution σ can be extended to a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ given by:

(1) $\widehat{\sigma}[x_i] := x_i \in X$,

(2) $\widehat{\sigma}[f_i(t_1,\ldots,t_{n_i})] := S_m^{n_i}(\sigma(f_i),\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_{n_i}])$.

To define a multiplication, denoted by $\circ_h$, on the set $Hyp(\tau)$ of all hypersubstitutions of type τ, we need this extension of mappings. Let σ_1, σ_2 be two hypersubstitutions. The operation $\circ_h$ can be defined by $\sigma_1 \circ_h \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$, where $\circ$ is the usual composition of functions. Generally, $\circ_h$ is associative. Furthermore, under the identity hypersubstitution σ_{id} which takes each n_i-ary operation symbol f_i to $f_i(x_1,\ldots,x_{n_i})$, we obtain the monoid $\mathcal{H}yp(\tau) := (Hyp(\tau),\circ_h,\sigma_{id})$. For research in this area, see [16, 19].

Now we introduce a concept of a specific class of hypersubstitutions which take the operation symbols to left variable terms.

Definition 4.1. A hypersubstitution σ of type τ is said to be *left variable hypersubstitution* if σ maps every n_i-ary operation symbol to an n_i-ary left variable term, i.e.,

$$\sigma : \{f_i \mid i \in I\} \to W_\tau^{lv}(X).$$

Let $Hyp^{lv}(\tau)$ be the set of all left variable hypersubstitutions of type τ.

Now we present an example of left variable hypersubstitutions.

Example 4.2. Let $\tau = (3,2)$ be the type with one ternary operation symbol g and one binary operation symbol f. Let σ be the hypersubstitution taking g to $f(x_3,f(x_1,x_2))$ and f to $g(x_1,x_1,x_2)$. Then $\sigma \in Hyp^{lv}(3,2)$.

Using the partial superposition operation $S^{lv}{}_m^n$, we can define the extension of each left variable hypersubstitution of type τ

$$\widehat{\sigma} : W^{lv}(X) \to W^{lv}(X)$$

by

(1) $\widehat{\sigma}[t] := t$ if t is a variable from X;

(2) $\widehat{\sigma}[t] := S^{lv}{}_n^{n_i}(\sigma(f_i),\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_{n_i}])$ if $t = f(t_1,\ldots,t_{n_i})$ and $\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_{n_i}]$ are already defined.

To demonstrate the process of computation by this definition, let $\tau = (2)$ be a type and $t = f(x_1,f(x_1,x_2))$ be a binary left variable term. Put $\sigma(f) = f(x_2,f(x_2,x_1))$, by applying the extension $\widehat{\sigma}$ of a left variable hypersubstitution σ, then we have

$$
\begin{aligned}
\widehat{\sigma}[t] &= \widehat{\sigma}[f(x_1, f(x_1, x_2))] \\
&= S^{lv}\,{}^2_2(\sigma(f), \widehat{\sigma}[x_1], \widehat{\sigma}[f(x_1, x_2)]) \\
&= S^{lv}\,{}^2_2(f(x_2, f(x_2, x_1)), x_1, f(x_2, f(x_2, x_1))).
\end{aligned}
$$

We see that the partial superposition $S^{lv}\,{}^2_2$ does not defined since $x_2 \in \overline{leftmost(\sigma(f))}$, but $\widehat{\sigma}[f(x_1, x_2)] \notin X_2$. This shows that $\widehat{\sigma}$ does not necessary maps a left variable term to a left variable term.

Generally, to make sure that the extension $\widehat{\sigma}$ of a left variable hypersubstitution sends left variable terms to left variable terms, the following lemmas are needed.

Lemma 4.3. *Let σ be a left variable hypersubstitution of type τ. Then $\widehat{\sigma}[x_i] \in W^{lv}(X)$ for every variable x_i.*

Proof. The proof is obvious. □

Lemma 4.4. *For any left variable hypersubstitution σ of type τ and a left variable term $t = f_i(t_1, \ldots, t_{n_i})$ with the conditions $\overline{leftmost(\sigma(f_i))} = \{x_{j_1}, \ldots, x_{j_k}\}$ and $\widehat{\sigma}[t_{j_q}] \in X$ for all $q = 1, \ldots, k$, then $\widehat{\sigma}[t] \in W^{lv}_\tau(X)$.*

Proof. Let $t = f_i(t_1, \ldots, t_{n_i}) \in W^{lv}_\tau(X)$ and let $\sigma \in Hyp^{lv}(\tau)$. Since, by our the assumption, $\overline{leftmost(\sigma(f_i))} = \{x_{j_1}, \ldots, x_{j_k}\}$ and $\widehat{\sigma}[t_{j_q}] \in X$ for all $q = 1, \ldots, k$, by Lemma 2.6, we have $S^{lv}\,{}^n_m(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]) \in W^{lv}_\tau(X)$. □

Then we develop the definition of $\widehat{\sigma}$ by setting to be a partial mapping

$$\widehat{\sigma} : W^{lv}(X) \multimap W^{lv}(X)$$

which is defined by

$$
\widehat{\sigma}[t] := \begin{cases}
t & \text{if } t \text{ is a variable from } X, \\
S^{lv}\,{}^n_m(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}]) & \text{if } t = f_i(t_1, \ldots, t_{n_i}) \text{ with} \\
& \overline{leftmost(\sigma(f_i))} = \{x_{j_1}, \ldots, x_{j_k}\} \\
& \text{and } \widehat{\sigma}[t_{j_q}] \in X \text{ for all } q = 1, \ldots, k, \\
\text{not defined} & \text{otherwise.}
\end{cases}
$$

Now the partial binary operation $\circ^{lv}_h$ on $Hyp^{lv}(\tau)$ is certainly defined by

$$
\sigma_1 \circ^{lv}_h \sigma_2 = \begin{cases}
\widehat{\sigma}_1 \circ \sigma_2 & \text{if } \sigma_2(f_i) \in dom(\widehat{\sigma}_1) \text{ for all } i \in I; \\
\text{not defined} & \text{otherwise.}
\end{cases}
$$

Consequently, we obtain the partial monoid $(Hyp^{lv}(\tau), \circ^{lv}_h, \sigma_{id})$.

Lemma 4.5. *If for all left variable hypersubstitutions σ_1 and σ_2 if holds that $\sigma_2(f_i) \in dom\widehat{\sigma}_1$ for all $i \in I$, then the partial monoid $(Hyp^{lv}(\tau), \circ^{lv}_h, \sigma_{id})$ is total.*

We write $\overline{Hyp^{lv}(\tau)}$ to designate the fact that the conditions of Lemma 4.5 hold for the set of all left variable hypersubstitutions of type τ, i.e., $(Hyp^{lv}(\tau), \circ^{lv}_h, \sigma_{id})$ is a (total) monoid.

Theorem 4.6. $(\overline{Hyp^{lv}(\tau)}, \circ^{lv}_h, \sigma_{id})$ *is a submonoid of* $(Hyp(\tau), \circ_h, \sigma_{id})$.

Proof. It is observed that σ_{id} is left variable hypersubstitution since $\sigma_{id}(f_i) = f_i(t_1, \ldots, t_{n_i})$ is left variable. Suppose that $\sigma_1, \sigma_2 \in \overline{Hyp^{lv}(\tau)}$. We show that $\sigma_1 \circ_h \sigma_2 \in \overline{Hyp^{lv}(\tau)}$. In fact, we have $(\sigma_1 \circ^{lv}_h \sigma_2)(f_i) = \widehat{\sigma}_1[\sigma_2(f_i)]$. Since $\sigma_2(f_i)$ is a left variable term and since σ_1 is a left variable hypersubstitution, by Lemma 4.3 and 4.4, $\widehat{\sigma}_1[\sigma_2(f_i)]$ is a left variable term. □

Finally we define some interesting subsets of $Hyp^{lv}(\tau)$ and investigate their algebraic properties. For more backgrounds, see [20].

Definition 4.7. Let $\tau = (n_i)_{i \in I}$ be a type with an operation symbol f_i having the arity n_i for each $i \in I$. A left variable hypersubstitution σ in $\overline{Hyp^{lv}(\tau)}$ is said to be

(1) a *projection left variable hypersubstitution* if $\sigma(f_i) \in X_{n_i}$. Let $P^{lv}(\tau)$ be the set of all projection left variable hypersubstitutions of type τ.

(2) a *pre-left variable hypersubstitution* if $\sigma(f_i) \in W_\tau^{lv}(X_{n_i}) \setminus X_{n_i}$. Let $Pre^{lv}(\tau)$ be the set of all pre-left variable hypersubstitutions of type τ.

Proposition 4.8. *$P^{lv}(\tau) \cup \{\sigma_{id}\}$ and $Pre^{lv}(\tau)$ are submonoids of $(\overline{Hyp^{lv}(\tau)}, \circ_h^{lv}, \sigma_{id})$.*

Proof. It is clear that σ_{id} belongs to the sets $P^{lv}(\tau) \cup \{\sigma_{id}\}$ and $Pre^{lv}(\tau)$. Now let $\sigma_1, \sigma_2 \in P^{lv}(\tau) \cup \{\sigma_{id}\}$. To show that the set $P^{lv}(\tau) \cup \{\sigma_{id}\}$ is closed under the binary operation $\circ_h$, we consider in four cases. Case 1: $\sigma_1, \sigma_2 \in P^{lv}(\tau)$. Then both $\sigma_1(f_i)$ and $\sigma_2(f_i)$ are variables for each $i \in I$. Hence $(\sigma_1 \circ_h \sigma_2)(f_i) \in X_{n_i}$. Case 2: $\sigma_1 \in P^{lv}(\tau)$ and $\sigma_2 \in \{\sigma_{id}\}$. Then $(\sigma_1 \circ_h^{lv} \sigma_2)(f_i) = \widehat{\sigma}[\sigma_2(f_i)] = \widehat{\sigma}[f_i(x_1, \ldots, x_{n_i})] = S^{lv}{}_{n_i}^{n_i}(\sigma_1(f_i), x_1, \ldots, x_{n_i}) \in X_{n_i}$. Case 3: $\sigma_1 \in \{\sigma_{id}\}$ and $\sigma_2 \in P^{lv}(\tau)$. Then $(\sigma_1 \circ_h^{lv} \sigma_2)(f_i) = \widehat{\sigma_1}[\sigma_2(f_i)] \in X_{n_i}$. Case 4: $\sigma_1, \sigma_2 \in \{\sigma_{id}\}$. The proof is obvious. Hence $P^{lv}(\tau) \cup \{\sigma_{id}\}$ is a submonoid of $\overline{Hyp^{lv}(\tau)}$. Finally, it is not hard to verify that the composition of two pre-left variable hypersubstitutions is also a pre-left variable hypersubstitution. Thus $Pre^{lv}(\tau)$ is a submonoid of $\overline{Hyp^{lv}(\tau)}$. □

5 Final Remarks and Conclusions

In this paper, we introduced a specific class of terms, called left variable terms. This concept can be applied to define an identity for classifying algebras of the same type to the varieties. It is evidently observed that the commutative law is a basic example of left variable identity which is used to classify the algebras of type (2) to the variety of commutative groupoids. In addition, a novel concept of left variable hypersubstitutions of type τ was introduced. There are several technique to construct the partial monoid of such notion. Finally, the defining of left variable hyperidentity and left variable solid variety via these ideas that mentioned completely in this paper are also challenging problems to do the research in the near future. Moreover, based on a number of papers concerning the freeness of algebras, for example in [27], free structures of left variable terms can be studied.

References

[1] E. Aichinger, N. Mudrinski, J. Opršal, Complexity of term representations of finitary functions, *Int. J. Algebra Comput.*, **28** (2018), no.6, 1101-1118.

[2] A. Boonmee, S. Leeratanavalee, All intra-regular generalized hypersubstitutions of type (2), *Acta Univ. Sapientiae, Mathematica*, **11** (2019), 29-39.

[3] S. Bozapalidis, Z. Fülöp, G. Rahonis, Equational tree transformations, *Theoret. Comput. Sci.*, **412** (2011), 3676-3692.

[4] T. Changphas, B. Pibaljommee, K. Denecke, The monoid of linear hypersubstitutions, *Kyungpook Math. J.,* **59** (2019), 617-629.

[5] N. Chansuriya, All maximal idempotent submonoids of generalized cohypersubstitutions of type $\tau = (2)$, *Discuss. Math. Gen. Algebra Appl.,* **41** (2021), 45-54.

[6] K. Denecke, Partial clones, *Asian-Eur. J. Math.,* **13** (2020), no. 8, Article ID: 2050161 (19 pages).

[7] K. Denecke, The partial clone of linear terms, *Sib. Math J.*, **57** (2016), 589-598.

[8] K. Denecke, SL. Wismath, *Hyperidentities and Clones,* Gordon and Breach Science Publishers, The Netherlands, 2000.

[9] K. Denecke, SL. Wismath, *Universal Algebra and Applications in Theoretical Computer Science*, Chapman and Hall/CRC (2002), Boca Raton-London-New York-Washington D.C., USA, 2002.

[10] W.A. Dudek, V.S. Trokhimenko, On (i, j)-commutativity in Menger algebras of n-place functions, *Quasigroups Relat. Syst.,* **24** (2016), no. 2, 219–230.

[11] Y. Guellouma, H. Cherroun, From tree automata to rational tree expressions, *Int. J. Found. Comput. Sci.*, **29** (2018), 1045-1062.

[12] S. Kerhoff, R. Poschel, F.M. Schneider, A short introduction to clones,*Electron. Notes Theoret. Comput. Sci.,* **303** (2014), 107-120.

[13] J. Koppitz, All Reg-solid varieties of commutative semigroups, *Semigroup Forum*, **78** (2009), 148-156.

[14] J. Koppitz, K. Denecke, *M-solid Varieties*, Springer-Verlag: New York, USA, 2006.

[15] T. Kumduang, S. Leeratanavalee, Left translations and isomorphism theorems for Menger algebras of rank n, *Kyungpook Math. J.*, **61** (2021), no. 2, 223-237.

[16] T. Kumduang, S. Leeratanavalee, Monoid of linear hypersubstitutions for algebraic systems of type $((n),(2))$ and its regularity, *Songklanakarin Journal of Science and Technology,* **41** (2019), no. 6, 1248-1259.

[17] T. Kumduang, S. Leeratanavalee, Semigroups of terms, tree languages, Menger algebra of n-ary functions and their embedding theorems, *Symmetry*, **13** (2021), no. 4, 558.

[18] T. Kumduang, S. Leeratanavalee, Menger systems of idempotent cyclic and weak near-unanimity multiplace functions, *Asian-Eur. J. Math.,* doi.org/10.1142/S1793557122501625.

[19] S. Leeratanavalee, Outermost-strongly solid variety of commutative semigroups, *Thai. J. Math.*, **14** (2016), no. 2, 305-313.

[20] S. Leeratanavalee, Submonoids of generalized hypersubstitutions, *Demonstr. Math.*, **XL** (2007), 13-22.

[21] S. Phuapong, T. Kumduang, Menger algebras of terms induced by transformations with restricted range, *Quasigroup and related system*, **29** (2021), no. 2, 255-268.

[22] D. Phusanga, J. Koppitz, The semigroup of linear terms, *Asian-Eur. J. Math.,* **13**, (2020), no. 1, Article ID: 20500059 (9 pages).

[23] L. Polak, All solid varieties of semigroups, *J. Algebra* **219** (1999), 421–436.

[24] S. Shtrakov, J. Koppitz, Stable varieties of semigroups and groupoids, *Algebra Univers.,* **75** (2016), 85–106.

[25] K. Wattanatripop, T. Kumduang, T. Changphas, S. Leeratanavalee, Power Menger algebras of terms induced by order-decreasing transformations and superpositions, *Int. J. Math. Comput. Sci.*, **16** (2021), no. 4, 1683-1696.

[26] K. Wattanatripop, T. Changphas, The Menger algebra of terms induced by order-decreasing transformations, *Comm. Algebra*, **49** (2021), 3114-3123.

[27] A.V. Zhuchok, J. Koppitz, Free products ofn-tuple semigroups. *Ukrainian Math. J.* **70** (2019), no. 11, 1710–1726 .

Author information

Nuthawud Sungtong, Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna, Chiang Mai 50200, Thailand.
E-mail: `nuthawud@gmail.com`

Thodsaporn Kumduang, Department of Mathematics, Faculty of Science Chiang Mai University, Chiang Mai 50200, Thailand.
E-mail: `kumduang01@gmail.com`

Sarawut Phuapong, Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna, Chiang Mai 50200, Thailand.
E-mail: `Corresponding author:phuapong.sa@gmail.com`

Received: December 20th, 2021
Accepted: February 21st, 2022