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# SINGULAR CONFORMABLE FRACTIONAL DIRAC SYSTEMS

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**Abstract** We establish the classical Weyl-Titchmarsh theory for conformable fractional Dirac system.

## 1 Introduction

Nowadays, conformable fractional calculus have become a very active area of research. Conformable fractional calculus were introduced by Khalil, Al Horani, Yousef and Sababheh in [12]. Later Abdeljawad [1] defined the right and left conformable fractional derivatives and conformable fractional integrals. Although classical fractional derivatives (Riemann–Liouville and Caputo) share some weaknesses, the conformable fractional derivative allows for many extensions of classical properties in ordinary calculus. For some recent works on the theory of conformable fractional calculus we refer the readers to [1, 2, 12, 8, 11, 10] and reference therein.

The theory of Titchmarsh-Weyl plays an important role in the spectral analysis of differential operators. This theory of was introduced by H. Weyl in 1910 ([16]). In [16], the author studied that the following Sturm-Liouville problem

$$-\left(py'\right)' + qy = \lambda y,\ 0 \leq x < \infty.$$

Weyl defined the limit-point, limit-circle classification. Since then many authors have expounded on various features of this theory (see [14, 13, 18, 5, 6, 7]). Recently, in [9], Baleanu et al. studied the singular conformable Sturm-Liouville problem. They investigated Titchmarsh-Weyl theory. Allahverdiev and Tuna studied one-dimensional conformable fractional Dirac systems ([3, 4]).

The purpose of this article is to develop Titchmarsh-Weyl theory for a conformable fractional (CF) Dirac system defined by

$$\begin{cases} -T_\alpha y_2 + p(x)y_1 = \lambda y_1 \\ T_\alpha y_1 + r(x)y_2 = \lambda y_2 \end{cases},\ x \in [0,\infty), \tag{1.1}$$

where $p$ and $r$ are real-valued functions defined on $[0,\infty)$ and $\lambda$ is a complex spectral parameter. To the best of authors knowledge such study has not been reported earlier in the literature.

In this context, we give some basic concepts of conformable fractional calculus (see [1, 2, 12] for more details). Let $\alpha$ be a positive number with $0 < \alpha < 1$.

**Definition 1.1.** Let $g : [0,\infty) \to \mathbb{R}$. The CF-derivative of $g$ is defined by

$$T_\alpha g(t) = \lim_{\varepsilon \to 0} \frac{g(t + \varepsilon t^{1-\alpha}) - g(t)}{\varepsilon}.$$

**Definition 1.2.** Let $g : [0,\infty) \to \mathbb{R}$. The CF-integral $g$ is defined by

$$\left(I_\alpha g\right)(t) = \int_0^t g(x)d\alpha\left(x\right) = \int_0^t x^{\alpha-1}g(x)dx.$$

Let $L_\alpha^2(0,\infty)$ denotes the Hilbert space (see [17]) consisting of all functions satisfying the following

$$\|g\| := \left( \int_0^\infty |g(t)|^2 \, d\alpha(t) \right)^{1/2} < \infty$$

with the inner product

$$(g,h) := \int_0^\infty g(t)\, \overline{h(t)} d\alpha(t),$$

where $g, h \in L_\alpha^2(0,\infty)$.

Let us consider the following system:

$$\tau y = \lambda y, \ 0 \le x < \infty, \tag{1.2}$$

where

$$\tau y := \begin{cases} -T_\alpha y_2 + p(x) y_1 \\ T_\alpha y_1 + r(x) y_2 \end{cases}, \ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \ \lambda \in \mathbb{C},$$

$p, r : [0,\infty) \to \mathbb{R}$, and $p, r \in L_{\alpha,loc}^1(0,\infty)$,

$$L_{\alpha,loc}^1(0,\infty) := \left\{ g : [0,\infty) \to \mathbb{C} : \int_0^b |g(t)| \, d\alpha(t) < \infty, \ \forall b \in (0,\infty) \right\}.$$

Now, we shall define an appropriate Hilbert space. Let us consider the following inner product in the Hilbert space $L_\alpha^2((0,\infty); E)$ $(E := \mathbb{C}^2)$

$$(g,h) := \int_0^\infty (g(t), h(t))_E d\alpha(t).$$

**Theorem 1.3** ([3]). *Let $c_i, i = 0,1,$ be given complex constants. Then, the CF-Dirac system (1.2) has a unique solution $\Psi(x,\lambda) = \begin{pmatrix} \Psi_1(x,\lambda) \\ \Psi_2(x,\lambda) \end{pmatrix}$ such that*

$$\Psi_1(0,\lambda) = c_0, \ \Psi_2(0,\lambda) = c_1, \ \lambda \in \mathbb{C}. \tag{1.3}$$

*Further, for each $x \in [0,\infty)$, the vector-valued function $\Psi(x,\lambda)$ is an entire function of $\lambda$.*

Let $D_\alpha$ be a subset of $L_\alpha^2((0,\infty); E)$ such that

$$D_\alpha = \left\{ y \in L_\alpha^2((0,\infty); E) : y \in AC_{loc}[0,\infty); E), \ \tau y \in L_\alpha^2((0,\infty); E) \right\},$$

where $AC_{loc}([0,\infty); E)$ denotes the collection of vector-valued functions $y$ which are absolutely continuous on all compact intervals $[0,b] \subset [0,\infty)$, $\forall b \in (0,\infty)$.

**Lemma 1.4.** *Following equality holds*

$$\int_0^x [((\tau y(t), u(t))_E - (y(t), \tau u(t))_E] \, d\alpha(t)$$

$$= [y,u]_x - [y,u]_0, \tag{1.4}$$

*where $x \in (0,\infty)$, $y, u \in D_\alpha$ and*

$$[y,u]_x := y_1(x) \overline{u_2(x)} - \overline{u_1(x)} y_2(x).$$

*Proof.* For $y, u \in D_\alpha$ one has

$$\int_0^x \left[(\tau y(t), u(t))_E - (y(t), \tau u(t))_E\right] d\alpha\,(t)$$

$$= \int_0^x (-T_\alpha y_2(t) + p\,(t)\, y_1(t))\overline{u_1(t)} d\alpha\,(t)$$

$$+ \int_0^x (T_\alpha y_1(t) + r\,(t)\, y_2(t))\,\overline{u_2(t)} d\alpha\,(t)$$

$$- \int_0^x y_1(t)\overline{(-T_\alpha u_2(t) + p\,(t)\, u_1(t))} d\alpha\,(t)$$

$$- \int_0^x y_2(t)\overline{(T_\alpha u_1(t) + r\,(t)\, u_2(t))} d\alpha\,(t)$$

$$= - \int_0^x [(T_\alpha y_2(t))\overline{u_1(t)} + y_2(t)\overline{T_\alpha u_1(t)}] d\alpha\,(t)$$

$$+ \int_0^x [(T_\alpha y_1(t))\,\overline{u_2(t)} + y_1(t)\overline{T_\alpha u_2(t))}] d\alpha\,(t)\,.$$

Thus we get

$$\int_0^x \left[(\tau y(t), u(t))_E - (y(t), \tau u(t))_E\right]\ d\alpha\,(t)$$

$$= \int_0^x \left[(T_\alpha y_1\,(t))\overline{u_2\,(t)} - (T_\alpha \overline{u_1\,(t)}) y_2\,(t)\right]\ d\alpha\,(t)$$

$$= \int_0^x T_\alpha [y_1\,(t)\,\overline{u_2\,(t)} - \overline{u_1\,(t)} y_2\,(t)]\ d\alpha\,(t)$$

$$= [y, u]_x - [y, u]_0\,.$$

□

Let $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be two solutions of (1.2) satisfying

$$\varphi_1\,(0, \lambda) = \cos\delta,\ \ \varphi_2\,(0, \lambda) = \sin\delta, \tag{1.5}$$

$$\psi_1\,(0, \lambda) = \sin\delta,\ \ \psi_2\,(0, \lambda) = -\cos\delta, \tag{1.6}$$

where $\delta \in \mathbb{R}$ be fixed.

**Lemma 1.5.** *For $x \in [0, \infty)$ and $\lambda \in \mathbb{C}$, following equalities hold*

$$\overline{\varphi(x, \lambda)} = \varphi\left(x, \overline{\lambda}\right),$$

$$\overline{\psi(x, \lambda)} = \psi\left(x, \overline{\lambda}\right).$$

*Proof.* By virtue of the fact that $\varphi(x, \lambda)$ is a solution of (1.2), we arrive that

$$-T_\alpha \varphi_2 + p(x)\varphi_1 = \lambda\varphi_1,$$

$$T_\alpha \varphi_1 + r(x)\varphi_2 = \lambda\varphi_2,$$

where $x \in [0, \infty)$. Thus, we obtain

$$-T_\alpha \overline{\varphi_2} + p(x)\overline{\varphi_1} = \overline{\lambda}\overline{\varphi_1},$$

$$T_\alpha \overline{\varphi_1} + r(x)\overline{\varphi_2} = \overline{\lambda}\overline{\varphi_2}.$$

It follows from (1.5) that the function $\overline{\varphi(x,\lambda)}$ is a solution of the following system

$$-T_\alpha z_2 + p(x)z_1 = \overline{\lambda}z_1,$$

$$T_\alpha z_1 + r(x)z_2 = \overline{\lambda}z_2.$$

On the other hand, $\varphi\left(x,\overline{\lambda}\right)$ is also a solution of (1.2)-(1.5). From the uniqueness of solutions, we conclude that

$$\overline{\varphi(x,\lambda)} = \varphi\left(x,\overline{\lambda}\right).$$

□

Now we define the *Wronskian* of $y$ and $z$ by

$$W\left(y,z\right)(x) = y_1(x)z_2\left(x\right) - z_1(x)y_2\left(x\right), \tag{1.7}$$

where $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$.

**Theorem 1.6.** *Let $y$ and $z$ be two solutions of Eq.(1.2). Then $W\left(y,z\right)(x)$ does not depend on $x$, $0 \leq x < \infty$, $\lambda \in \mathbb{C}$.*

*Proof.* From (1.4), we conclude that

$$\int_0^x \left[\left(\tau y(t,\lambda), \overline{u(t,\lambda)}\right)_E - \left(y(t,\lambda), \tau\overline{u(t,\lambda)}\right)_E\right] d\alpha\left(t\right)$$

$$= \left[y,\overline{u}\right]_x - \left[y,\overline{u}\right]_0 = W\left(y,u\right)(x,\lambda) - W\left(y,u\right)(0,\lambda).$$

Since $\tau y = \lambda y$ and $\tau\overline{u} = \overline{\lambda}\overline{u}$, we conclude that

$$\int_0^x \left[(\lambda y(t,\lambda), \overline{u(t,\lambda)})_E - (y(t,\lambda), \overline{\lambda}\overline{u(t,\lambda)})_E\right] d\alpha\left(t\right)$$

$$= (\lambda - \lambda)\int_0^x (y(t,\lambda), \overline{u(t,\lambda)})_E d\alpha\left(t\right) = 0$$

$$= W\left(y,u\right)(x,\lambda) - W\left(y,u\right)(0,\lambda). \tag{1.8}$$

This proves the assertion. □

**Lemma 1.7.** *Let $y$ is a solution of Eq. (1.2). Then the following equation holds*

$$\int_0^\xi \left\|y(x,\lambda)\right\|_E^2 d\alpha\left(x\right) = \frac{\left[W\left(y,\overline{y}\right)(\xi,\lambda) - W\left(y,\overline{y}\right)(0,\lambda)\right]}{2iv}, \tag{1.9}$$

$$\xi > 0,\ v = \operatorname{Im}\lambda,\ \lambda \in \mathbb{C}.$$

*Proof.* Writing $u(x,\lambda) = \overline{y(x,\lambda)}$ in (1.8), we can get the desired result. □

## 2 Main Results

Let us consider the general solution

$$\psi + l\varphi$$

of (1.2) satisfying

$$(\psi_1(b,\lambda) + l\varphi_1(b,\lambda))\cos\beta + (\psi_2(b,\lambda) + l\varphi_2(b,\lambda))\sin\beta = 0, \tag{2.1}$$

where $b \in (0,\infty)$ and $\beta \in \mathbb{R}$. Then, we see that

$$l = -\frac{\psi_{1,b}\cot\beta + \psi_{2,b}}{\varphi_{1,b}\cot\beta + \varphi_{2,b}}, \tag{2.2}$$

where

$$\psi_{1,b} = \psi_1(b,\lambda), \psi_{2,b} = \psi_2(b,\lambda),$$
$$\varphi_{1,b} = \varphi_1(b,\lambda), \varphi_{2,b} = \varphi_2(b,\lambda).$$

Since $\psi(x,\lambda)$ and $\varphi(x,\lambda)$ are entire functions of $\lambda$, $l$ is a meromorphic function of $\lambda$. Since the regular CF Dirac system has real eigenvalue, all poles of $l$ are real and simple. Replacing $\cot\beta$ by $z$ yields

$$l = -\frac{\psi_{1,b}z + \psi_{2,b}}{\varphi_{1,b}z + \varphi_{2,b}}. \tag{2.3}$$

From Theorem 1.6, we obtain

$$\psi_{1,b}\varphi_{2,b} - \varphi_{1,b}\psi_{2,b} = W(\psi,\varphi)(b,\lambda)$$

$$= W(\psi,\varphi)(0,\lambda) = 1 \neq 0.$$

The real axis in the $z$-plane is associated with a circle $C_b(\lambda)$ in the $l$-plane. These circles are called the Weyl's circles.

The task is now to find $P_b(\lambda)$ and $R_b(\lambda)$, where $P_b(\lambda)$ and $R_b(\lambda)$ are the circle's center and radius, respectively.

**Theorem 2.1.** *Following equalities hold*

$$P_b(\lambda) = -\frac{W(\psi,\overline{\varphi})(b,\lambda)}{W(\varphi,\overline{\varphi})(b,\lambda)}, \tag{2.4}$$

$$R_b(\lambda) = \left(2v\int_0^b \|\varphi(x,\lambda)\|_E^2\, d\alpha(x)\right)^{-1}, \tag{2.5}$$

*where* $\lambda \in \mathbb{C}$ *and* $v = \operatorname{Im}\lambda \neq 0$.

*Proof.* An easy computation shows that $l(\lambda, z') = \infty$ if and only if

$$z' = -\frac{\varphi_{2,b}}{\varphi_{1,b}}.$$

Hence, $P_b(\lambda)$ is given by the formula

$$\begin{aligned} P_b(\lambda) &= l\left(\lambda, -\frac{\varphi_{2,b}}{\varphi_{1,b}}\right) \\ &= -\frac{\psi_{1,b}\left(-\frac{\overline{\varphi_{2,b}}}{\overline{\varphi_{1,b}}}\right) + \psi_{2,b}}{\varphi_{1,b}\left(-\frac{\overline{\varphi_{2,b}}}{\overline{\varphi_{1,b}}}\right) + \varphi_{2,b}} \\ &= -\frac{\psi_{1,b}\varphi_2(b,\overline{\lambda}) - \varphi_1(b,\overline{\lambda})\psi_{2,b}}{\varphi_{1,b}\varphi_2(b,\overline{\lambda}) - \varphi_1(b,\overline{\lambda})\varphi_{2,b}} \\ &= -\frac{W(\psi,\overline{\varphi})(b,\lambda)}{W(\varphi,\overline{\varphi})(b,\lambda))}. \end{aligned}$$

Moreover, we may write

$$\begin{aligned} R_b(\lambda) &= \left| \frac{\psi_2(b,\lambda)}{\varphi_2(b,\lambda)} - \frac{W(\psi,\overline{\varphi})(b,\lambda)}{W(\varphi,\overline{\varphi})(b,\lambda)} \right| \\ &= \left| \frac{\varphi_2\left(b,\overline{\lambda}\right) W(\psi,\varphi)(b,\lambda)}{\varphi_2(b,\lambda) W(\varphi,\overline{\varphi})(b,\lambda)} \right| \\ &= \left| \frac{W(\psi,\varphi)(b,\lambda)}{W(\varphi,\overline{\varphi})(b,\lambda)} \right| = \frac{1}{|W(\varphi,\overline{\varphi})(b,\lambda)|}, \end{aligned}$$

since $W_q(\psi,\varphi)(b) = 1$. By Lemma 1.7, we have

$$W(\varphi,\overline{\varphi})(b,\lambda) = 2iv \int_0^b \|\varphi(x,\lambda)\|_E^2 \, d\alpha(x).$$

Thus, we get

$$|W(\varphi,\overline{\varphi})(b,\lambda)| = 2|v| \int_0^b \|\varphi(x,\lambda)\|_E^2 \, d\alpha(x),$$

and the theorem follows. □

**Theorem 2.2.** *For all $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda > 0$, the exterior of the circle $C_b$ is associated with the upper half-plane.*

*Proof.* Let $\lambda = u + iv$, where $v > 0$. From (1.7), we have

$$\begin{aligned} \operatorname{Im}\left\{ \frac{\varphi_{2,b}}{\varphi_{1,b}} \right\} &= \frac{1}{2} i \left\{ -\frac{\varphi_{2,b}}{\varphi_{1,b}} + \frac{\varphi_2\left(b,\overline{\lambda}\right)}{\varphi_1\left(b,\overline{\lambda}\right)} \right\} \\ &= \frac{1}{2} i \frac{W(\overline{\varphi},\varphi)(b,\lambda)}{|\varphi_{1,b}|^2} \\ &= -\frac{1}{2} i \frac{W(\varphi,\overline{\varphi})(b,\lambda)}{|\varphi_{1,b}|^2} \\ &= \frac{v}{|\varphi_{1,b}|^2} \int_0^b \|\varphi(x,\lambda)\|_E^2 \, d\alpha(x) > 0. \end{aligned}$$

□

**Theorem 2.3.** *Let $\psi(x,\lambda)$ and $\varphi(x,\lambda)$ be the solutions of (1.2) satisfying*

$$\varphi_1(0,\lambda) = \cos\delta, \ \varphi_2(0,\lambda) = \sin\delta,$$

$$\psi_1(0,\lambda) = \sin\delta, \ \psi_2(0,\lambda) = -\cos\delta,$$

*where $\operatorname{Im}\lambda \neq 0$. Then, the solution $\chi = \psi + l\varphi$ satisfies the following condition*

$$\{\psi_1(b,\lambda) + l\varphi_1(b,\lambda)\}\cos\beta + \{\psi_2(b,\lambda) + l\varphi_2(b,\lambda)\}\sin\beta = 0$$

*if and only if $l$ is on $C_b$ with*

$$W(\chi,\overline{\chi})(b,\lambda) = 0.$$

*Letting $b \to \infty$, the circles $C_b$ may converge either to a circle $C_\infty$ or a point $m_\infty$. In the limit-circle (LC) case, all solutions of (1.2) belong to $L_\alpha^2((0,\infty);E)$. In that case, a point is on $C_\infty$ $\Leftrightarrow$*

$$\lim_{b\to\infty} W(\chi,\overline{\chi})(b,\lambda) = 0.$$

*In the limit-point (LP) case, precisely one linearly independent solution in $L_\alpha^2((0,\infty);E)$, if $\operatorname{Im}\lambda \neq 0$.*

*Proof.* Let $\lambda \in \mathbb{C}$. Then, we find

$$W(\psi + l\varphi, \overline{\psi + l\varphi})(0, \lambda)$$

$$= W\left(\psi, \overline{\psi}\right)(0, \lambda) + lW\left(\varphi, \overline{\psi}\right)(0, \lambda) + \overline{l}W\left(\psi, \overline{\varphi}\right)(0, \lambda)$$

$$+ |l|^2 W\left(\varphi, \overline{\varphi}\right)(0, \lambda) = -l + \overline{l} = -2i \operatorname{Im} l.$$

Using Lemma 1.7,

$$2v \int_0^b \|\psi(x, \lambda) + l\varphi(x, \lambda)\|_E^2 \, d\alpha\,(x)$$

$$= \frac{1}{i}(W(\psi + l\varphi, \psi + l\varphi)(b, \lambda) + 2i \operatorname{Im} l). \tag{2.6}$$

From Theorem 2.2, $l$ is inside $C_b$ for $v > 0$ if $\operatorname{Im} z < 0$. It follows from (2.3) that

$$z = -\frac{\psi_{2,b} + l\varphi_{2,b}}{\psi_{1,b} + l\varphi_{1,b}}.$$

and

$$i\left(z - \overline{z}\right) = i\left\{-\frac{\psi_{2,b} + l\varphi_{2,b}}{\psi_{1,b} + l\varphi_{1,b}} + \frac{\overline{\psi_{2,b}} + \overline{l\varphi_{2,b}}}{\overline{\psi_{1,b}} + \overline{l\varphi_{1,b}}}\right\}$$

$$= i\frac{W\left(\psi + l\varphi, \overline{\psi + l\varphi}\right)(b, \lambda)}{\left|\psi_{1,b} + l\varphi_{1,b}\right|^2}.$$

Thus

$$\operatorname{Im} z < 0 \Leftrightarrow iW\left(\psi + l\varphi, \overline{\psi + l\varphi}\right)(b, \lambda) > 0. \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$\int_0^b \|\psi(x, \lambda) + l\varphi(x, \lambda)\|_E^2 \, d\alpha\,(x) < \frac{\operatorname{Im} l}{v}. \tag{2.8}$$

Moreover, $\operatorname{Im} z = 0 \Leftrightarrow$ the point $l$ is on $C_b$. Hence

$$W\left(\psi + l\varphi, \overline{\psi + l\varphi}\right)(b, \lambda) = 0.$$

and

$$\int_0^b \|\psi(x, \lambda) + l\varphi(x, \lambda)\|_E^2 \, d\alpha\,(x) = \frac{\operatorname{Im} l}{v}.$$

Moreover, if the point $l$ is inside $C_b$ and $0 < k < b$, then we have

$$\int_0^k \|\psi(x, \lambda) + l\varphi(x, \lambda)\|_E^2 \, d\alpha\,(x)$$

$$< \int_0^b \|\psi(x, \lambda) + l\varphi(x, \lambda)\|_E^2 \, d\alpha\,(x) < \frac{\operatorname{Im} l}{v},$$

i.e., the circle $C_k$ contains the circle $C_b$ if $k < b$. Consequently, these circles are nested and may converge to the limit-circle $C_\infty$ or to limit-point $m_\infty$ as $b \to \infty$. It follows from (2.5) that $\varphi(., \lambda) \in L_\alpha^2((0, \infty)\,; E)$ If $C_b \to C_\infty$. Let $\xi \in C_\infty$. From (2.8), we deduce that

$$\int_0^b \|\psi(x, \lambda) + l\varphi(x, \lambda)\|_E^2 \, d\alpha\,(x) < \frac{\operatorname{Im} \xi}{v}.$$

Thus, we obtain $\psi(.,\lambda)+l\varphi(.,\lambda)\in L^2_\alpha((0,\infty);E)$. If $C_b\to C_\infty,\ b\to\infty$ and $\operatorname{Im}\lambda\neq 0$, then all solutions of (1.2) are in the space $L^2_\alpha((0,\infty);E)$. If $C_b\to m_\infty,\ b\to\infty$, then $\lim_{b\to\infty}R_b(\lambda)=0$. From (2.5), we get $\varphi(.,\lambda)\notin L^2_\alpha((0,\infty);E)$ i.e., only one solution belongs to $L^2_\alpha((0,\infty);E)$. Furthermore, $l$ is on $C_\infty \Leftrightarrow$

$$\int_0^\infty \|\psi(x,\lambda)+l\varphi(x,\lambda)\|_E^2\, d\alpha(x)=\frac{\operatorname{Im} l}{v}. \tag{2.9}$$

By virtue of (2.9), (2.6) and (2.8), we see that $l$ is on $C_\infty \Leftrightarrow$

$$\lim_{b\to\infty} W\left(\psi+l\varphi,\overline{\psi+l\varphi}\right)(b,\lambda)=0,$$

which completes the proof. □

Now, we study the behavior of the solutions of (1.2) around $\infty$.

**Theorem 2.4.** *The equation (1.2) is LP at infinity.*

*Proof.* Let

$$\theta(t)=\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix} \text{ and } \phi(t)=\begin{pmatrix}\phi_1(t)\\ \phi_2(t)\end{pmatrix}$$

be the linearly independent solutions of

$$\tau y=0,$$

where $t\in[0,\infty)$. It is easy to check that

$$W(\theta,\phi)(t)=\kappa\neq 0,\ t\in[0,\infty).$$

Then, for

$$\rho(t)=\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix},\ \omega(t)=\begin{pmatrix}\overline{\phi_2(t)}\\ -\overline{\phi_1(t)}\end{pmatrix},$$

we obtain

$$\begin{aligned}|\kappa| &= |W(\theta,\phi)(t)|\\ &= |\theta_1(t)\phi_2(t)-\phi_1(t)\theta_2(t)|\\ &= |(\rho(t),\omega(t))_E|\leq \|\rho t)\|_E\,\|\omega(t)\|_E\\ &\leq \frac{1}{2}(\|\rho(t)\|_E^2+\|\omega(t)\|_E^2).\end{aligned}$$

It immediately that $\rho$ and $\omega$ (also $\theta$ and $\phi$) can not both lie right in the space $L^2_\alpha((0,\infty);E)$, which proves the theorem. □

**Remark 2.5.** For $\alpha=1$, Theorem 2.4 was introduced in [13, 15].

**Conclusion.** In this study, the Weyl-Titchmarsh theory for a CF-Dirac system is studied. This paper extends classical Weyl-Titchmarsh theory.

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