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A NEW CONVERGENCE THEOREM FOR A SYSTEM OF VARIATIONAL-LIKE INCLUSIONS AND FIXED POINT PROBLEMS

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Abstract: In this manuscript, using $(H-\eta)-$ accretive mappings we study the existence of common solution of a system of generalized mixed variational-like inclusion problems and the set of fixed point problems in q-uniformly smooth Banach spaces. The method used in this paper can be considered as an extension of methods for studying the existence of common solution for various classes of variational inclusions considered and studied by many authors in q-uniformly smooth Banach spaces.

1. Introduction

A widely studied problem known as variational inclusion problem have many applications in the fields of optimization and control, economics and transportation equilibrium, engineering sciences, etc. Several researchers used different approaches to develop iterative algorithms for solving various classes of variational inequality and variational inclusion problems. For details, we refer,[7,11,13] and the references therein.

Equally important for the variational inequalities and variational inclusion problems, we also have the problem of finding the fixed points of the nonlinear mappings, which is a subject of current interest. In this direction, several authors have introduced some iterative schemes for finding a common element of a set of the solutions of the variational problems and a set of the fixed points of nonlinear mappings, see [4,9,12] and the references therein.

Motivated and inspired by the above works and by the ongoing research in this direction, in this paper, we introduce and study a system of generalized mixed variational-like inclusion problems and the set of fixed point problems in q-uniformly smooth Banach spaces.

2. Resolvent Operator and Formulation of Problem

We need the following definitions and results from the literature.

Let X be a real Banach space equipped with norm $\|.\|$ and $X^\star$ be the topological dual space of X. Let $< .,. >$ be the dual pair between X and $X^\star$ and 2^X be the power set of X.

Definition 0.1. Definition 2.1[13]. For $q>1$, a mapping $J_q : X \to 2^{X^\star}$ is said to be generalized duality mapping, if it is defined by

$$J_q(x) = \{f \in X^\star : \langle x, f\rangle = \|x\|^q, \|x\|^{q-1} = \|f\|\}, \ \ \forall x \in X.$$

In particular, J_2 is the usual normalized duality mapping on X. It is well known (see, e.g., [13]) that

$$J_q(x) = \|x\|^{q-2} J_2(x), \ \ \forall x (\neq 0) \in X.$$

Note that if $X \equiv H$, a real Hilbert space, then J_2 becomes the identity mapping on X.

Definition 0.2. Definition 2.2[13]. A Banach space X is said to be smooth if, for every $x \in X$ with $\|x\| = 1$, there exists a unique $f \in X^\star$ such that $\|f\| = f(x) = 1$.

The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$, defined by

$$\rho_X(\sigma) = \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in X, \|x\| = 1, \|y\| = \sigma \right\}$$

Definition 0.3. Definition 2.3[13]. A Banach space X is said to be

(i) uniformly smooth if $\lim_{\sigma \to 0} \dfrac{\rho_X(\sigma)}{\sigma} = 0$,

(ii) q-uniformly smooth, for $q > 1$, if there exists a constant $c > 0$ such that $\rho_X(\sigma) \leq c\sigma^q,\ \sigma \in [0, \infty)$.

It is well known (see,e.g.,[14]) that

$$L_q(or\ l_q) \text{ is } \begin{cases} q-\text{uniformlysmooth}, & \text{if } 1 < q \leq 2, \\ \\ 2-\text{uniformlysmooth}, & \text{if } q \geq 2. \end{cases}$$

Note that if X is uniformly smooth, J_q becomes single-valued. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [13] established the following lemma.

Lemma 0.4. Lemma 2.4[13]. *Let $q > 1$ be a real number and let X be a smooth Banach space. Then the following statements are equivalent:*

(i) X is q-uniformly smooth.

(ii) There is a constant $c_q > 0$ such that for every $x, y \in X$, the following inequality holds

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Definition 0.5. Definition 2.5. Let X be a q-uniformly smooth Banach space. Let $H : X \to X$, $\eta : X \times X \to X$ be single-valued mappings and $M : X \times X \to 2^X$ be multi-valued mapping. Then

(i) H is said to be η-accretive, if

$$\Big\langle Hx - Hy, J_q\Big(\eta(x, y)\Big)\Big\rangle \geq 0, \ \ \forall x, y \in X.$$

(ii) H is said to be strictly η-accretive, if H is η-accretive and equality holds if and only if $x = y$.

(iii) H is said to be k-strongly η-accretive if there exists a constant $k > 0$ such that

$$\Big\langle Hx - Hy, J_q(\eta(x, y))\Big\rangle \geq k\|x - y\|^q, \ \ \forall x, y \in X.$$

(iv) η is said to be τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau\|x - y\|, \ \ \forall x, y \in X.$$

(v) M is said to be η-accretive in the first argument if

$$\Big\langle u - v, J_q(\eta(x, y))\Big\rangle \geq 0, \ \ \forall x, y \in X, \ \ \forall u \in M(x, t), v \in M(y, t), \ \text{for each fixed } t \in X.$$

(vi) M is said to be strictly η-accretive, if M is η-accretive in the first argument and equality holds if and only if $x = y$.

Definition 0.6. Definition 2.6. Let X be a q-uniformly smooth Banach space. Let $T : X \to X$, $N, \eta : X \times X \to X$ be single-valued mappings and $S_1, S_2 : X \to X$ be nonlinear mappings. Then

(i) T is said to be μ-η-cocoercive if there exists a constant $\mu > 0$ such that

$$\Big\langle Tx - Ty, J_q(\eta(x, y))\Big\rangle \geq \mu\|Tx - Ty\|^q, \ \ \forall x, y \in X.$$

(ii) N is said to be (r, δ)-mixed Lipschitz continuous if there exist constants $r > 0, \delta > 0$ such that

$$\|N(x, z) - N(y, t)\| \leq r\|x - y\| + \delta\|z - t\|, \ \forall x, y, z, t \in X.$$

(iii) S_1, S_2 is said to be m_1, m_2-Lipschitz continous, respectively, if there exists a constant $m_1, m_2 > 0$ such that

$$\|S_1(x_1) - S_1(x_2)\| \leq m_1\|x_1 - x_2\|, \ \forall x_1, x_2 \in X.$$

$$\|S_2(y_1) - S_2(y_2)\| \leq m_2\|y_1 - y_2\|, \ \forall y_1, y_2 \in X.$$

We denote by $K(S_1, S_2)$ the set of fixed points of S_1, S_2 such that $K(S_1, S_2) = \{(x, y) \in X \times X : S_1(x) = x, S_2(y) = y.\}$

Throughout the rest of the paper unless otherwise stated, we assume X to be q-uniformly smooth Banach space.

Definition 0.7. Definition 2.7. Let $H : X \to X, \eta : X \times X \to X$ be single-valued mappings, $M : X \times X \to 2^X$ be a multi-valued mapping, then M is said to be $(H - \eta)-$accretive mapping if for each fixed $t \in X, M(., t)$ is η-accretive in the first argument and $(H + \lambda M(., t))X = X$, $\forall \lambda > 0$.

Lemma 0.8. Lemma 2.8[6]. *Let $\{\zeta^n\}, \{\nu^n\}$ and $\{c^n\}$ be nonnegative sequences satisfying*

$$\zeta^{n+1} \leq (1 - \omega^n)\zeta^n + \omega^n\nu^n + c^n, \ \forall n \geq 0,$$

where $\{\omega^n\}_{n=0}^{\infty} \subset [0, 1]$, $\sum\limits_{n=0}^{\infty} \omega^n = +\infty$, $\lim\limits_{n\to\infty} \nu^n = 0$ and $\sum\limits_{n=0}^{\infty} c^n < \infty$. Then $\lim\limits_{n\to\infty} \zeta^n = 0$.

Definition 0.9. Definition 2.9. The Hausdorff metric $\mathcal{D}(\cdot, \cdot)$ on $CB(X)$, is defined by

$$\mathcal{D}(B, P) = \max\left\{\sup_{u\in B}\inf_{v\in P} d(u, v), \ \sup_{v\in P}\inf_{u\in B} d(u, v)\right\}, \ B, P \in CB(X),$$

where $d(\cdot, \cdot)$ is the induced metric on X and $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X.

Definition 0.10. Definition 2.10[3]. A set-valued mapping $P : X \to CB(X)$ is said to be γ-$\mathcal{D}$-Lipschitz continuous, if there exists a constant $\gamma > 0$ such that

$$\mathcal{D}(P(x), P(y)) \leq \gamma\|x - y\|, \ \forall x, y \in X.$$

Theorem 0.11. Theorem 2.11(Nadler [8]). *Let $P : X \to CB(X)$ be a set-valued mapping on X and (X, d) be a complete metric space. Then:*

(i) For any given $\mu > 0$ and for any given $x, y \in X$ and $u \in P(x)$, there exists $v \in P(y)$ such that

$$d(u, v) \ \leq \ (1 + \mu)\mathcal{D}(P(x), P(y)).$$

(ii) If $P : X \to C(X)$, then (i) holds for $\mu = 0$, (where $C(X)$ denotes the family of all nonempty compact subsets of X).

Theorem 0.12. Theorem 2.12. *Let $H : X \to X$, $\eta : X \times X \to X$ be single-valued mappings. Let $H : X \to X$ be k-strongly η-accretive, $M : X \times X \to 2^X$ be $(H - \eta)-$accretive mapping. If the following inequality : $\Big\langle u - v, J_q(\eta(x, y))\Big\rangle \geq 0$, holds $\forall (y, v) \in$ Graph $(M(., t))$, then $(x, u) \in$ Graph$(M(., t))$, where Graph $(M(., t)) := \{(x, u) \in X \times X : u \in M(x, t)\}$.*

Theorem 0.13. Theorem 2.13. *Let $H : X \to X$, $\eta : X \times X \to X$ be single-valued mappings. Let $H : X \to X$ be k-strongly η-accretive, $M : X \times X \to 2^X$ be $(H-\eta)-$accretive mappings. Then the mapping $(H + \lambda M(.,t))^{-1}$ is single-valued,* $\forall\ \lambda > 0$.

Definition 0.14. Definition 2.14. Let $H : X \to X$, $\eta : X \times X \to X$ be single-valued mappings. Let $H : X \to X$ be k-strongly η-accretive, $M : X \times X \to 2^X$ be $(H-\eta)-$accretive mappings. Then for each fixed $t \in X$, the resolvent operator $R^{H,\eta}_{M(.,t),\lambda} : X \to X$ is defined by

$$R^{H,\eta}_{M(.,t),\lambda}(x) = (H + \lambda M(.,t))^{-1}(x), \quad \forall x \in X. \tag{2.1}$$

Now, we prove the following result which guarentees the Lipschitz continuity of the resolvent operator $R^{H,\eta}_{M(.,t),\lambda}$.

Theorem 0.15. Theorem 2.15. *Let $H : X \to X$ be k-strongly η-accretive and $\eta : X \times X \to X$ be τ-Lipschitz continuous. Let $M : X \times X \to 2^X$ be $(H-\eta)-$accretive mappings. Then for each fixed $t \in X$, the resolvent operator $R^{H,\eta}_{M(.,t),\lambda} : X \to X$ is Lipschitz continuous with constant L, that is,*

$$\|R^{H,\eta}_{M(.,t),\lambda}(x) - R^{H,\eta}_{M(.,t),\lambda}(y)\| \le L\|x-y\|, \quad \forall x, y \in X, \text{ where } L := \frac{\tau^{q-1}}{k}. \tag{2.2}$$

Definition 0.16. Definition 2.16. Let $H : X \to X, \eta : X \times X \to X$ be single-valued mappings, let $\{M^n\}, M^n : X \to 2^X$ be a sequence of $(H-\eta)-$accretive mappings. A sequence $\{M^n\}_{n\ge 0}$ is said to be graph convergent to M, denoted by $M^n \xrightarrow{G} M$, if for each $(x,u) \in graph(M)$, there is a sequence $\{(x^n, u^n)\}_{n \ge 0} \subseteq graph(M^n)$ such that $x^n \to x, u^n \to u$ as $n \to \infty$.

Lemma 0.17. Lemma 2.17. *Let $H : X \to X$ be k-strongly η-accretive and s-Lipschitz continuous, $\eta : X \times X \to X$ be τ-Lipschitz continuous and $\{M^n\}, M^n : X \times X \to 2^X$ be a sequence of $(H-\eta)-$accretive mappings for $n = 0, 1, 2, ...$ If $M^n(., t^n) \xrightarrow{G} M(., t)$ then* $\lim\limits_{n\to\infty} R^{H,\eta}_{M^n(.,t^n),\lambda}(u) = R^{H,\eta}_{M(.,t),\lambda}(u), \ \forall u \in X.$

Proof. **Proof.** Since $(H + \lambda M(.,t))(X) = X, \ \forall z \in X.$
Hence there exists $(x,u) \in graph(M(.,t))$ such that $z = H(x) + \lambda u$.
Since $M^n(.,t^n) \longrightarrow M(.,t)$, therefore there exists $\{x^n, u^n\} \subset graph(M^n(.,t^n))$ such that $x^n \to x, u^n \to u$ as $n \to \infty$.

Let $z^n = H(x^n) + \lambda u^n$ and noting that

$$R^{H,\eta}_{M(.,t),\lambda}(H(x) + \lambda u) = x, \quad \text{and } R^{H,\eta}_{M^n(.,t^n),\lambda}(H(x^n) + \lambda u^n) = x^n.$$

Using Lipschitz continuity of $R^{H,\eta}_{M(.,t),\lambda}$, we have

$$\begin{aligned}
&\left\|R^{H,\eta}_{M^n(.,t^n),\lambda}(z) - R^{H,\eta}_{M(.,t),\lambda}(z)\right\| \\
&\quad \le \left\|R^{H,\eta}_{M^n(.,t^n),\lambda}(z^n) - R^{H,\eta}_{M(.,t),\lambda}(z)\right\| + \left\|R^{H,\eta}_{M^n(.,t^n),\lambda}(z^n) - R^{H,\eta}_{M^n(.,t^n),\lambda}(z)\right\| \\
&\quad \le \|x^n - x\| + \frac{\tau^{q-1}}{k}\|z^n - z\| \\
&\quad = \|x^n - x\| + \frac{\tau^{q-1}}{k}\|(H(x^n) + \lambda u^n) - (H(x) + \lambda u)\|
\end{aligned}$$

$$\leq \quad \|x^n - x\| + \frac{\tau^{q-1}}{k}\{\|H(x^n) - H(x)\| + \lambda\|u^n - u\|\}$$

$$\leq \quad \|x^n - x\| + \frac{\tau^{q-1}}{k}\{s\|x^n - x\| + \lambda\|u^n - u\|\}$$

$$\longrightarrow \quad 0 \text{ as } n \to \infty.$$

This completes the proof. □

Now, we formulate our main problem.

For each $i = 1, 2, j \in \{1, 2\} \setminus i$, let X_i be a q_i-uniformly smooth Banach space with norm $\|.\|_i$. Let $T_i, f_i : X_i \to X_i, p_i, N_i : X_i \times X_j \to X_i$ be single-valued mappings, $M_i : X_i \times X_i \to 2^{X_i}$ be $(H_i - \eta_i)-$ accretive mappings. Let $B_i, P_i, G_i : X_i \to C(X_i)$ be set-valued mappings. We consider the following system of generalized mixed variational-like inclusion problems (SGMVLIP): Find (x_i, u_i, v_i, t_i) where $x_i \in X_i, u_i \in B_i(x_i), v_i \in P_i(x_i), t_i \in G_i(x_i)$ such that

$$\left.\begin{aligned} 0 &\in T_1\Big(f_1(x_1) + p_1(x_1, x_2)\Big) + \lambda_1\Big(N_1(u_1, v_2) + M_1(x_1, t_1)\Big), \\ 0 &\in T_2\Big(f_2(x_2) + p_2(x_2, x_1)\Big) + \lambda_2\Big(N_2(u_2, v_1) + M_2(x_2, t_2)\Big). \end{aligned}\right\} \tag{2.3}$$

Special Cases:

I. If in problem (2.3), $T_1 = T_2 \equiv I$, (an identity mapping), then problem (2.3) reduces to the following problem: Find (x_i, u_i, v_i, t_i) such that

$$\left.\begin{aligned} 0 &\in f_1(x_1) + p_1(x_1, x_2) + \lambda_1\Big(N_1(u_1, v_2) + M_1(x_1, t_1)\Big), \\ 0 &\in f_2(x_2) + p_2(x_2, x_1) + \lambda_2\Big(N_2(u_2, v_1) + M_2(x_2, t_2)\Big), \end{aligned}\right\} \tag{2.4}$$

which is an important generalization of the problem considered and studied by Peng and Zhu [10].

II. If in problem (2.3) $X_i \equiv H_i$ (a real Hilbert space), $T_1 = T_2 \equiv 0$,(a zero mapping), $\lambda_1 = \lambda_2 = 1$, then problem (2.3) reduces to the following problem: Find (x_i, u_i, v_i, t_i) such that

$$\left.\begin{aligned} 0 &\in N_1(u_1, v_2) + M_1(x_1, t_1), \\ 0 &\in N_2(u_2, v_1) + M_2(x_2, t_2). \end{aligned}\right\} \tag{2.5}$$

This type of problem has been considered and studied by Zeng *et al.* [15].

We remark that for appropriate and suitable choices of the above defined mappings, SGMVLIP (2.3) includes a number of variational and variational-like inclusions as special cases, see for example [1,2,5] and the related references cited therein.

3. Iterative Algorithm

First, we give the following technical lemma:

Lemma 0.18. Lemma 3.1. *Let X_i be a real q_i-uniformly smooth Banach space. Let $T_i, f_i : X_i \to X_i, p_i, N_i : X_i \times X_j \to X_i$ be single-valued mappings, $M_i : X_i \times X_i \to 2^{X_i}$ be $(H_i - \eta_i)-$ accretive mappings. Then (x_i, u_i, v_i, t_i) is a solution of SGMVLIP (2.3) where $x_i \in X_i, u_i \in B_i(x_i), v_i \in P_i(x_i), t_i \in G_i(x_i)$ if and only if*

$$\left.\begin{aligned} x_1 &= R^{H_1,\eta_1}_{M_1(.,t_1),\lambda_1}\Big[H_1(x_1) - T_1\Big(f_1(x_1) + p_1(x_1, x_2)\Big) - \lambda_1 N_1(u_1, v_2)\Big], \\ x_2 &= R^{H_2,\eta_2}_{M_2(.,t_2),\lambda_2}\Big[H_2(x_2) - T_2\Big(f_2(x_2) + p_2(x_2, x_1)\Big) - \lambda_2 N_2(u_2, v_1)\Big]. \end{aligned}\right\} \tag{3.1}$$

where $R^{H_1,\eta_1}_{M_1(.,t_1),\lambda_1} = (H_1+\lambda_1 M_1(.,t_1))^{-1}$, $R^{H_2,\eta_2}_{M_2(.,t_2),\lambda_2} = (H_2+\lambda_2 M_2(.,t_2))^{-1}$ *are the resolvent operators.*

Notation 3.2. For $A \in X_1 \times X_2$, the symbol $A \cap K(S_1, S_2) \neq \emptyset$ means that there exists $(x_1, x_2) \in X_1 \times X_2$ such that $(x_1, x_2) \in A$ and $\{x_1, x_2\} \subset K(S_1, S_2)$, where S_1, S_2 are Lipschitz continuous.

Now, we suggest the following Remark for finding a common element of two different sets namely, the set of solutions of the system of generalized mixed variational-like inclusion problems and the set of fixed points of Lipschitz mappings S_1, S_2.

Remark 0.19. Remark 3.3. If $(x_1, x_2) \in$ SGMVLIP (2.3) and $\{x_1, x_2\} \subset K(S_1, S_2)$, then it follows from Lemma 3.1 that

$$x_1 = S_1(x_1) = S_1\Big\{R^{H_1,\eta_1}_{M_1(.,t_1),\lambda_1}\Big[H_1(x_1) - T_1\Big(f_1(x_1) + p_1(x_1, x_2)\Big) - \lambda_1 N_1(u_1, v_2)\Big]\Big\}, \ \ \lambda_1 > 0,$$

$$x_2 = S_2(x_2) = S_2\Big\{R^{H_2,\eta_2}_{M_2(.,t_2),\lambda_2}\Big[H_2(x_2) - T_2\Big(f_2(x_2) + p_2(x_2, x_1)\Big) - \lambda_2 N_2(u_2, v_1)\Big]\Big\}, \ \ \lambda_2 > 0.$$

Lemma 3.1 and Remark 3.3 are very important from the numerical point of view as it along with Nadler [8] allows us to suggest the following iterative algorithm for finding the approximate solution of SGMVLIP (2.3).

Algorithm 0.20. Iterative Algorithm 3.4. For each $i = 1, 2$, given $(x_i^0, u_i^0, v_i^0, t_i^0)$ where $x_i^0 \in X_i, u_i^0 \in B_i(x_i^0), v_i^0 \in P_i(x_i^0)$ and $t_i^0 \in G_i(x_i^0)$ such that $B_i, P_i, G_i : X_i \to C(X_i)$, compute the sequences $\{x_i^n\}, \{u_i^n\}, \{v_i^n\}, \{t_i^n\}$ defined by the iterative schemes:

$$x_1^{n+1} = (1-\alpha^n)x_1^n + \alpha^n S_1\Big\{R^{H_1,\eta_1}_{M_1^n(.,t_1^n),\lambda_1}\Big[H_1(x_1^n) - T_1\Big(f_1(x_1^n) + p_1(x_1^n, x_2^n)\Big) - \lambda_1 N_1(u_1^n, v_2^n)\Big]\Big\}$$

$$x_2^n = S_2\Big\{R^{H_2,\eta_2}_{M_2^n(.,t_2^n),\lambda_2}\Big[H_2(x_2^n) - T_2\Big(f_2(x_2^n) + p_2(x_2^n, x_1^n)\Big) - \lambda_2 N_2(u_2^n, v_1^n)\Big]\Big\}$$

$$u_i^n \in B_i(x_i^n) : ||u_i^{n+1} - u_i^n|| \leq \mathcal{D}(B_i(x_i^{n+1}), B_i(x_i^n))$$

$$v_i^n \in P_i(x_i^n) : ||v_i^{n+1} - v_i^n|| \leq \mathcal{D}(P_i(x_i^{n+1}), P_i(x_i^n))$$

$$t_i^n \in G_i(x_i^n) : ||t_i^{n+1} - t_i^n|| \leq \mathcal{D}(G_i(x_i^{n+1}), G_i(x_i^n))$$

where $M_i^n : X_i \times X_i \to 2^{X_i}$ are $(H_i - \eta_i)-$accretive mappings for $i \in \{1, 2\}, n = 0, 1, 2, ...,$ and

$$R^{H_1,\eta_1}_{M_1^n(.,t_1^n),\lambda_1} = (H_1 + \lambda_1 M_1^n(., t_1^n))^{-1}, \qquad R^{H_2,\eta_2}_{M_2^n(.,t_2^n),\lambda_2} = (H_2 + \lambda_2 M_2^n(., t_2^n))^{-1},$$

and α^n be a sequence of real numbers such that $\alpha^n \in [0, 1]$ and $\sum_{n=0}^{\infty} \alpha^n = +\infty$.

4. Existence of Solution and Convergence Analysis

Now, we prove the existence of common element of solutions of SGMVLIP (2.3) and the set of fixed points of Lipschitz mappings S_1 and S_2.

Theorem 0.21. Theorem 4.1. *Let X_i be a real q_i-uniformly smooth Banach space. Suppose for each $i = 1, 2, j \in \{1, 2\} \setminus i$, $H_i : X_i \to X_i$ be k_i-strongly-η_i-accretive, $\eta_i : X_i \times X_i \to X_i$ be τ_i-Lipschitz continuous, $H_i, S_i, T_i : X_i \to X_i$ be Lipschitz continuous with constants s_i, m_i, γ_i, respectively. Let $N_i : X_i \times X_j \to X_i$ be (r_i, δ_i)-mixed Lipschitz continuous, $p_i : X_i \times X_j \to X_i$ be ξ_i-Lipschitz continuous in the second argument. Suppose, $M_i^n : X_i \times X_i \to 2^{X_i}$ be $(H_i - \eta_i)-$accretive mappings such that $M_i^n(., x_i^n) \xrightarrow{G} M_i(., x_i)$ as $n \to \infty$. Further, suppose $H_i, T_i, f_i : X_i \to X_i, p_i : X_i \times X_j \to X_i$ be such that $\Big[H_i(.) - T_i\Big(f_i(.) + p_i(., x_j^n)\Big)\Big]$ be $\mu_i - \eta_i-$ cocoercive. Let $B_i, P_i, G_i : X_i \to C(X_i)$ be set-valued mappings such that B_i is*

$L_{B_i}-\mathcal{D}-$Lipschitz continuous, P_i is $L_{P_i}-\mathcal{D}-$Lipschitz continuous and G_i is $L_{G_i}-\mathcal{D}-$Lipschitz continuous. In addition, if

$$\left.\begin{array}{l}\left\{1-m_2L_2\left[\dfrac{\tau_2}{\mu_2^{q_2-1}}+\lambda_2r_2L_{B_2}\right]\right\}>0,\\[2ex] 0<m_1L_1\left[\dfrac{\tau_1}{\mu_1^{q_1-1}}+\lambda_1r_1L_{B_1}\right]+m_1L_1\left[\gamma_1\xi_1+\lambda_1\delta_1L_{P_2}\right]\\[2ex] \qquad\times\left(\dfrac{m_2L_2\left[\gamma_2\xi_2+\lambda_2\delta_2L_{P_1}\right]}{\left\{1-m_2L_2\left[\dfrac{\tau_2}{\mu_2^{q_2-1}}+\lambda_2r_2L_{B_2}\right]\right\}}\right)<1,\end{array}\right\}\tag{4.1}$$

where $L_i:=\dfrac{\tau_i^{q_i-1}}{k_i}$. Then the sequences $\{x_i^n\},\{u_i^n\},\{v_i^n\},\{t_i^n\}$ generated by Iterative Algorithm 3.4 converges strongly to x_i,u_i,v_i,t_i a solution of SGMVLIP (2.3) where $x_i\in X_i,u_i\in B_i(x_i),v_i\in P_i(x_i),t_i\in G_i(x_i)$ such that $(x_1,x_2)\in$ SGMVLIP (2.3) and $\{x_1,x_2\}\subset K(S_1,S_2)$.

Proof. **Proof.** From Lemma 3.1, we have

$$x_1=R^{H_1,\eta_1}_{M_1(.,t_1),\lambda_1}\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2)\Big)-\lambda_1N_1(u_1,v_2)\Big].$$

Therefore, from Lemma 3.1, Iterative Algorithm 3.4 and fixed point property of S_1, it follows that

$$\begin{aligned}&\|x_1^{n+1}-x_1\|_1\\ =&\ \Big\|(1-\alpha^n)x_1^n+\alpha^nS_1\Big\{R^{H_1,\eta_1}_{M_1^n(.,t_1^n),\lambda_1}\Big[H_1(x_1^n)-T_1\Big(f_1(x_1^n)+p_1(x_1^n,x_2^n)\Big)-\lambda_1N_1(u_1^n,v_2^n)\Big]\Big\}\\ &-(1-\alpha^n)x_1-\alpha^nS_1\Big\{R^{H_1,\eta_1}_{M_1(.,t_1),\lambda_1}\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2)\Big)-\lambda_1N_1(u_1,v_2)\Big]\Big\}\Big\|_1\\ \le&\ (1-\alpha^n)\|x_1^n-x_1\|_1\\ &+\alpha^nm_1\Big\|R^{H_1,\eta_1}_{M_1^n(.,t_1^n),\lambda_1}\Big[H_1(x_1^n)-T_1\Big(f_1(x_1^n)+p_1(x_1^n,x_2^n)\Big)-\lambda_1N_1(u_1^n,v_2^n)\Big]\\ &-R^{H_1,\eta_1}_{M_1(.,t_1),\lambda_1}\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2)\Big)-\lambda_1N_1(u_1,v_2)\Big]\Big\|_1\\ \le&\ (1-\alpha^n)\|x_1^n-x_1\|_1\\ &+\alpha^nm_1\Big\|R^{H_1,\eta_1}_{M_1^n(.,t_1^n),\lambda_1}\Big[H_1(x_1^n)-T_1\Big(f_1(x_1^n)+p_1(x_1^n,x_2^n)\Big)-\lambda_1N_1(u_1^n,v_2^n)\Big]\\ &-R^{H_1,\eta_1}_{M_1^n(.,t_1^n),\lambda_1}\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2)\Big)-\lambda_1N_1(u_1,v_2)\Big]\Big\|_1\\ &+\alpha^nm_1\Big\|R^{H_1,\eta_1}_{M_1^n(.,t_1^n),\lambda_1}\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2)\Big)-\lambda_1N_1(u_1,v_2)\Big]\\ &-R^{H_1,\eta_1}_{M_1(.,t_1),\lambda_1}\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2)\Big)-\lambda_1N_1(u_1,v_2)\Big]\Big\|_1.\end{aligned}\tag{4.2}$$

Using Theorem 2.15, we have

$$\Big\|R^{H_1,\eta_1}_{M_1^n(.,t_1^n),\lambda_1}\Big[H_1(x_1^n)-T_1\Big(f_1(x_1^n)+p_1(x_1^n,x_2^n)\Big)-\lambda_1N_1(u_1^n,v_2^n)\Big]$$

$$
\begin{aligned}
& -R^{H_1,\eta_1}_{M_1^n(.,t_1^n),\lambda_1}\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2)\Big)-\lambda_1 N_1(u_1,v_2)\Big]\Big\|_1 \\
\leq\ & L_1\Big\|\Big[H_1(x_1^n)-T_1\Big(f_1(x_1^n)+p_1(x_1^n,x_2^n)\Big)-\lambda_1 N_1(u_1^n,v_2^n)\Big] \\
& -\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2)\Big)-\lambda_1 N_1(u_1,v_2)\Big]\Big\|_1 \\
\leq\ & L_1\Big\|\Big[H_1(x_1^n)-T_1\Big(f_1(x_1^n)+p_1(x_1^n,x_2^n)\Big)\Big] \\
& -\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2^n)\Big)\Big]\Big\|_1 \\
& +L_1\Big\|\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2^n)\Big)\Big] \\
& -\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2)\Big)\Big]\Big\|_1 \\
& +L_1\lambda_1\Big\|N_1(u_1^n,v_2^n)-N_1(u_1,v_2)\Big\|_1. \qquad (4.3)
\end{aligned}
$$

Since $\Big[H_1(.)-T_1\Big(f_1(.)+p_1(.,x_2^n)\Big)\Big]$ is $\mu_1-\eta_1-$cocoercive, we have

$$
\begin{aligned}
& \Big\|\Big[H_1(x_1^n)-T_1\Big(f_1(x_1^n)+p_1(x_1^n,x_2^n)\Big)\Big] \\
& -\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2^n)\Big)\Big]\Big\|_1 \Big\|\eta_1(x_1^n,x_1)\Big\|_1^{q_1-1} \\
\geq\ & \Big\langle\Big[H_1(x_1^n)-T_1\Big(f_1(x_1^n)+p_1(x_1^n,x_2^n)\Big)\Big] \\
& -\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2^n)\Big)\Big], J_{q_1}\Big(\eta_1(x_1^n,x_1)\Big)\Big\rangle_1 \\
\geq\ & \mu_1\Big\|\Big[H_1(x_1^n)-T_1\Big(f_1(x_1^n)+p_1(x_1^n,x_2^n)\Big)\Big] \\
& -\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2^n)\Big)\Big]\Big\|_1^{q_1}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
& \mu_1\Big\|\Big[H_1(x_1^n)-T_1\Big(f_1(x_1^n)+p_1(x_1^n,x_2^n)\Big)\Big] \\
& -\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2^n)\Big)\Big]\Big\|_1^{q_1} \\
\leq\ & \Big\|\Big[H_1(x_1^n)-T_1\Big(f_1(x_1^n)+p_1(x_1^n,x_2^n)\Big)\Big] \\
& -\Big[H_1(x_1)-T_1\Big(f_1(x_1)+p_1(x_1,x_2^n)\Big)\Big]\Big\|_1 \Big\|\eta_1(x_1^n,x_1)\Big\|_1^{q_1-1}
\end{aligned}
$$

$$\Longrightarrow \quad \Big\| \Big[H_1(x_1^n) - T_1\Big(f_1(x_1^n) + p_1(x_1^n, x_2^n)\Big)\Big] - \Big[H_1(x_1) - T_1\Big(f_1(x_1) + p_1(x_1, x_2^n)\Big)\Big]\Big\|_1^{q_1-1} \leq \frac{\tau_1^{q_1-1}}{\mu_1} \|x_1^n - x_1\|_1^{q_1-1}$$

$$\Longrightarrow \Big\| \Big[H_1(x_1^n) - T_1\Big(f_1(x_1^n) + p_1(x_1^n, x_2^n)\Big)\Big] - \Big[H_1(x_1) - T_1\Big(f_1(x_1) + p_1(x_1, x_2^n)\Big)\Big]\Big\|_1 \leq \frac{\tau_1}{\mu_1^{q_1-1}} \|x_1^n - x_1\|_1. \tag{4.4}$$

Also, since $T_1 : X_1 \to X_1$ is γ_1-Lipschitz continuous, $p_1 : X_1 \times X_2 \to X_1$ is ξ_1-Lipschitz continuous in the second argument, we have

$$\begin{aligned}
&\Big\| \Big[H_1(x_1) - T_1\Big(f_1(x_1) + p_1(x_1, x_2^n)\Big)\Big] - \Big[H_1(x_1) - T_1\Big(f_1(x_1) + p_1(x_1, x_2)\Big)\Big]\Big\|_1 \\
&\leq \Big\| T_1\Big(f_1(x_1) + p_1(x_1, x_2^n)\Big) - T_1\Big(f_1(x_1) + p_1(x_1, x_2)\Big)\Big\|_1 \\
&\leq \gamma_1 \Big\| p_1(x_1, x_2^n) - p_1(x_1, x_2)\Big\|_1 \\
&\leq \gamma_1 \xi_1 \|x_2^n - x_2\|_2.
\end{aligned} \tag{4.5}$$

Again using (r_1, δ_1)-mixed Lipschitz continuity of $N_1 : X_1 \times X_2 \to X_1$, L_{B_1} and $L_{P_2} - \mathcal{D}$-Lipschitz continuity of B_1 and P_2 respectively, we have

$$\begin{aligned}
\Big\| N_1(u_1^n, v_2^n) - N_1(u_1, v_2)\Big\|_1 &\leq r_1 \|u_1^n - u_1\|_1 + \delta_1 \|v_2^n - v_2\|_2 \\
&\leq r_1 L_{B_1} \|x_1^n - x_1\|_1 + \delta_1 L_{P_2} \|x_2^n - x_2\|_2.
\end{aligned} \tag{4.6}$$

Using (4.3)-(4.6) in (4.2), we have

$$\begin{aligned}
&\|x_1^{n+1} - x_1\|_1 \\
&\leq (1-\alpha^n)\|x_1^n - x_1\|_1 + \alpha^n m_1 L_1 \Big[\frac{\tau_1}{\mu_1^{q_1-1}} + \lambda_1 r_1 L_{B_1}\Big] \|x_1^n - x_1\|_1 \\
&\quad + \alpha^n m_1 L_1 \Big[\gamma_1 \xi_1 + \lambda_1 \delta_1 L_{P_2}\Big] \|x_2^n - x_2\|_2 + \alpha^n m_1 b_1^n,
\end{aligned}$$

where

$$\begin{aligned}
b_1^n = \Big\| & R^{H_1,\eta_1}_{M_1^n(.,t_1^n),\lambda_1} \Big[H_1(x_1) - T_1\Big(f_1(x_1) + p_1(x_1, x_2)\Big) - \lambda_1 N_1(u_1, v_2)\Big] \\
& - R^{H_1,\eta_1}_{M_1(.,t_1),\lambda_1} \Big[H_1(x_1) - T_1\Big(f_1(x_1) + p_1(x_1, x_2)\Big) - \lambda_1 N_1(u_1, v_2)\Big]\Big\|_1,
\end{aligned}$$

and $b_1^n \to 0$ as $n \to \infty$. Therefore, we have

$$\begin{aligned}
&\|x_1^{n+1} - x_1\|_1 \\
&\leq (1-\alpha^n)\|x_1^n - x_1\|_1 + \alpha^n m_1 L_1 \Big[\frac{\tau_1}{\mu_1^{q_1-1}} + \lambda_1 r_1 L_{B_1}\Big] \|x_1^n - x_1\|_1 \\
&\quad + \alpha^n m_1 L_1 \Big[\gamma_1 \xi_1 + \lambda_1 \delta_1 L_{P_2}\Big] \|x_2^n - x_2\|_2.
\end{aligned} \tag{4.7}$$

Again, using Lemma 3.1, Iterative Algorithm 3.4 and fixed point property of S_2, we have

$$\|x_2^n - x_2\|_2$$

$$\begin{aligned}
&= \Big\|S_2\Big\{R^{H_2,\eta_2}_{M_2^n(.,t_2^n),\lambda_2}\Big[H_2(x_2^n)-T_2\Big(f_2(x_2^n)+p_2(x_2^n,x_1^n)\Big)-\lambda_2N_2(u_2^n,v_1^n)\Big]\Big\}\\
&\quad -S_2\Big\{R^{H_2,\eta_2}_{M_2(.,t_2),\lambda_2}\Big[H_2(x_2)-T_2\Big(f_2(x_2)+p_2(x_2,x_1)\Big)-\lambda_2N_2(u_2,v_1)\Big]\Big\}\Big\|_2\\
&\le m_2\Big\|R^{H_2,\eta_2}_{M_2^n(.,t_2^n),\lambda_2}\Big[H_2(x_2^n)-T_2\Big(f_2(x_2^n)+p_2(x_2^n,x_1^n)\Big)-\lambda_2N_2(u_2^n,v_1^n)\Big]\\
&\quad -R^{H_2,\eta_2}_{M_2^n(.,t_2^n),\lambda_2}\Big[H_2(x_2)-T_2\Big(f_2(x_2)+p_2(x_2,x_1)\Big)-\lambda_2N_2(u_2,v_1)\Big]\Big\|_2\\
&\quad +m_2\Big\|R^{H_2,\eta_2}_{M_2^n(.,t_2^n),\lambda_2}\Big[H_2(x_2)-T_2\Big(f_2(x_2)+p_2(x_2,x_1)\Big)-\lambda_2N_2(u_2,v_1)\Big]\\
&\quad -R^{H_2,\eta_2}_{M_2(.,t_2),\lambda_2}\Big[H_2(x_2)-T_2\Big(f_2(x_2)+p_2(x_2,x_1)\Big)-\lambda_2N_2(u_2,v_1)\Big]\Big\|_2. \qquad (4.8)
\end{aligned}$$

Now, using Theorem 2.15, we have

$$\begin{aligned}
&\Big\|R^{H_2,\eta_2}_{M_2^n(.,t_2^n),\lambda_2}\Big[H_2(x_2^n)-T_2\Big(f_2(x_2^n)+p_2(x_2^n,x_1^n)\Big)-\lambda_2N_2(u_2^n,v_1^n)\Big]\\
&\quad -R^{H_2,\eta_2}_{M_2^n(.,t_2^n),\lambda_2}\Big[H_2(x_2)-T_2\Big(f_2(x_2)+p_2(x_2,x_1)\Big)-\lambda_2N_2(u_2,v_1)\Big]\Big\|_2\\
&\le L_2\Big\|\Big[H_2(x_2^n)-T_2\Big(f_2(x_2^n)+p_2(x_2^n,x_1^n)\Big)-\lambda_2N_2(u_2^n,v_1^n)\Big]\\
&\quad -\Big[H_2(x_2)-T_2\Big(f_2(x_2)+p_2(x_2,x_1)\Big)-\lambda_2N_2(u_2,v_1)\Big]\Big\|_2\\
&\le L_2\Big\|\Big[H_2(x_2^n)-T_2\Big(f_2(x_2^n)+p_2(x_2^n,x_1^n)\Big)\Big]\\
&\quad -\Big[H_2(x_2)-T_2\Big(f_2(x_2)+p_2(x_2,x_1^n)\Big)\Big]\Big\|_2\\
&\quad +L_2\Big\|\Big[H_2(x_2)-T_2\Big(f_2(x_2)+p_2(x_2,x_1^n)\Big)\Big]\\
&\quad -\Big[H_2(x_2)-T_2\Big(f_2(x_2)+p_2(x_2,x_1)\Big)\Big]\Big\|_2\\
&\quad +L_2\lambda_2\Big\|N_2(u_2^n,v_1^n)-N_2(u_2,v_1)\Big\|_2. \qquad (4.9)
\end{aligned}$$

Since $\Big[H_2(.)-T_2\Big(f_2(.)+p_2(.,x_1^n)\Big)\Big]$ is $\mu_2-\eta_2-$cocoercive, therefore following the same steps as in (4.4), we have

$$\begin{aligned}
&\Big\|\Big[H_2(x_2^n)-T_2\Big(f_2(x_2^n)+p_2(x_2^n,x_1^n)\Big)\Big]-\Big[H_2(x_2)-T_2\Big(f_2(x_2)+p_2(x_2,x_1^n)\Big)\Big]\Big\|_2\\
&\quad \le \frac{\tau_2}{\mu_2^{q_2-1}}\|x_2^n-x_2\|_2. \qquad (4.10)
\end{aligned}$$

Again, as $T_2 : X_2 \to X_2$ is γ_2-Lipschitz continuous, $p_2 : X_2 \times X_1 \to X_2$ is ξ_2-Lipschitz continuous in the second argument, therefore proceeding as in (4.5), we have

$$\begin{aligned}
&\Big\|\Big[H_2(x_2)-T_2\Big(f_2(x_2)+p_2(x_2,x_1^n)\Big)\Big]-\Big[H_2(x_2)-T_2\Big(f_2(x_2)+p_2(x_2,x_1)\Big)\Big]\Big\|_2\\
&\quad \le \gamma_2\xi_2\|x_1^n-x_1\|_1. \qquad (4.11)
\end{aligned}$$

Now, using (r_2,δ_2)-mixed Lipschitz continuity of $N_2 : X_2 \times X_1 \to X_2$, L_{B_2} and $L_{P_1}-\mathcal{D}$-Lipschitz continuity of B_2 and P_1 respectively, we have

$$\Big\|N_2(u_2^n,v_1^n)-N_2(u_2,v_1)\Big\|_2 \le r_2L_{B_2}\|x_2^n-x_2\|_2+\delta_2L_{P_1}\|x_1^n-x_1\|_1. \qquad (4.12)$$

Using (4.9)-(4.12) in (4.8), we have

$$\|x_2^n - x_2\|_2 \leq m_2 L_2 \Big[\frac{\tau_2}{\mu_2^{q_2-1}} + \lambda_2 r_2 L_{B_2} \Big] \|x_2^n - x_2\|_2 + m_2 L_2 \Big[\gamma_2 \xi_2 + \lambda_2 \delta_2 L_{P_1} \Big] \|x_1^n - x_1\|_1 + m_2 b_2^n.$$

This implies

$$\Big\{ 1 - m_2 L_2 \Big[\frac{\tau_2}{\mu_2^{q_2-1}} + \lambda_2 r_2 L_{B_2} \Big] \Big\} \|x_2^n - x_2\|_2 \leq m_2 L_2 \Big[\gamma_2 \xi_2 + \lambda_2 \delta_2 L_{P_1} \Big] \|x_1^n - x_1\|_1 + m_2 b_2^n$$

or,

$$\|x_2^n - x_2\|_2 \leq \left(\frac{m_2 L_2 \Big[\gamma_2 \xi_2 + \lambda_2 \delta_2 L_{P_1} \Big]}{\Big\{ 1 - m_2 L_2 \Big[\frac{\tau_2}{\mu_2^{q_2-1}} + \lambda_2 r_2 L_{B_2} \Big] \Big\}} \right) \|x_1^n - x_1\|_1 + \left(\frac{m_2}{\Big\{ 1 - m_2 L_2 \Big[\frac{\tau_2}{\mu_2^{q_2-1}} + \lambda_2 r_2 L_{B_2} \Big] \Big\}} \right) b_2^n \tag{4.13}$$

where,

$$b_2^n = \Big\| R^{H_2,\eta_2}_{M_2^n(.,t_2^n),\lambda_2} \Big[H_2(x_2) - T_2\Big(f_2(x_2) + p_2(x_2, x_1) \Big) - \lambda_2 N_2(u_2, v_1) \Big] - R^{H_2,\eta_2}_{M_2(.,t_2),\lambda_2} \Big[H_2(x_2) - T_2\Big(f_2(x_2) + p_2(x_2, x_2) \Big) - \lambda_2 N_2(u_2, v_1) \Big] \Big\|_2,$$

and $b_2^n \to 0$ as $n \to \infty$.

Using (4.13) in (4.7), we have

$$\begin{aligned}
\|x_1^{n+1} - x_1\|_1 &\leq (1 - \alpha^n)\|x_1^n - x_1\|_1 + \alpha^n m_1 L_1 \Big[\frac{\tau_1}{\mu_1^{q_1-1}} + \lambda_1 r_1 L_{B_1} \Big] \|x_1^n - x_1\|_1 \\
&+ \alpha^n m_1 L_1 \Big[\gamma_1 \xi_1 + \lambda_1 \delta_1 L_{P_2} \Big] \left(\frac{m_2 L_2 \Big[\gamma_2 \xi_2 + \lambda_2 \delta_2 L_{P_1} \Big]}{\Big\{ 1 - m_2 L_2 \Big[\frac{\tau_2}{\mu_2^{q_2-1}} + \lambda_2 r_2 L_{B_2} \Big] \Big\}} \right) \|x_1^n - x_1\|_1 \\
&+ \alpha^n m_1 L_1 \Big[\gamma_1 \xi_1 + \lambda_1 \delta_1 L_{P_2} \Big] \left(\frac{m_2}{\Big\{ 1 - m_2 L_2 \Big[\frac{\tau_2}{\mu_2^{q_2-1}} + \lambda_2 r_2 L_{B_2} \Big] \Big\}} \right) b_2^n \\
&\leq \Bigg[1 - \alpha^n \Bigg\{ 1 - m_1 L_1 \Big[\frac{\tau_1}{\mu_1^{q_1-1}} + \lambda_1 r_1 L_{B_1} \Big] \\
&- m_1 L_1 \Big[\gamma_1 \xi_1 + \lambda_1 \delta_1 L_{P_2} \Big] \left(\frac{m_2 L_2 \Big[\gamma_2 \xi_2 + \lambda_2 \delta_2 L_{P_1} \Big]}{\Big\{ 1 - m_2 L_2 \Big[\frac{\tau_2}{\mu_2^{q_2-1}} + \lambda_2 r_2 L_{B_2} \Big] \Big\}} \right) \Bigg\} \Bigg] \|x_1^n - x_1\|_1 \\
&+ \alpha^n m_1 L_1 \Big[\gamma_1 \xi_1 + \lambda_1 \delta_1 L_{P_2} \Big] \left(\frac{m_2}{\Big\{ 1 - m_2 L_2 \Big[\frac{\tau_2}{\mu_2^{q_2-1}} + \lambda_2 r_2 L_{B_2} \Big] \Big\}} \right) b_2^n \\
&\leq [1 - \alpha^n (1 - h)] \|x_1^n - x_1\|_1 + \alpha^n l^n,
\end{aligned} \tag{4.14}$$

where

$$h = m_1 L_1 \Big[\frac{\tau_1}{\mu_1^{q_1-1}} + \lambda_1 r_1 L_{B_1} \Big] + m_1 L_1 \Big[\gamma_1 \xi_1 + \lambda_1 \delta_1 L_{P_2} \Big] \left(\frac{m_2 L_2 \Big[\gamma_2 \xi_2 + \lambda_2 \delta_2 L_{P_1} \Big]}{\Big\{ 1 - m_2 L_2 \Big[\frac{\tau_2}{\mu_2^{q_2-1}} + \lambda_2 r_2 L_{B_2} \Big] \Big\}} \right),$$

$$l^n = L_1\left(\frac{m_1m_2\Big[\gamma_1\xi_1 + \lambda_1\delta_1 L_{P_2}\Big]}{\Big\{1 - m_2L_2\Big[\frac{\tau_2}{\mu_2^{q_2-1}} + \lambda_2 r_2 L_{B_2}\Big]\Big\}}\right) b_2^n.$$

Since $h < 1$ from assumption (4.1), therefore we can rewrite inequality (4.14) as

$$\|x_1^{n+1} - x_1^n\|_1 \leq [1 - \alpha^n(1-h)]\|x_1^n - x_1\|_1 + \alpha^n(1-h)\frac{l^n}{(1-h)}. \tag{4.15}$$

If $\zeta^n = ||x_1^n - x_1||_1,\ \nu^n = \dfrac{l^n}{(1-h)}$ and $\omega^n = \alpha^n(1-h)$, then we have

$$\zeta^{n+1} \leq (1-\omega^n)\zeta^n + \omega^n\nu^n.$$

Using Lemma 2.8, we have $\zeta^n \to 0$ as $n \to \infty$ and thus $x_1^n \to x_1$ as $n \to \infty$, and therefore from (4.13) it follows that $x_2^n \to x_2$ as $n \to \infty$. Since B_i is $L_{B_i} - \mathcal{D}$-Lipschitz continuous, it follows from Iterative Algorithm 3.4 that

$$\begin{aligned} ||u_i^n - u_i||_i &\leq \mathcal{D}(B_i(x_i^n), B_i(x_i)) \\ &\leq L_{B_i}||x_i^n - x_i||_i. \end{aligned}$$

This implies that

$$u_i^n \to u_i \text{ as } n \to \infty.$$

Further we claim that $u_i \in B_i(x_i)$

$$\begin{aligned} d(u_i, B_i(x_i)) &\leq ||u_i - u_i^n||_i + d(u_i^n, B_i(x_i)) \\ &\leq ||u_i - u_i^n||_i + \mathcal{D}(B_i(x_i^n), B_i(x_i)) \\ &\leq ||u_i - u_i^n||_i + L_{B_i}||x_i^n - x_i||_i \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $B_i(x_i)$ is compact, we have $u_i \in B_i(x_i)$.
Similarly, we can prove that $v_i \in P_i(x_i)$ and $t_i \in G_i(x_i)$.
Thus the approximate solution $(x_i^n, u_i^n, v_i^n, t_i^n)$ generated by Iterative Algorithm 3.4 converges strongly to (x_i, u_i, v_i, t_i) a solution of SGMVLIP (2.3) where $x_i \in X_i, u_i \in B_i(x_i), v_i \in P_i(x_i), t_i \in G_i(x_i)$ such that $(x_1, x_2) \in$ SGMVLIP (2.3) and $\{x_1, x_2\} \subset K(S_1, S_2)$. This completes the proof. □

Conclusion 4.2. In the present study it has been concluded that the convergence criteria of an iterative algorithm helps to find the common solution of a system of generalized mixed variational-like inclusion problems and fixed point problems of nonlinear Lipschitz mappings. Moreover, it has been proved that the sequences generated by the iterative algorithm converge strongly to a common element of the two systems.

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