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SOME UNIFIED INTEGRAL FORMULAE ASSOCIATED WITH HURWITZ-LERCH ZETA FUNCTION

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Abstract In the present work, we investigate five new generalised integral formulae by involving the extension form of the Hurwitz-Lerch zeta function and obtain the results in the form of a hypergeometric function in product form by using the properties of the Hadamard product, from which two power series emerge. Furthermore, we also address their special cases by making suitable substitutions. The results obtained here are of a general nature and far more auspicious in the study of applied science, engineering and technology problems.

1 Introduction and Preliminaries

In the field of science and technology, integral formulae are very useful couse of the implementation of the relevant problems. As we know that several integral mechanisms have already been developed but due to time requirements, we are also contributing to the development of new integral formulae associated with Hurwitz-Lerch zeta function.
The Hurwitz-Lerch Zeta function (1.1) and its integral assertion (1.2) are respectively concrete by (see [1] pp. 27, [5] pp. 121 and [6] pp. 194) as below:

$$\phi(z,\varepsilon,\rho)=\sum_{l=0}^{\infty}\frac{z^l}{(n+\rho)^{\varepsilon}},\ (\rho\in C\backslash z_0^-,\varepsilon\in C, for\ \ |z|<1;\Re(\varepsilon)>1, when\,|z|=1). \tag{1.1}$$

Besides

$$\phi(z,\varepsilon,\rho)=\frac{1}{\Gamma(\varepsilon)}\int_0^{\infty}\frac{t^{\varepsilon-1}e^{-\rho t}}{1-ze^{-t}}dt=\frac{1}{\Gamma(\varepsilon)}\int_0^{\infty}\frac{t^{\varepsilon-1}e^{-(\rho-1)t}}{e^t-z}dt, \tag{1.2}$$

$(\Re(\varepsilon)>0,\Re(\rho)>0\ \text{ for } |z|\leq 1(z\neq 1);\Re(\varepsilon)>1, \text{ when } z=1)$.

In this sequel, Goyal and Laddha [4] and Garg et al. [3] defined the new extension formula of Hurwitz-Lerch Zeta function in (1.3) and (1.5) respectively and also defined their integral representation as in (1.4) and (1.6) respectively.

$$\phi_{\gamma}^{*}(z,\varepsilon,\rho)=\sum_{l=0}^{\infty}\frac{(\gamma)_n}{l!}\frac{z^l}{(n+\rho)^{\varepsilon}}, \tag{1.3}$$

$(\gamma\in C,\rho\in C\backslash z_0^-,\varepsilon\in C \text{ when } |z|<1;\Re(\varepsilon-\gamma)>1 \text{ when } |z|=1)$,

$$\phi_{\gamma}^{*}(z,\varepsilon,\rho)=\frac{1}{\Gamma(\varepsilon)}\int_0^{\infty}\frac{t^{\varepsilon-1}e^{-\rho t}}{(1-ze^{-t})^{\gamma}}dt=\frac{1}{\Gamma(\varepsilon)}\int_0^{\infty}\frac{t^{\varepsilon-1}e^{-(\rho-\gamma)t}}{(e^t-z)^{\gamma}}dt, \tag{1.4}$$

$(\Re(\varepsilon)>0,\Re(\rho)>0 \text{ when } |z|\leq 1(z\neq 1);\Re(\varepsilon)>1 \text{ when } z=1)$.

And

$$\phi_{\gamma,u;v}(z,\varepsilon,\rho)=\sum_{l=0}^{\infty}\frac{(\gamma)_l(u)_l}{(v)_l l!}\frac{z^l}{(l+\rho)^{\varepsilon}}, \tag{1.5}$$

$(\gamma, u, v \in C, \rho \in C\backslash z_0^-, \varepsilon \in C$ when $|z| < 1; \Re(\varepsilon + v - \gamma - u) > 1$ when $|z| = 1)$,

$$\phi_{\gamma,u;v}(z, \varepsilon, \rho) = \frac{1}{\Gamma(\varepsilon)} \int_0^\infty t^{\varepsilon-1} e^{-\rho t} {}_2F_1(\gamma, u; v; ze^{-t}) dt, \tag{1.6}$$

$(\Re(\varepsilon) > 0, \Re(\rho) > 0$ when $|z| \leq 1 (z \neq 1); \Re(\varepsilon) > 1$ when $z = 1)$.

In addition, Parmar [13] introduced and investigated the new extension of the Hurwitz-Lerch Zeta function in the form of beta function as

$$\phi_{\gamma,u;v}(z, \varepsilon, \rho; p) = \sum_{l=0}^{\infty} \frac{(\gamma)_l B(u + l, v - u; p)}{B(u, v - u) l!} \frac{z^l}{(l + \rho)^\varepsilon}, \tag{1.7}$$

where $p \geq 0, \gamma, u, v \in C, \rho \in C\backslash Z_0^-, \varepsilon \in C$ when $|z| < 1, \Re(\varepsilon + v - \gamma - u) > 1$ when $|z| = 1$.

Where $B(\vartheta, \varphi; p)$ is the extended beta function which is investigated by Chaudhry et al. [1] as follow

$$B(\vartheta, \varphi; p) = B_p(\vartheta, \varphi) = \int_0^1 t^{\vartheta-1}(1 - t)^{\varphi-1} e^{-\frac{p}{t(1-t)}} dt, \tag{1.8}$$

where $\Re(p) > 0, \Re(\vartheta) > 0, \Re(\varphi) > 0$.

Shadab et al. [17] recently developed a new and updated version of beta function extension,which goes like this:

$$B_p^\epsilon(\vartheta, \varphi) = B(\vartheta, \varphi; p, \epsilon) = \int_0^1 t^{\vartheta-1}(1 - t)^{\varphi-1} E_\epsilon \left(-\frac{p}{t(1 - t)}\right) dt, \tag{1.9}$$

where $\Re(r) > 0, \Re(s) > 0$ and $E_\epsilon\,(.)$ is the Mittag-Leffler function given as

$$E_\epsilon\,(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(\epsilon l + 1)}. \tag{1.10}$$

Recently, Rahman et al. [15] have created a new extension of the Hurwitz-Lerch zeta function in the form of extended beta function (1.9) as

$$\phi_{\gamma,u;v}[z, \varepsilon, \rho; p, \delta] = \phi^\delta_{\gamma,u;v}[z, \varepsilon, \rho; p] = \sum_{l=0}^{\infty} \frac{(\gamma)_l B_p^\delta(u + l, v - u)}{B(u, v - u) \;\; l!} \frac{z^l}{(l + \rho)^\varepsilon}, \tag{1.11}$$

where $[\gamma, u, v \in C, p \geq 0, \delta > 0, \rho \in C\backslash z_0^-, \varepsilon \in C$ when $|z| < 1, \Re(\varepsilon + v - \gamma - u) > 1$ when $|z| = 1]$.

For our present investigation, we need some integral formulae which are given by Mac Robert [6], Oberhettinger [12] and Lavoie-Trottier [5] in equation (1.12), (1.13) and (1.14) respectively are as follows:

$$\int_0^1 y^{\varsigma-1}(1 - y)^{\epsilon-1}[cy + d(1 - y)]^{-\varsigma-\epsilon} dy = \frac{1}{c^\varsigma d^\epsilon} \frac{\Gamma(\varsigma)\Gamma(\epsilon)}{\Gamma(\varsigma + \epsilon)}, \tag{1.12}$$

provided that $\Re(\varsigma) > 0, \Re(\epsilon) > 0$, c and d are nonzero constants so the expression $cy + d(1 - y)$, where $0 \leq y \leq 1$.

$$\int_0^\infty \theta^{\epsilon-1} \left(\theta + c + \sqrt{(\theta^2 + 2c\theta)}\right)^{-\varsigma} d\theta = 2\varsigma c^{-\varsigma} \left(\frac{c}{2}\right)^\epsilon \frac{\Gamma(2\epsilon)\Gamma(\varsigma - \epsilon)}{\Gamma(1 + \epsilon + \varsigma)}, \tag{1.13}$$

provided that $0 < \Re(\epsilon) < \Re(\varsigma)$.

$$\int_0^1 \theta^{\varsigma-1} (1 - \theta)^{2\epsilon-1} \left(1 - \frac{\theta}{3}\right)^{2\varsigma-1} \left(1 - \frac{\theta}{4}\right)^{\epsilon-1} d\theta = \left(\frac{2}{3}\right)^{2\varsigma} \frac{\Gamma(\varsigma)\Gamma(\epsilon)}{\Gamma(\varsigma + \epsilon)}, \tag{1.14}$$

provided that $\Re(\varsigma) > 0, \Re(\epsilon) > 0$.

Here we also recall Hadamard product of two analytic functions which are helpful in our current exploration. This will help us to ablate the function which has emerged into the product of two known functions. Let's have 2 power series be

$f(x)=\sum_{n=0}^{\infty}a_nx^n\ (|x|<R_f)$ and $g(x)=\sum_{n=0}^{\infty}b_nx^n\ (|x|<R_g)$,
where R_f and R_g are radii of convergence respectively. Then their Hadamard product [6],[14],[16] is describes by the power series as

$$(f*g)(x)=\sum_{n=0}^{\infty}a_nb_nx^n=(g*f)(x)\ (|x|<R), \tag{1.15}$$

where $R=\lim_{n\to\infty}\left|\frac{a_nb_n}{a_{n+1}b_{n+1}}\right|=\left(\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|\right).\left(\lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|\right)=R_f.R_g$,
in general $R\geq R_f.R_g$.
Several authors contributed in the field of special functions and also associated with Baskakov-Durrmeyer-Stancu type operators see ([7]-[11]). We also recall the generalized hypergeometric function [2] with r and s are numerator and denominator respectively demarcated as

$${}_rF_s\left(\begin{matrix}u_1,u_2,...,u_r;\\ v_1,v_2,...,v_s;\end{matrix}\ z\right)=\sum_{l=0}^{\infty}\frac{(u_1)_l(u_2)_l...(u_r)_l}{(v_1)_l(v_2)_l...(v_s)_l}\frac{z^l}{l!}, \tag{1.16}$$

where $z,u_i,v_j\in C, i=1,2,...r, j=1,2,...,s$ and v_j is nonzero, non-negative integer.

2 Main Results

In this section we define five generalized integral formulae by inserting the extension form of Hurwitz-Lerch zeta function (1.11) into the integral formulae (1.12), (1.13) and (1.14) by taking suitable argument into the integrand.

Theorem 1: Let us suppose that $\Re(\varsigma)>0,\Re(\epsilon)>0,\gamma,u,v\in C,p\geq 0,\delta>0,\rho\in C\backslash z_0^-$, c and d are nonzero constants and $0\leq y\leq 1$, then

$$\int_0^1 y^{\varsigma-1}(1-y)^{\epsilon-1}[cy+d(1-y)]^{-\varsigma-\epsilon}\phi^{\delta}_{\gamma,u;v}\left(\frac{2cdy(1-y)}{\{cy+d(1-y)\}^2},\varepsilon,\rho;p\right)dy$$
$$=\frac{B(\varsigma,\epsilon)}{c^{\varsigma}d^{\epsilon}}\phi^{\delta}_{\gamma,u;v}\left(\frac{1}{2},\varepsilon,\rho;p\right)*{}_3F_2\left[\varsigma,\epsilon,1;\frac{\varsigma+\epsilon}{2},\frac{\varsigma+\epsilon+1}{2};\frac{1}{2}\right]. \tag{2.1}$$

Proof: For our convenience L.H.S. is denoted by I_1, and by making the use of equation (1.11), then we have

$$I_1=\int_0^1 y^{\varsigma-1}(1-y)^{\epsilon-1}[cy+d(1-y)]^{-\varsigma-\epsilon}\sum_{l=0}^{\infty}\frac{(\gamma)_lB_p^{\delta}(u+l,v-u)}{B(u,v-u)(l+\rho)^{\varepsilon}l!}\left[\frac{2cdy(1-y)}{\{cy+d(1-y)\}^2}\right]^l dy$$

now we are adjusting the order of integration and summation,

$$I_1=\sum_{l=0}^{\infty}\frac{(\gamma)_lB_p^{\delta}(u+l,v-u)(2cd)^l}{B(u,v-u)(l+\rho)^{\varepsilon}l!}\int_0^1 y^{\varsigma+l-1}(1-y)^{\epsilon+l-1}[cy+d(1-y)]^{-\varsigma-\epsilon-2l}dy$$

by making the use of equation (1.12), after some arrangements and simplification, we get

$$I_1=\sum_{l=0}^{\infty}\frac{(\gamma)_lB_p^{\delta}(u+l,v-u)2^l}{B(u,v-u)(l+\rho)^{\varepsilon}l!}\frac{\Gamma(\varsigma)(\varsigma)_l\Gamma(\epsilon)(\epsilon)_l}{c^{\varsigma}d^{\epsilon}2^{2l}\left(\frac{\varsigma+\epsilon}{2}\right)_l\left(\frac{\varsigma+\epsilon+1}{2}\right)_l\Gamma(\varsigma+\epsilon)} \tag{2.2}$$

now we are applying the Hadamard product (1.15) in (2.2) and making the use of (1.11) and (1.16), then we get the wanted outcome.

Theorem 2: Let us suppose that $0<\Re(\epsilon)<\Re(\varsigma),c\in N,\gamma,u,v\in C,p\geq 0,\delta>0,\rho\in C\backslash z_0^-$, then

$$\int_0^{\infty}\theta^{\epsilon-1}\left(\theta+c+\sqrt{(\theta^2+2c\theta)}\right)^{-\varsigma}\phi^{\delta}_{\gamma,u;v}\left(\frac{y}{\left(\theta+c+\sqrt{(\theta^2+2c\theta)}\right)},\varepsilon,\rho;p\right)d\theta$$

$$= 2^{1-\epsilon}c^{\epsilon-\varsigma}\varsigma\,\Gamma(2\epsilon)\frac{\Gamma(\varsigma-\epsilon)}{\Gamma(\epsilon+\varsigma+1)}$$

$$\times\phi^{\delta}_{\gamma,u;v}\left(\frac{y}{c},\varepsilon,\rho;p\right)*{}_3F_2\left[\varsigma+1,\varsigma-\epsilon,1;\varsigma,\varsigma+\epsilon+1;\frac{y}{c}\right]. \tag{2.3}$$

Proof: For our convenience L.H.S. is denoted by I_2, and by making the use of equation (1.11), then we have

$$I_2=\int_0^\infty\theta^{\epsilon-1}\left(\theta+c+\sqrt{(\theta^2+2c\theta)}\right)^{-\varsigma}\sum_{l=0}^{\infty}\frac{(\gamma)_l B^{\delta}_p(u+l,v-u)}{B(u,v-u)(l+\rho)^{\varepsilon}l!}$$

$$\times\left[\frac{y}{\left(\theta+c+\sqrt{(\theta^2+2c\theta)}\right)}\right]^l d\theta$$

now we are adjusting the order of integration and summation,

$$I_2=\sum_{l=0}^{\infty}\frac{(\gamma)_l B^{\delta}_p(u+l,v-u)y^l}{B(u,v-u)(l+\rho)^{\varepsilon}l!}\int_0^\infty\theta^{\epsilon-1}\left(\theta+c+\sqrt{(\theta^2+2c\theta)}\right)^{-\varsigma-l}d\theta$$

by making the use of equation (1.13), after some simplification and rearranging the terms, we get

$$I_2=2^{1-\epsilon}c^{\epsilon-\varsigma}\varsigma\,\Gamma(2\epsilon)\frac{\Gamma(\varsigma-\epsilon)}{\Gamma(\epsilon+\varsigma+1)}\sum_{l=0}^{\infty}\frac{(\gamma)_l B^{\delta}_p(u+l,v-u)(\varsigma+1)_l(\varsigma-\epsilon)_l}{B(u,v-u)(l+\rho)^{\varepsilon}(\varsigma)_l(\epsilon+\varsigma+1)_l l!}\left(\frac{y}{c}\right)^l \tag{2.4}$$

now we apply the Hadamard product (1.15) in (2.4), and making the use of (1.11) and (1.16), then we get the wanted consequence.

Theorem 3: Let us suppose that $0<\Re(\epsilon)<\Re(\varsigma), c\in N,\gamma,u,v\in C,p\geq 0,\delta>0,\rho\in C\backslash z_0^-$, then

$$\int_0^\infty\theta^{\epsilon-1}\left(\theta+c+\sqrt{(\theta^2+2c\theta)}\right)^{-\varsigma}\phi^{\delta}_{\gamma,u;v}\left(\frac{y\theta}{\left(\theta+c+\sqrt{(\theta^2+2c\theta)}\right)},\varepsilon,\rho;p\right)d\theta$$

$$=2^{1-\epsilon}c^{\epsilon-\varsigma}\varsigma\,\Gamma(2\epsilon)\frac{\Gamma(\varsigma-\epsilon)}{\Gamma(\epsilon+\varsigma+1)}\phi^{\delta}_{\gamma,u;v}\left(\frac{y}{2},\varepsilon,\rho;p\right)$$

$$*{}_4F_3\left[\varsigma+1,\epsilon,\epsilon+\frac{1}{2},1;\varsigma,\frac{\epsilon+\varsigma+1}{2},\frac{\epsilon+\varsigma+2}{2};\frac{y}{2}\right]. \tag{2.5}$$

Proof: For our convenience L.H.S. is denoted by I_3, and by making the use of equation (1.11), then we have

$$I_3=\int_0^\infty\theta^{\epsilon-1}\left(\theta+c+\sqrt{(\theta^2+2c\theta)}\right)^{-\varsigma}\sum_{l=0}^{\infty}\frac{(\gamma)_l B^{\delta}_p(u+l,v-u)}{B(u,v-u)(l+\rho)^{\varepsilon}l!}$$

$$\times\left[\frac{y\theta}{\left(\theta+c+\sqrt{(\theta^2+2c\theta)}\right)}\right]^l d\theta$$

now we are adjusting the order of integration and summation,

$$I_3=\sum_{l=0}^{\infty}\frac{(\gamma)_l B^{\delta}_p(u+l,v-u)y^l}{B(u,v-u)(l+\rho)^{\varepsilon}l!}\int_0^\infty\theta^{\epsilon+l-1}\left(\theta+c+\sqrt{(\theta^2+2c\theta)}\right)^{-\varsigma-l}d\theta$$

by making the use of equation (1.13), after some simplification and rearranging the terms, we get

$$I_3 = 2^{1-\epsilon}c^{\epsilon-\varsigma}\varsigma\,\Gamma(2\epsilon)\frac{\Gamma(\varsigma-\epsilon)}{\Gamma(\epsilon+\varsigma+1)}\sum_{l=0}^{\infty}\frac{(\gamma)_l B_p^{\delta}(u+l,v-u)(\varsigma+1)_l(\epsilon)_l\left(\epsilon+\frac{1}{2}\right)_l}{B(u,v-u)(l+\rho)^{\varepsilon}(\varsigma)_l\left(\frac{\epsilon+\varsigma+1}{2}\right)_l\left(\frac{\epsilon+\varsigma+2}{2}\right)_l l!}\left(\frac{y}{2}\right)^l \tag{2.6}$$

now we put on the Hadamard product (1.15) in (2.6), and making the use of (1.11) and (1.16), then we get the awaited result.

Theorem 4: Let us suppose that $\Re(\varsigma) > 0, \Re(\epsilon) > 0, \gamma, u, v \in C, p \geq 0, \delta > 0, \rho \in C\backslash z_0^-$, then

$$\int_0^1 \theta^{\varsigma-1}(1-\theta)^{2\epsilon-1}\left(1-\frac{\theta}{3}\right)^{2\varsigma-1}\left(1-\frac{\theta}{4}\right)^{\epsilon-1}\phi_{\gamma,u;v}^{\delta}\left[y\left(1-\frac{\theta}{4}\right)(1-\theta)^2,\varepsilon,\rho;p\right]d\theta$$

$$= \left(\frac{2}{3}\right)^{2\varsigma} B(\varsigma,\epsilon)\phi_{\gamma,u;v}^{\delta}(y,\varepsilon,\rho;p) * {}_2F_1[\epsilon,1;\varsigma+\epsilon;y]. \tag{2.7}$$

Proof: For our convenience L.H.S. is denoted by I_4, and by making the use of equation (1.11), then we have

$$I_4 = \int_0^1 \theta^{\varsigma-1}(1-\theta)^{2\epsilon-1}\left(1-\frac{\theta}{3}\right)^{2\varsigma-1}\left(1-\frac{\theta}{4}\right)^{\epsilon-1}\sum_{l=0}^{\infty}\frac{(\gamma)_l B_p^{\delta}(u+l,v-u)}{B(u,v-u)(l+\rho)^{\varepsilon}l!}$$

$$\times y^l\left(1-\frac{\theta}{4}\right)^l(1-\theta)^{2l}\,d\theta$$

now we are adjusting the order of integration and summation,

$$I_4 = \sum_{l=0}^{\infty}\frac{(\gamma)_l B_p^{\delta}(u+l,v-u)y^l}{B(u,v-u)(l+\rho)^{\varepsilon}l!}\int_0^1 \theta^{\varsigma-1}(1-\theta)^{2(\epsilon+l)-1}\left(1-\frac{\theta}{3}\right)^{2\varsigma-1}\left(1-\frac{\theta}{4}\right)^{\epsilon+l-1}d\theta$$

by making the use of equation (1.14), and further simplification and rearranging the terms, we get

$$I_4 = \left(\frac{2}{3}\right)^{2\varsigma}B(\varsigma,\epsilon)\sum_{l=0}^{\infty}\frac{(\gamma)_l B_p^{\delta}(u+l,v-u)(\epsilon)_l y^l}{B(u,v-u)(l+\rho)^{\varepsilon}(\varsigma+\epsilon)_l l!} \tag{2.8}$$

now we apply the Hadamard product (1.15) in (2.8), and making the use of (1.11) and (1.16), then we get the wanted outcome.

Theorem 5: Let us suppose that $\Re(\varsigma) > 0, \Re(\epsilon) > 0, \gamma, u, v \in C, p \geq 0, \delta > 0, \rho \in C\backslash z_0^-$, then

$$\int_0^1 \theta^{\varsigma-1}(1-\theta)^{2\epsilon-1}\left(1-\frac{\theta}{3}\right)^{2\varsigma-1}\left(1-\frac{\theta}{4}\right)^{\epsilon-1}\phi_{\gamma,u;v}^{\delta}\left[y\theta\left(1-\frac{\theta}{3}\right)^2,\varepsilon,\rho;p\right]d\theta$$

$$= \left(\frac{2}{3}\right)^{2\varsigma}B(\varsigma,\epsilon)\phi_{\gamma,u;v}^{\delta}\left(\frac{4y}{9},\varepsilon,\rho;p\right) * {}_2F_1\left[\varsigma,1;\varsigma+\epsilon;\frac{4y}{9}\right]. \tag{2.9}$$

Proof: For our convenience L.H.S. is denoted by I_5, and by making the use of equation (1.11), then we have

$$I_5 = \int_0^1 \theta^{\varsigma-1}(1-\theta)^{2\epsilon-1}\left(1-\frac{\theta}{3}\right)^{2\varsigma-1}\left(1-\frac{\theta}{4}\right)^{\epsilon-1}\sum_{l=0}^{\infty}\frac{(\gamma)_l B_p^{\delta}(u+l,v-u)}{B(u,v-u)(l+\rho)^{\varepsilon}l!}y^l\theta^l$$

$$\times\left(1-\frac{\theta}{3}\right)^{2l}d\theta$$

now we are adjusting the order of integration and summation,

$$I_5 = \sum_{l=0}^{\infty} \frac{(\gamma)_l B_p^{\delta}(u+l, v-u) y^l}{B(u, v-u)(l+\rho)^{\varepsilon} l!} \int_0^1 \theta^{\varsigma+l-1} (1-\theta)^{2\epsilon-1} \left(1-\frac{\theta}{3}\right)^{2(\varsigma+l)-1} \left(1-\frac{\theta}{4}\right)^{\epsilon-1} d\theta$$

by making the use of equation (1.14), and further simplification and rearranging the terms, we get

$$I_5 = \left(\frac{2}{3}\right)^{2\varsigma} B(\varsigma, \epsilon) \sum_{l=0}^{\infty} \frac{(\gamma)_l B_p^{\delta}(u+l, v-u)(\varsigma)_l}{B(u, v-u)(l+\rho)^{\varepsilon}(\varsigma+\epsilon)_l l!} \left(\frac{4y}{9}\right)^l \tag{2.10}$$

now we apply the Hadamard product (1.15) in (2.10), and making the use of (1.11) and (1.16), then we get the anticipated consequence.

3 Special cases

In this section we are going to find some integral formulae by substituting particular values, If we put $\delta = p = 1$ in (2.1),(2.3),(2.5),(2.7) and (2.9), then we have our results in the form Hadamard product of Hurwitz-Lerch zeta function investigated by Garg et al. [3] with hypergeometric function, which are defined in the following Corollaries. As if we put $\varsigma = \epsilon = c = d = \delta = p = 1$, in (2.1) then we have Corollary 1 as below:

Corollary 1: Let us suppose that $\Re(\varsigma) > 0, \Re(\epsilon) > 0, \gamma, u, v \in C, \rho \in C \backslash z_0^-$, c and d are nonzero constants and $0 \leq y \leq 1$. then

$$\int_0^1 \phi_{\gamma,u;v} \left[2y(1-y), \varepsilon, \rho\right] dy = \phi_{\gamma,u;v} \left(\frac{1}{2}, \varepsilon, \rho\right) * {}_2F_1 \left[1, 1; \frac{3}{2}; \frac{1}{2}\right]. \tag{3.1}$$

Similarly, as if we put $\epsilon = \delta = p = 1, \varsigma = 2$, in (2.3) and (2.5), then we have Corollary 2 and Corollary 3 as follows:

Corollary 2: Let us suppose that $0 < \Re(\epsilon) < \Re(\varsigma), c \in N, \gamma, u, v \in C, \rho \in C \backslash z_0^-$, then

$$\int_0^{\infty} \left(\theta + c + \sqrt{(\theta^2 + 2c\theta)}\right)^{-2} \phi_{\gamma,u;v} \left(\frac{y}{\left(\theta + c + \sqrt{(\theta^2 + 2c\theta)}\right)}, \varepsilon, \rho\right) d\theta$$

$$= \frac{1}{3c} \phi_{\gamma,u;v} \left(\frac{y}{c}, \varepsilon, \rho\right) * {}_3F_2 \left[3, 1, 1; 2, 4; \frac{y}{c}\right]. \tag{3.2}$$

Corollary 3: Let us suppose that $0 < \Re(\epsilon) < \Re(\varsigma), c \in N, \gamma, u, v \in C, \rho \in C \backslash z_0^-$, then

$$\int_0^{\infty} \left(\theta + c + \sqrt{(\theta^2 + 2c\theta)}\right)^{-2} \phi_{\gamma,u;v} \left(\frac{y\theta}{\left(\theta + c + \sqrt{(\theta^2 + 2c\theta)}\right)}, \varepsilon, \rho\right) d\theta$$

$$= \frac{1}{3c} \phi_{\gamma,u;v} \left(\frac{y}{2}, \varepsilon, \rho\right) * {}_4F_3 \left[3, 1, \frac{3}{2}, 1; 2, 2, \frac{5}{2}; \frac{y}{2}\right]. \tag{3.3}$$

In this manner if we substitute $\varsigma = 1, \epsilon = \delta = p = 1$, in (2.7) and (2.9), then we have Corollary 4 and Corollary 5 as follows:

Corollary 4: Let us suppose that $\Re(\varsigma) > 0, \Re(\epsilon) > 0, \gamma, u, v \in C, \rho \in C \backslash z_0^-$, then

$$\int_0^1 (1-\theta) \left(1-\frac{\theta}{3}\right) \phi_{\gamma,u;v} \left[y \left(1-\frac{\theta}{4}\right) (1-\theta)^2, \varepsilon, \rho\right] d\theta$$

$$= \frac{4}{9} \phi_{\gamma,u;v} (y, \varepsilon, \rho) * {}_2F_1 [1, 1; 2; y]. \tag{3.4}$$

Corollary 5: Let us suppose that $\Re(\varsigma) > 0, \Re(\epsilon) > 0, \gamma, u, v \in C, \rho \in C \backslash z_0^-$, then

$$\int_0^1 (1-\theta)\left(1-\frac{\theta}{3}\right)\phi_{\gamma,u;v}\left[y\theta\left(1-\frac{\theta}{3}\right)^2,\varepsilon,\rho\right]d\theta$$
$$=\frac{4}{9}\phi_{\gamma,u;v}\left(\frac{4y}{9},\varepsilon,\rho\right)*{}_2F_1\left[1,1;2;\frac{4y}{9}\right]. \tag{3.5}$$

4 Conclusion

In this present investigation, we defined five new generalised integral formulae by involving the extension form of the Hurwitz-Lerch zeta function and deduced the results in the form of hypergeometric functions in product form by using the properties of the Hadamard product of two power series. Furthermore, we also discussed their special cases by making suitable substitutions. The future scope of these integrals is that one can define many other impressive integrals by using different kinds of Hurwitz-Lerch zeta function, trigonometric and hyperbolic functions, after appropriate parametric replacements, special functions product with different kinds of polynomials or multivariable polynomials, which gives remarkable results. The reported findings are general in nature and useful in the study of science and technology.

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