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FORCING WEAK EDGE TRIANGLE FREE DETOUR NUMBER OF A GRAPH

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Abstract For any two vertices u and v in a connected graph $G = (V, E)$, the *triangle free detour distance* $D_{\triangle f}(u, v)$ is the length of a longest $u - v$ triangle free path in G. A $u - v$ path of length $D_{\triangle f}(u, v)$ is called a $u - v$ *triangle free detour*. A subset T of a weak edge triangle free detour basis $S \subseteq V$ of G is a forcing weak edge triangle free detour subset for S if S is the unique weak edge triangle free detour basis containing T. The forcing weak edge triangle free detour number of G is $fwdn_{\triangle f}(G) = min\{fwdn_{\triangle f}(S)\}$, where the minimum is taken over all weak edge triangle free detour bases S in G. It is shown that for any two positive integers a and b with $0 \leq a \leq b$, there exists a connected graph G such that $fwdn_{\triangle f}(G) = a$, $wdn_{\triangle f}(G) = b$.

1 Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected connected simple graph. For basic definitions and terminologies, we refer to Chartrand et al. [3].

The concept of geodetic number was introduced by Harary et al. [4]. For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. A set $S \subseteq V$ is called *geodetic set* of G if every vertex of G lies on a geodesic joining a pair of vertices of S. The *geodetic number* $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is called a geodetic basis of G. Let S be a geodetic basis of G. A subset $T \subseteq S$ is called a *forcing geodetic subset* for S if S is the unique geodetic basis containing T. A forcing geodetic subset for S of minimum cardinality is a minimum forcing geodetic subset of S. The *forcing geodetic number* $f_g(S)$ in G is the cardinality of a minimum forcing geodetic subset of S. The *forcing geodetic number* of G is $f_g(G) = min\{f_g(S)\}$, where the minimum is taken over all geodetic bases S in G.

The concept of detour number was introduced by Chartrand et al. [2]. The *detour distance* $D(u, v)$ is the length of a longest $u - v$ path in G. A $u - v$ path of length $D(u, v)$ is called a $u - v$ *detour*. A set $S \subseteq V$ is called *detour set* of G if every vertex of G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of its detour sets and any detour set of order $dn(G)$ is called a detour basis of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique detour basis containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The *forcing detour number* $f_{dn}(S)$ in G is the cardinality of a minimum forcing subset of S. The *forcing detour number* of G is $f_{dn}(G) = min\{f_{dn}(S)\}$, where the minimum is taken over all detour bases S in G.

The concept of weak edge detour number was introduced by Santhakumaran and Athisayanathan [6]. A set $S \subseteq V$ is called an *weak edge detour set* of G if every edge of G has both ends in S or it lies on a detour joining a pair of vertices of S. The *weak edge detour number* $dn_w(G)$ of G is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is a *weak edge detour basis* of G. Let S be a weak edge detour basis of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique weak edge detour basis containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The *forcing weak edge detour number* of G is $fwdn_{\triangle f}(G) = min\{f_{dn}(S)\}$, where the minimum is taken over all weak edge detour bases S in G.

The concept of triangle free detour distance was introduced by Keerthi Asir and Athisayanathan

[5]. A path P is called a *triangle free path* if no three vertices of P induces a triangle. The *triangle free detour distance* $D_{\triangle f}(u, v)$ is the length of a longest $u - v$ triangle free path in G. A $u - v$ path of length $D_{\triangle f}(u, v)$ is called a $u - v$ *triangle free detour*. The *triangle free detour eccentricity* $e_{\triangle f}(v)$ of a vertex in G is the maximum triangle free detour distance from v to a vertex of G. The *triangle free detour radius*, $R_{\triangle f}(G)$ of G is the minimum triangle free detour eccentricity among the vertices of G, while the *triangle free detour diameter*, $D_{\triangle f}(G)$ of G is the maximum triangle free detour eccentricity among the vertices of G.

The concept of triangle free detour number was introduced by Sethu Ramalingam, Keerthi Asir and Athisayanathan [7]. A set $S \subseteq V$ is called a triangle free detour set of G if every vertex of G lies on a triangle free detour joining a pair of vertices of S. The triangle free detour number $dn_{\triangle f}(G)$ of G is the minimum order of its triangle free detour sets and any triangle free detour set of order $dn_{\triangle f}(G)$ is called a triangle free detour basis of G. A subset T of a triangle free detour basis S of G is a forcing triangle free detour subset for S if S is the unique triangle free detour basis containing T. A forcing triangle free detour subset for S of minimum cardinality is a minimum forcing triangle free detour subset of S. The forcing triangle free detour number $f_{dn_{\triangle f}}(S)$ in G is the cardinality of a minimum forcing triangle free detour subset of S. The forcing triangle free detour number of G is $f_{dn_{\triangle f}}(G) = min\{f_{dn_{\triangle f}}(S)\}$, where the minimum is taken over all triangle free detour bases S in G.

The concept of weak edge triangle free detour number was introduced by Sethu Ramalingam, Keerthi Asir and Athisayanathan [8]. A set $S \subseteq V$ is called a *weak edge triangle free detour set* of G if every edge of G has both ends in S or it lies on a detour joining a pair of vertices of S. The *weak edge triangle free detour number* $wdn_{\triangle f}(G)$ of G is the minimum order of its weak edge triangle free detour sets and any weak edge triangle free detour set of order $wdn_{\triangle f}(G)$ is a *weak edge triangle free detour basis* of G. In this paper, we introduce a forcing weak edge triangle free detour number in a connected graph G. The following theorems will be used in the sequel.

Theorem 1.1. [8] *For any connected graph G of order $n \geq 2$, $2 \leq wdn_{\triangle f}(G) \leq n$.*

Theorem 1.2. [8] *For the complete graph $K_n (n \geq 3)$, then $wdn_{\triangle f}(K_n) = n$.*

Theorem 1.3. [8] *Let G be an even cycle of order $n \geq 4$. Then a set $S \subseteq V$ is a weak edge triangle free detour basis of G if and only if S consists of any two adjacent vertices or two antipodal vertices of G.*

Theorem 1.4. [8] *Let G be an odd cycle of order $n \geq 5$. Then a set $S \subseteq V$ is a weak edge triangle free detour basis of G if and only if S consists of any two adjacent vertices of G.*

Theorem 1.5. [8] *If T is a tree with k end-vertices, then $dn_{\triangle f}(T) = wdn_{\triangle f}(T) = k$.*

Theorem 1.6. [8] *Let G be a complete bipartite graph $K_{r,s} (2 \leq r \leq s)$. Then a set $S \subseteq V$ is a weak edge triangle free detour basis of G if and only if S consists of any two vertices of G.*

Theorem 1.7. [8] *Every extreme-vertex of a connected graph G belongs to every weak edge triangle free detour set of G. Also, if the set S of all extreme-vertices of G is a weak edge triangle free detour set, then S is the unique weak edge triangle free detour basis for G.*

Theorem 1.8. [8] *Let G be a connected graph with cut-vertices and S a weak edge triangle free detour set of G. Then for any cut-vertex v of G, every component of $G - v$ contains an element of S.*

Throughout this paper G denotes a connected graph with at least two vertices.

2 Forcing Weak Edge Triangle Free Detour Number of a Graph

Definition 2.1. Let G be a connected graph and S a weak edge triangle free detour basis of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique weak edge triangle free detour basis containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing weak edge triangle free detour number of S, denoted by $fwdn_{\triangle f}(S)$, is the cardinality of a minimum forcing subset for S. The forcing weak edge triangle free detour number of G, denoted by $fwdn_{\triangle f}(G)$, is $fwdn_{\triangle f}(G) = min\{fwdn_{\triangle f}(S)\}$, where the minimum is taken over all minimum weak edge triangle free detour bases S in G.

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{u, v, x\}$, $S_2 = \{u, v, y\}$ and $S_3 = \{u, v, z\}$ are the only weak edge triangle free detour bases of G so that $fwdn_{\triangle f}(G) = 1$ and $wdn_{\triangle f}(G) = 3$.

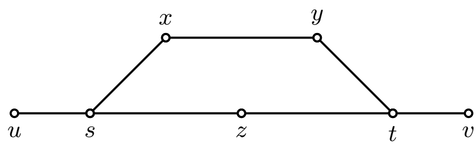


Figure 2.1 : G

Theorem 2.3. *For any connected graph G, $0 \leq fwdn_{\triangle f}(G) \leq wdn_{\triangle f}(G)$.*

Proof. It is clear from the definition of $fwdn_{\triangle f}(G)$ that $fwdn_{\triangle f}(G) \geq 0$. Let S be any weak edge triangle free detour basis of G. Since $fwdn_{\triangle f}(S) \leq fwdn_{\triangle f}(G)$ and since $fwdn_{\triangle f}(G) = min\{fwdn_{\triangle f}(S)$: S is a weak edge triangle free detour basis of $G\}$, it follows that $fwdn_{\triangle f}(G) \leq wdn_{\triangle f}(G)$. Thus $0 \leq fwdn_{\triangle f}(G) \leq wdn_{\triangle f}(G)$.

If $fwdn_{\triangle f}(G) = 0$, then by definition, $fwdn_{\triangle f}(S) = 0$ for some weak edge triangle free detour basis S of G so that empty set Φ is the minimum forcing subset of S. Since the empty set Φ is a subset of every set, it follows that S is the unique weak edge triangle free detour basis of G. The converse is clear.

If $fwdn_{\triangle f}(G) = wdn_{\triangle f}(G)$, then $fwdn_{\triangle f}(S) = wdn_{\triangle f}(G)$ for every weak edge triangle free detour basis S in G. Also by Theorem 1.1, $wdn_{\triangle f}(G) \geq 2$ and hence $fwdn_{\triangle f}(G) \geq 2$. Then by proof of first part of this theorem , G has at least two weak edge triangle free detour bases and so the empty set Φ is not a forcing subset of any weak edge triangle free detour basis of G. Since $fwdn_{\triangle f}(S) = wdn_{\triangle f}(G)$, no proper subset of S is a forcing subset of S. Thus no weak edge triangle free detour basis of G is the unique weak edge triangle free detour basis containing any of its proper subsets.

Conversely, the data implies that G contains more than one weak edge triangle free detour basis and no subset of any weak edge triangle free detour basis S other than S is a forcing subset for S. Hence it follows that $fwdn_{\triangle f}(G) = wdn_{\triangle f}(G)$. □

Theorem 2.4. *Let G be connected graph. Then $fwdn_{\triangle f}(G) = 1$ if and only if G has at least two weak edge triangle free detour bases, one of which is a unique weak edge triangle free detour basis containing one of its elements.*

Proof. Let $fwdn_{\triangle f}(G) = 1$. Then G has at least two weak edge triangle free detour bases. Also, since $fwdn_{\triangle f}(G) = 1$, there is a singleton subset T of a weak edge triangle free detour basis S of G such that T is not a subset of any other weak edge triangle free detour basis of G. Thus S is the unique weak edge triangle free detour basis containing one of its elements. The converse is clear. □

Theorem 2.5. *Let G be a connected graph G and let $\Im$ be the set of relative complements of the minimum forcing subsets in their respective weak edge triangle free detour basis in G. Then $\cap_{F \in \Im} F$ is the set of weak edge triangle free detour edges of G.*

Proof. Let W be the set of all weak edge triangle free detour vertices of G. We claim that $W = \cap_{F \in \Im} F$. Let $v \in W$. Then v is a weak edge triangle free detour vertex of G so that v belongs to every weak edge triangle free detour basis S of G. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S is the unique weak edge triangle free detour basis containing T' so that T' is a forcing subset for S with $|T'| < |T|$, which is a contradiction to T a minimum forcing subset for S_x. Thus $v \notin T$ and so $v \in F$, where F is the relative complement of T in S. Hence $v \in \cap_{F \in \Im} F$ so that $W \subseteq \cap_{F \in \Im} F$.

Conversely, let $v \in \cap_{F \in \Im} F$. Then v belongs to the relative complement of T in S for every T and every S such that $T \subseteq S$, where T is a minimum forcing subset for S. Since F is the relative complement of T in S, $F \subseteq S$ and so $v \in S$ for every S so that v is a weak edge triangle free detour vertex of G. Thus $v \in W$ and so $\cap_{F \in \Im} F \subseteq W$. Hence $W = \cap_{F \in \Im} F$. □

Corollary 2.6. *Let G be a connected graph and S a weak edge triangle free detour basis of G. Then no weak edge triangle free detour vertex of G belongs to any minimum forcing set of S.*

Proof. The proof is contained in the proof of the first part of Theorem 2.5. □

Theorem 2.7. *Let G be the a connected graph and W be the set of all weak edge triangle free detour vertices of G. Then $fwdn_{\triangle f}(G) \leq wdn_{\triangle f}(G) - |W|$.*

Proof. Let S be any weak edge triangle free detour basis of G. Then $wdn_{\triangle f}(G) = |S|, W \subseteq S$ and S is the unique weak edge triangle free detour basis containing $S - W$. Thus $fwdn_{\triangle f}(S) \leq |S - W| = |S| - |W| = wdn_{\triangle f}(G) - |W|$. □

Proposition 2.8. a) If G is the complete bipartite graph $K_{r,s}(2 \leq r \leq s)$, then $wdn_{\triangle f}(G) = fwdn_{\triangle f}(G) = 2$.

b) If G is the cycle $C_n(n \geq 4)$, then $wdn_{\triangle f}(G) = fwdn_{\triangle f}(G) = 2$.

c) If G is a tree of order $n(n \geq 2)$, with k end-vertices, then $wdn_{\triangle f}(G) = k$, $fwdn_{\triangle f}(G) = 0$.

d) If G is a complete graph K_n, then $dn_{\triangle f}(G) = n$ and $f_{dn_{\triangle f}}(G) = 0$.

Proof. a) By Theorem 1.6, a set S of vertices is a weak edge triangle free detour basis if and only if S consists of any two vertices of G. For each vertex v in G there are two or more vertices adjacent with v. Thus the vertex v belongs to more than one weak edge triangle free detour basis of G. Hence it follows that no set consisting of a single vertex is a forcing subset for any weak edge triangle free detour basis of G. Thus the result follows.

b) By Theorem 1.3 or 1.4 (according as G even or odd), a set S of two adjacent vertices of G is a weak edge triangle free detour basis. For each vertex v in G there are two vertices adjacent with v. Thus the vertex v belongs to more than one weak edge triangle free detour basis of G. Hence it follows that no set consisting of a single vertex is a forcing subset for any weak edge triangle free detour basis of G. Thus the result follows.

c) By Theorem 1.5, $wdn_{\triangle f}(G) = k$. Since the set of all end-vetices of a tree is the unique weak edge triangle free detour basis, the result follows from Theorem 2.3.

d) For K_n, it follows from Theorem 1.4 that the set of all vertices of G is the unique triangle free detour basis of G. Now, it follows from Theorem 2.3 that $f_{dn_{\triangle f}}(G) = 0$. □

Theorem 2.9. *For any two positive integers a, b with $0 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G such that $fwdn_{\triangle f}(G) = a$, $wdn_{\triangle f}(G) = b$.*

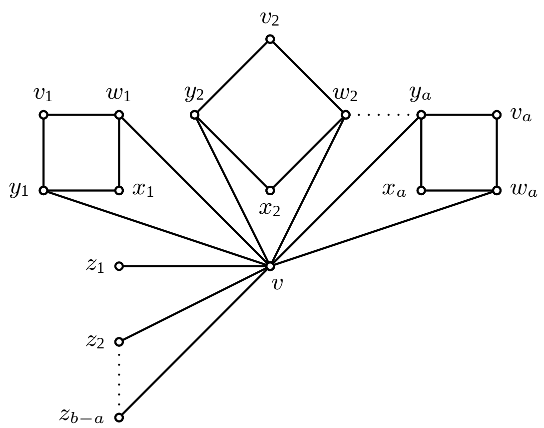
Proof. **Case 1.** $a = 0$. For each $b \geq 2$, let G be a tree with b end-vertices. Then $fwdn_{\triangle f}(G) = 0$ and $wdn_{\triangle f}(G) = b$ by Theorem 2.8(c).

Case 2. $a \geq 1$. For each $i(1 \leq i \leq a)$, let $F_i : u_i, v_i, w_i, x_i, u_i$ be the cycle of order 4 and let $H = K_{1,b-a}$ be a star at v whose set of end-vertices is $\{z_1, z_2, ..., z_{b-a}\}$. Let G be the graph obtained by joining the central vertex v of H to both vertices u_i, w_i of each $F_i(1 \leq i \leq a)$. Clearly the graph G is connected and is shown in Figure 2.3. Let $W = \{z_1, z_2, ..., z_{b-a}\}$ be the set of all $(b - a)$ end-vertices of G.

First, we show that $wdn_{\triangle f}(G) = b$. Then by Theorems 1.7 and 1.8 every weak edge triangle free detour basis contains W and at least one vertex from each $F_i(1 \leq i \leq a)$. Thus

$wdn_{\triangle f}(G) \geq (b-a)+a=b$. On the other hand, since the set $S_1 = W \cup \{v_1, v_2, ..., v_a\}$ is a weak edge triangle free detour set of G, it follows that $wdn_{\triangle f}(G) \leq |S_1| = b$. Therefore $wdn_{\triangle f}(G) = b$.

Next we show that $fwdn_{\triangle f}(G) = a$. It is clear that W is the set of all weak edge triangle free detour vertices of G. Hence it follows from Theorem 2.7 that $fwdn_{\triangle f}(G) \leq wdn_{\triangle f}(G) - |W| = b-(b-a) = a$. Now, since $wdn_{\triangle f}(G) = b$, it is easily seen that a set S is a weak edge triangle free detour basis of G if and only if S is of the form $S = W \cup \{y_1, y_2, ..., y_a\}$, where $y_i \in \{v_i, x_i\} \subseteq V(F_i)(1 \leq i \leq a)$. Let T be a subset of S with $|T| < a$. Then there is a vertex $y_j(1 \leq j \leq a)$ such that $y_j \notin T$. Let $s_j \in \{v_j, x_j\} \subseteq V(F_j)$ disjoint from y_j. Then $S^{'} = (S - \{y_j\}) \cup \{s_j\}$ is a weak edge triangle free detour basis that contains T. Thus S is not the unique weak edge triangle free detour basis containing T. Thus $fwdn_{\triangle f}(S) \geq a$. Since this is true for all weak edge triangle free detour basis of G, it follows that $fwdn_{\triangle f}(G) \geq a$ and so $fwdn_{\triangle f}(G) = a$.



□

Figure 2.3 : G

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