Palestine Journal of Mathematics

Vol. 6(Special Issue: II) (2017) , 174–185

CUBIC WEAK BI-IDEALS OF NEAR RINGS

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 08A72, 20N25; Secondary 20M12, 97H40.

Keywords and phrases: Near-rings, weak bi-ideals, fuzzy weak bi-ideals, cubic weak bi-ideals, Homomorphisam of cubic weak bi-ideals.

Abstract: In this paper, we introduced the new notion of cubic weak bi-ideals of near-rings, which is the generalized concept of fuzzy weak bi-ideals of near rings. we also investigated some of its properties with examples.

1 Introduction

Zadeh [17] initiated the concept of fuzzy sets in 1965. Abou-Zaid [1] first made the study of fuzzy subnear-rings and ideals of near-rings. The concept of bi-ideals was applied to near rings in [14]. The idea of fuzzy ideals of near-rings was first proposed by Kim et al.[5]. Jun et al.[6] defined the concept of fuzzy R-subgroups of near-rings. Moreover, Manikantan [7] introduced the notion of fuzzy bi-ideals of near-rings and discussed some of its properties. Yong Uk Cho et al.[16] introduced the concept of weak bi-ideals applied to near-rings. N. Thillaigovindan et al.[15] introduced interval valued fuzzy ideals of near rings. Chinnadurai et al.[4] introduced fuzzy weak bi-ideals of near-rings. Jun et al.[10] introduced the concept of cubic sets. This structure encompasses interval-valued fuzzy set and fuzzy set. Also Jun et al.[12] introduced the notion of cubic ideals of semigroups. Chinnadurai et al.[3] introduced the notion of cubic ring. In this paper, we defined a new notion of cubic weak bi-ideals of near-rings, we also discussed some of its properties with examples.

2 Preliminaries

In this section, we listed some basic definitions related to cubic weak bi-ideals of near-rings.

Throughout this paper R denotes a left near-ring.

Definition 2.1. [1] A near-ring is an algebraic system $(R,+,\cdot)$ consisting of a non-empty set R together with two binary operations called $+$ and $\cdot$ such that $(R,+)$ is a group not necessarily abelian and $(R,\cdot)$ is a semigroup connected by the following distributive law: $x\cdot(y+z) = x\cdot y + x\cdot z$ valid for all $x,y,z \in R$. We use the word 'near-ring' to means 'left near-ring'. We denote xy insted of $x\cdot y$. An ideal I of a near-ring R is a subset of R such that (i) (I,+) is a normal subgroup of (R,+) (ii) $RI \subseteq I$ (iii) $(x+a)y - xy \in I$ for any $a \in I$ and $x,y \in R$. A R-subgroup H of a near-ring R is the subset of R such that (i) (H,+) is a subgroup of (R, +) (ii) $RH \subseteq H$ (iii) $HR \subseteq H$.

Note that H is a left R-subgroup of R if H satisfies (i) and (ii) and a right R-subgroup of R if H satisfies (i) and (iii).

Definition 2.2. [7] Let R be a near ring . Given two subsets A and B of R, we define the following products $AB = \{ab \mid a \in A, b \in B\}$ and $A \star B = \{(a'+b)a - a'a \mid a, a' \in A, b \in B\}$.

Definition 2.3. [14] A subgroup B of $(R,+)$ is said to be bi-ideal of R if $BRB \cap B \star RB \subseteq B$.

Definition 2.4. [16] A subgroup B of $(R,+)$ is said to be weak bi-ideal of R if $BBB \subseteq B$.

Definition 2.5. [2] A fuzzy subset μ of a set X is a function $\mu : X \to [0,1]$.

Definition 2.6. [2] Let μ and λ be any two fuzzy subsets of R. Then $\mu\lambda$ is fuzzy subset of R defined by

$$(\mu\lambda)(x) = \begin{cases} \sup\limits_{x=yz} \min\{\mu(y), \lambda(z)\} & \text{if } x = yz \text{ for all } x, y, z \in R \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.7. [7] A fuzzy subgroup μ of $(R, +)$ is said to be fuzzy bi-ideal of R if $\mu R\mu \cap \mu \star R\mu \subseteq \mu$

Definition 2.8. [1] Let R be a near-ring and μ be a fuzzy subset of R. We say μ is a fuzzy subnear-ring of R if

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

Definition 2.9. [1] Let R be a near-ring and μ be a fuzzy subset of R. Then μ is called a fuzzy ideal of R, if

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(y + x - y) \geq \mu(x)$

(iii) $\mu(xy) \geq \mu(y)$

(iv) $\mu((x + z)y - xy) \geq \mu(z)$ for all $x, y \in R$.

A fuzzy subset with (i) to (iii) is called a fuzzy left ideal of R, whereas a fuzzy subset with (i),(ii) and (iv) are called a fuzzy right ideal of R.

Definition 2.10. [1] A fuzzy subset μ of a near-ring R is called a fuzzy R-subgroup of R if

(i) μ is a fuzzy subgroup of $(R, +)$

(ii) $\mu(xy) \geq \mu(y)$

(iii) $\mu(xy) \geq \mu(x)$ for all $x, y \in R$.

A fuzzy subset with (i) and (ii) is called a fuzzy left R-subgroup of R, whereas a fuzzy subset with (i) and (iii) is called a fuzzy right R-subgroup of R.

Definition 2.11. [4] A fuzzy subgroup μ of R is called fuzzy weak bi-ideal of R, if

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}.$$

Definition 2.12. [2] Let X be a non-empty set. A mapping $\overline{\mu} : X \to D[0, 1]$ is called an interval-valued (in short i-v) fuzzy subset of X, if for all $x \in X, \overline{\mu}(x) = [\mu^-(x), \mu^+(x)]$, where μ^- and μ^+ are fuzzy subsets of X such that $\mu^-(x) \leq \mu^+(x)$. Thus $\overline{\mu}(x)$ is an interval (a closed subset of [0,1]) and not a number from the interval [0,1] as in the case of fuzzy set.

3 Cubic weak bi-ideals of near-rings

In this section, we introduced the notion of cubic weak bi-ideals of near-rings and dicuss some of its properties.

Definition 3.1. A cubic set $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ of R is called cubic subgroup of R, if

(i) $\overline{\mu}(x - y) \geq \min\{\overline{\mu}(x), \overline{\mu}(y)\}$

(ii) $\omega(x - y) \leq \max\{\omega(x), \omega(y)\} \quad \forall x, y \in R$.

Definition 3.2. A cubic subgroup $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ of R is called cubic weak bi-ideal of R, if

(i) $\overline{\mu}(xyz) \geq \min\{\overline{\mu}(x), \overline{\mu}(y), \overline{\mu}(z)\}$

(ii) $\omega(xyz) \leq \max\{\omega(x), \omega(y), \omega(z)\} \quad \forall x, y, z \in R$.

Example 3.3. Let $R = \{a, b, c, d\}$ be a near-ring with two binary operations $+$ and $\cdot$ are defined as follows:

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	b	c	d

Then $(R, +, \cdot)$ is a near-ring.

Let $\overline{\mu}$: $R \to D[0,1]$ be an interval valued fuzzy subset defined by $\overline{\mu}(a) = [0.8, 0.9]$, $\overline{\mu}(b) = [0.6, 0.7]$ and $\overline{\mu}(c) = [0.4, 0.5] = \overline{\mu}(d)$. Then $\overline{\mu}$ is an interval-valued fuzzy weak bi-ideal of R.

Let $\omega : R \to [0,1]$ be a fuzzy subset defined by $\omega(a) = 0.2, \omega(b) = 0.4$ and $\omega(c) = 0.8 = \omega(d)$. Then ω is a fuzzy weak bi-ideal of R.

Hence $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic weak bi-ideal of R.

Definition 3.4. Let $\mathscr{A}_i$ be cubic weak bi-ideals of near-rings R_i for $i = 1, 2, 3, \ldots, n$. Then the cubic direct product of $\mathscr{A}_i (i = 1, 2, \ldots, n)$ is a function $\bar{\mu}_1 \times \bar{\mu}_2 \times \cdots \times \bar{\mu}_n : R_1 \times R_2 \times \cdots \times R_n \to D[0,1], \omega_1 \times \omega_2 \times \cdots \times \omega_n : R_1 \times R_2 \times \cdots \times R_n \to [0,1]$ defined by $(\bar{\mu}_1 \times \bar{\mu}_2 \times \cdots \times \bar{\mu}_n)(x_1, x_2, \ldots, x_n) = \min\{\bar{\mu}_1(x_1), \bar{\mu}_2(x_2), \ldots, \bar{\mu}_n(x_n)\}$ and $(\omega_1 \times \omega_2 \times \cdots \times \omega_n)(x_1, x_2, \ldots, x_n) = \max\{\omega_1(x_1), \omega_2(x_2), \ldots, \omega_n(x_n)\}$.

Definition 3.5. Let $\mathscr{A}_1 = \langle \bar{\mu}_1, \omega_1 \rangle$ *and* $\mathscr{A}_2 = \langle \bar{\mu}_2, \omega_2 \rangle$ be any two cubic subsets of R. Then $\mathscr{A}_1 \mathscr{A}_2$ is cubic subsets of R defined by:

$$(\mathscr{A}_1\mathscr{A}_2)(x) = \begin{cases} (\bar{\mu}_1\bar{\mu}_2)(x) = \begin{cases} \sup\limits_{x=yz} \min\{\overline{\mu}(y), \overline{\mu}(z)\} & \text{if } x = yz \text{ for all } x, y, z \in R \\ [0,0] & \text{otherwise} \end{cases} \\ (\omega_1\omega_2)(x) = \begin{cases} \inf\limits_{x=yz} \max\{\omega(y), \omega(z)\} & \text{if } x = yz \text{ for all } x, y, z \in R \\ 1 & \text{otherwise} \end{cases} \end{cases}$$

Theorem 3.6. *Let $A = \langle \overline{\mu}, \omega \rangle$ be a cubic subgroup of R. Then $A = \langle \overline{\mu}, \omega \rangle$ is a cubic weak bi-ideal of $R \Leftrightarrow AAA \sqsubseteq A$. (i.e., $\overline{\mu}\,\overline{\mu}\,\overline{\mu} \subseteq \overline{\mu}$ and $\omega\,\omega\,\omega \supseteq \omega$)*

Proof. Assume that $A = \langle \overline{\mu}, \omega \rangle$ is a cubic weak bi-ideal of R. Let $x, y, z, p, q \in R$ such that $x = yz$ and $y = pq$. Then

$$\begin{aligned} (\overline{\mu}\,\overline{\mu}\,\overline{\mu})(x) &= \sup_{x=yz} \{\min\{(\overline{\mu}\,\overline{\mu})(y), \overline{\mu}(z)\}\} \\ &= \sup_{x=yz} \left\{\min\left\{\sup_{y=pq} \min\{\overline{\mu}(p), \overline{\mu}(q)\}, \overline{\mu}(z)\right\}\right\} \\ &= \sup_{x=yz} \sup_{y=pq} \{\min\{\min\{\overline{\mu}(p), \overline{\mu}(q)\}, \overline{\mu}(z)\}\} \\ &= \sup_{x=pqz} \{\min\{\overline{\mu}(p), \overline{\mu}(q), \overline{\mu}(z)\} \\ &\le \sup_{x=pqz} \overline{\mu}(pqz) \\ &= \overline{\mu}(x) \end{aligned}$$

If x can not be expressed as $x = yz$ then $(\overline{\mu}\,\overline{\mu}\,\overline{\mu})(x) = \overline{0} \le \overline{\mu}(x)$.

In both cases $\overline{\mu}\,\overline{\mu}\,\overline{\mu} \subseteq \overline{\mu}$.

$$\begin{aligned}(\omega\,\omega\,\omega)(x) &= \inf_{x=yz}\{\max\{(\omega\,\omega)(y), \omega(z)\}\}\\ &= \inf_{x=yz}\left\{\max\left\{\inf_{y=pq}\max\{\omega(p), \omega(q)\}, \omega(z)\right\}\right\}\\ &= \inf_{x=yz}\ \inf_{y=pq}\{\max\{\max\{\omega(p), \omega(q)\}, \omega(z)\}\}\\ &= \inf_{x=pqz}\{\max\{\omega(p), \omega(q), \omega(z)\}\\ &\geq \inf_{x=pqz}\omega(pqz)\\ &= \omega(x)\end{aligned}$$

If x can not be expressed as $x = yz$ then $(\,\omega\,\omega\,\omega)(x) = 1 \geq \omega(x)$.

In both cases $\omega\,\omega\,\omega \supseteq \omega$.

Hence $AAA \sqsubseteq A$.

Conversly, assume that $AAA \sqsubseteq A$ holds. To prove that $A = \langle\overline{\mu}, \omega\rangle$ is a cubic weak bi-ideal of R.

For any $x, y, z, a \in R$ such that $a = xyz$ then

$$\begin{aligned}\overline{\mu}(xyz) &= \overline{\mu}(a) \geq (\overline{\mu}\,\overline{\mu}\,\overline{\mu})(a)\\ &= \sup_{a=bc}\min\{(\overline{\mu}\,\overline{\mu})(b),\ \overline{\mu}(c)\}\\ &= \sup_{a=bc}\left\{\min\left\{\sup_{b=pq}\min\{\overline{\mu}(p), \overline{\mu}(q)\}, \overline{\mu}(c)\right\}\right\}\\ &= \sup_{a=pqc}\{\min\{\overline{\mu}(p), \overline{\mu}(q)\}, \overline{\mu}(c)\}\}\\ \overline{\mu}(xyz) &\geq \min\{\overline{\mu}(x),\ \overline{\mu}(y),\ \overline{\mu}(z)\}\end{aligned}$$

$$\begin{aligned}\omega(xyz) &= \omega(a) \leq (\omega\,\omega\,\omega)(a)\\ &= \inf_{a=bc}\max\{(\omega\omega)(b), \omega(c)\}\\ &= \inf_{a=bc}\left\{\max\left\{\inf_{b=pq}\max\{\omega(p), \omega(q)\}, \omega(c)\right\}\right\}\\ &= \inf_{a=pqc}\{\max\{\omega(p), \omega(q), \omega(c)\}\}\\ \omega(xyz) &\leq \max\{\omega(x), \omega(y), \omega(z)\}\end{aligned}$$

Hence $A = \langle\overline{\mu}, \omega\rangle$ is a cubic weak bi-ideal of R. □

Theorem 3.7. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two cubic weak bi-ideals of R then the product $\mathscr{A}_1\mathscr{A}_2$ is a cubic weak bi-ideal of R.*

Proof. Let $\mathscr{A}_1 = \langle\overline{\mu}_1, \omega_1\rangle$ and $\mathscr{A}_2 = \langle\overline{\mu}_2, \omega_2\rangle$ be two cubic weak bi-deals of R.

Since $\overline{\mu}_1$ and $\overline{\mu}_2$ are interval-valued fuzzy weak bi-ideals of R then

$$\begin{aligned}(\overline{\mu}_1\overline{\mu}_2)(x-y) &= \sup_{x-y=pq}\min\{\overline{\mu}_1(p), \overline{\mu}_2(q)\}\\ &\geq \sup_{x-y=p_1q_1-p_2q_2\leq(p_1-p_2)(q_1-q_2)}\min\{\overline{\mu}_1(p_1-p_2), \overline{\mu}_2(q_1-q_2)\}\\ &\geq \sup\min\{\min\{\overline{\mu}_1(p_1), \overline{\mu}_1(p_2)\}, \min\{\overline{\mu}_2(q_1), \overline{\mu}_2(q_2)\}\}\\ &= \sup\min\{\min\{\overline{\mu}_1(p_1), \overline{\mu}_2(q_1)\}, \min\{\overline{\mu}_1(p_2), \overline{\mu}_2(q_2)\}\\ &= \min\left\{\sup_{x=p_1q_1}\min\left\{\overline{\mu}_1(p_1), \overline{\mu}_2(q_1)\}, \sup_{y=p_2q_2}\min\{\overline{\mu}_1(p_2), \overline{\mu}_2(q_2)\right\}\right\}\\ &= \min\{(\overline{\mu}_1\overline{\mu}_2)(x), (\overline{\mu}_1\overline{\mu}_2)(y)\}\end{aligned}$$

It follows that $(\overline{\mu}_1\overline{\mu}_2)$ is an interval-valued fuzzy subgroup of R. Further

$$\begin{aligned}(\overline{\mu}_1\overline{\mu}_2)(\overline{\mu}_1\overline{\mu}_2)(\overline{\mu}_1\overline{\mu}_2) &= \overline{\mu}_1\overline{\mu}_2(\overline{\mu}_1\overline{\mu}_2\overline{\mu}_1)\overline{\mu}_2\\ &\subseteq \overline{\mu}_1\overline{\mu}_2(\overline{\mu}_2\overline{\mu}_2\overline{\mu}_2)\overline{\mu}_2\\ &\subseteq \overline{\mu}_1(\overline{\mu}_2\overline{\mu}_2\overline{\mu}_2)\\ &\subseteq (\overline{\mu}_1\overline{\mu}_2)\end{aligned}$$

Therefore $(\overline{\mu}_1\overline{\mu}_2)$ is an interval-valued fuzzy weak bi-ideals of R.
Since ω_1, ω_2 are fuzzy weak bi-ideals of R, then

$$\begin{aligned}(\omega_1\omega_2)(x-y) &= \inf_{x-y=pq} \max\{\omega_1(p), \omega_2(q)\}\\ &\leq \inf_{x-y=p_1q_1-p_2q_2\leq(p_1-p_2)(q_1-q_2)} \max\{\omega_1(p_1-p_2), \omega_2(q_1-q_2)\}\\ &\leq \inf \max\{\max\{\omega_1(p_1), \omega_1(p_2)\}, \max\{\omega_2(q_1), \omega_2(q_2)\}\}\\ &= \inf \max\{\max\{\omega_1(p_1), \omega_2(q_1)\}, \max\{\omega_1(p_2), \omega_2(q_2)\}\}\\ &= \max\left\{\inf_{x=p_1q_1} \max\left\{\omega_1(p_1), \omega_2(q_1)\}, \inf_{y=p_2q_2} \max\{\omega_1(p_2), \omega_2(q_2)\right\}\right\}\\ &= \max\{(\omega_1\omega_2)(x), (\omega_1\omega_2)(y)\}\end{aligned}$$

It follows that $(\omega_1\omega_2)$ is a fuzzy subgroup of R. Further

$$\begin{aligned}(\omega_1\omega_2)(\omega_1\omega_2)(\omega_1\omega_2) &= \omega_1\omega_2(\omega_1\omega_2\omega_1)\omega_2\\ &\supseteq \omega_1\omega_2(\omega_2\omega_2\omega_2)\omega_2\\ &\supseteq \omega_1(\omega_2\omega_2\omega_2)\\ &\supseteq (\omega_1\omega_2)\end{aligned}$$

Thus $(\omega_1\omega_2)$ is a fuzzy weak bi-ideals of R. Hence $\mathscr{A}_1\mathscr{A}_2 = \langle(\overline{\mu}_1\overline{\mu}_2), (\omega_1\omega_2)\rangle$ is a cubic weak bi-ideal of R. □

Remark 3.8. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two cubic weak bi-ideals of R then the product $\mathscr{A}_2\mathscr{A}_1$ is also a cubic weak bi-ideal of R.

Theorem 3.9. *Let $\mathscr{A} = \langle\overline{\mu}, \omega\rangle$ be a cubic weak bi-ideal of R, then the set $R_{\mathscr{A}} = \{x \in R \mid \mathscr{A}(x) = \mathscr{A}(0)\}$ (i.e., $R_{\mathscr{A}} = \{x \in R \mid \overline{\mu}(x) = \overline{\mu}(0)$ and $\omega(x) = \omega(0)\}$) is a weak bi-ideal of R.*

Proof. Let $\mathscr{A} = \langle\overline{\mu}, \omega\rangle$ be a cubic weak bi-ideal of R. Let $x, y \in R_{\mathscr{A}}$. Then $\mathscr{A}(x) = \mathscr{A}(0)$ and $\mathscr{A}(y) = \mathscr{A}(0)$ (i.e.,) $\overline{\mu}(x) = \overline{\mu}(0), \omega(x) = \omega(0)$ and $\overline{\mu}(y) = \overline{\mu}(0), \omega(y) = \omega(0)$ Since $\overline{\mu}$ is an interval-valued fuzzy weak bi-ideal of R. we have $\overline{\mu}(x) = \overline{\mu}(0)$ and $\overline{\mu}(y) = \overline{\mu}(0)$
$\overline{\mu}(x-y) \geq \min\{\overline{\mu}(x), \overline{\mu}(y)\} = \min\{\overline{\mu}(0), \overline{\mu}(0)\} = \overline{\mu}(0)$ and ω is a fuzzy weak bi-ideal of R, we have $\omega(x) = \omega(0)$ and $\omega(y) = \omega(0)$ then $\omega(x-y) \leq \max\{\omega(x), \omega(y)\} = \max\{\omega(0), \omega(0)\} = \omega(0)$. Thus $x - y \in R_{\mathscr{A}}$

For every $x, y, z \in R_{\mathscr{A}}$. Then $\mathscr{A}(x) = \mathscr{A}(0)$, $\mathscr{A}(y) = \mathscr{A}(0)$ and $\mathscr{A}(z) = \mathscr{A}(0)$. Since $\overline{\mu}$ is an interval-valued fuzzy weak bi-ideal of R, we have $\overline{\mu}(x) = \overline{\mu}(0)$, $\overline{\mu}(y) = \overline{\mu}(0)$ and $\overline{\mu}(z) = \overline{\mu}(0)$ then $\overline{\mu}(xyz) \geq \min\{\overline{\mu}(x), \overline{\mu}(y), \overline{\mu}(y)\} = \min\{\overline{\mu}(0), \overline{\mu}(0), \overline{\mu}(0)\} = \overline{\mu}(0)$ and ω is a fuzzy weak bi-ideal of R, we have $\omega(x) = \omega(0), \omega(y) = \omega(0), \omega(z) = \omega(0)$ and $\omega(xyz) \leq \max\{\omega(x), \omega(y), \omega(z)\} = \max\{\omega(0), \omega(0), \omega(0)\} = \omega(0)$. Thus $xyz \in R_{\mathscr{A}}$.

Hence $R_{\mathscr{A}}$ is a cubic weak bi-ideal of R. □

Theorem 3.10. *Let $\{\mathscr{A}_i\}_{i\in\curlywedge} = \langle\overline{\mu}_i, \omega_i : i \in \curlywedge\rangle$ be a family of cubic weak bi-ideals of R, then $\bigcap_{i\in\curlywedge} \mathscr{A}_i = \left\langle \bigcap_{i\in\curlywedge} \overline{\mu}_i, \bigcup_{i\in\curlywedge} \omega_i \right\rangle$ is also a family of cubic weak bi-ideal of R, where $\curlywedge$ is any index set.*

Proof. Let $\{\mathscr{A}_i\}_{i\in\curlywedge} = \langle\overline{\mu}_i, \omega_i : i \in \curlywedge\rangle$ be a family of cubic weak bi-ideals of R.

Let $x, y, z \in R$ and $\bigcap_{i\in\curlywedge} \overline{\mu}_i(x) = (\inf_{i\in\curlywedge} \overline{\mu}_i)(x) = \inf_{i\in\curlywedge} \overline{\mu}_i(x)$, $\bigcup_{i\in\curlywedge} \omega_i(x) = (\sup_{i\in\curlywedge} \omega_i)(x) = \sup_{i\in\curlywedge} \omega_i(x)$

Since $\overline{\mu}_i$ is a family of interval-valued fuzzy weak bi-ideals of R, we have

$$\begin{aligned}\bigcap_{i\in\curlywedge}\overline{\mu}_i(x-y) &= \inf_{i\in\curlywedge}\overline{\mu}_i(x-y)\\ &\geq \inf_{i\in\curlywedge}\min\{\overline{\mu}_i(x),\overline{\mu}_i(y)\}\\ &= \min\left\{\inf_{i\in\curlywedge}\overline{\mu}_i(x),\inf_{i\in\curlywedge}\overline{\mu}_i(y)\right\}\\ &= \min\left\{\bigcap_{i\in\curlywedge}\overline{\mu}_i(x),\bigcap_{i\in\curlywedge}\overline{\mu}_i(y)\right\}\end{aligned}$$

and ω_i is a family of fuzzy weak bi-ideals of R. we have

$$\begin{aligned}\bigcup_{i\in\curlywedge}\omega_i(x-y) &= \sup_{i\in\curlywedge}\omega_i(x-y)\\ &\leq \sup_{i\in\curlywedge}\max\{\omega_i(x),\omega_i(y)\}\\ &= \max\left\{\sup_{i\in\curlywedge}\omega_i(x),\sup_{i\in\curlywedge}\omega_i(y)\right\}\\ &= \max\left\{\bigcup_{i\in\curlywedge}\omega_i(x),\bigcup_{i\in\curlywedge}\omega_i(y)\right\}\end{aligned}$$

Thus $\bigcap_{i\in\curlywedge}\mathscr{A}_i$ is a cubic subgroup of R.

Again

$$\begin{aligned}\bigcap_{i\in\curlywedge}\overline{\mu}_i(xyz) &= \inf_{i\in\curlywedge}\overline{\mu}_i(xyz)\\ &\geq \inf_{i\in\curlywedge}\min\{\overline{\mu}_i(x),\overline{\mu}_i(y),\overline{\mu}_i(z)\}\\ &= \min\left\{\inf_{i\in\curlywedge}\overline{\mu}_i(x),\inf_{i\in\curlywedge}\overline{\mu}_i(y),\inf_{i\in\curlywedge}\overline{\mu}_i(z)\right\}\\ &= \min\left\{\bigcap_{i\in\curlywedge}\overline{\mu}_i(x),\bigcap_{i\in\curlywedge}\overline{\mu}_i(y),\bigcap_{i\in\curlywedge}\overline{\mu}_i(z)\right\}\end{aligned}$$

and

$$\begin{aligned}\bigcup_{i\in\curlywedge}\omega_i(xyz) &= \sup_{i\in\curlywedge}\omega_i(xyz)\\ &\leq \sup_{i\in\curlywedge}\max\{\omega_i(x),\omega_i(y),\omega_i(z)\}\\ &= \max\left\{\sup_{i\in\curlywedge}\omega_i(x),\sup_{i\in\curlywedge}\omega_i(y),\sup_{i\in\curlywedge}\omega_i(z)\right\}\\ &= \max\left\{\bigcup_{i\in\curlywedge}\omega_i(x),\bigcup_{i\in\curlywedge}\omega_i(y),\bigcup_{i\in\curlywedge}\omega_i(z)\right\}\end{aligned}$$

Hence $\bigcap_{i\in\curlywedge}\mathscr{A}_i = \left\langle\bigcap_{i\in\curlywedge}\overline{\mu}_i,\bigcup_{i\in\curlywedge}\omega_i\right\rangle$ is also a family of cubic weak bi-ideal of R. □

Theorem 3.11. *Let H be a non empty subset of R and $\mathscr{A} = \langle\overline{\mu},\omega\rangle$ be a cubic subset of R defined by*

$$\mathscr{A}(x) = \begin{cases}\bar{\mu}(x) = \begin{cases}[p_1,p_2] & \text{if } x\in H\\ [q_1,q_2] & \text{otherwise}\end{cases}\\ \omega(x) = \begin{cases}1-p & \text{if } x\in H\\ 1-q & \text{otherwise}\end{cases}\end{cases}$$

for all $x \in R, [p_1, p_2], [q_1, q_2] \in D[0,1]$ and $p, q \in [0,1]$ with $[p_1, p_2] > [q_1, q_2], p > q$. Then H is a weak bi-ideal of $R \Leftrightarrow \mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic weak bi-ideal of R.

Proof. Assume that H is a weak bi-ideal of R. Let $x, y \in H$ we consider four cases:

(i) $x \in H$ and $y \in H$

(ii) $x \in H$ and $y \notin H$

(iii) $x \notin H$ and $y \in H$

(iv) $x \notin H$ and $y \notin H$

Case (i) If $x \in H$ and $y \in H$. Then $\overline{\mu}(x) = [p_1, p_2] = \overline{\mu}(y)$ and $\omega(x) = 1 - p = \omega(y)$. Since H is a weak bi-deal of R, then $x - y \in R$. Thus $\overline{\mu}(x - y) = [p_1, p_2] = \min\{[p_1, p_2], [p_1, p_2]\} = \min\{\overline{\mu}(x), \overline{\mu}(y)\}$ and $\omega(x - y) = 1 - p = \max\{1 - p, 1 - p\} = \max\{\omega(x), \omega(y)\}$.

Case (ii) If $x \in H$ and $y \notin H$. Then $\overline{\mu}(x) = [p_1, p_2], \overline{\mu}(y) = [q_1, q_2]$ and $\omega(x) = 1 - p, \omega(y) = 1 - q$. Clearly $\overline{\mu}(x - y) \geq \min\{\overline{\mu}(x), \overline{\mu}(y)\} = \min\{[p_1, p_2], [q_1, q_2]\} = [q_1, q_2]$ and $\omega(x - y) \leq \max\{\omega(x), \omega(y)\} = \max\{1 - p, 1 - q\} = 1 - q$. Now $\overline{\mu}(x - y) = [p_1, p_2]$ *or* $[q_1, q_2]$ according as $x - y \in H$ or $x - y \notin H$. By assumption that $[p_1, p_2] > [q_1, q_2]$ and $p > q$, we have $\overline{\mu}(x - y) \geq \min\{\overline{\mu}(x), \overline{\mu}(y)\}$ and $\omega(x - y) \leq \max\{\omega(x), \omega(y)\}$.

Similarly we can prove that case(iii).

Case(iv) If $x \notin H$ and $y \notin H$. Then $\overline{\mu}(x) = [q_1, q_2] = \overline{\mu}(y)$ and $\omega(x) = 1 - q = \omega(y)$. So, $\min\{\overline{\mu}(x), \overline{\mu}(y)\} = [q_1, q_2]$ and $\max\{\omega(x), \omega(y)\} = 1 - q$. Next $\overline{\mu}(x - y) = [p_1, P_2]$ or $[q_1, q_2]$ and $\omega(x - y) = 1 - p$ or $1 - q$, according as $x - y \in H$ or $x - y \notin H$. So $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic subgroup of R. Now let $x, y, z \in H$. We have the following eight cases:

(i) $x \in H, y \in H$ and $z \in H$

(ii) $x \notin H, y \in H$ and $z \in H$

(iii) $x \in H, y \notin H$ and $z \in H$

(iv) $x \in H, y \in H$ and $z \notin H$

(v) $x \notin H, y \notin H$ and $z \in H$

(vi) $x \in H, y \notin H$ and $z \notin H$

(vii) $x \notin H, y \in H$ and $z \notin H$

(viii) $x \notin H, y \notin H$ and $z \notin H$

These cases can be proved by similar arguments of the cubic cases above.

Hence, $\overline{\mu}(xyz) \geq \min\{\overline{\mu}(x), \overline{\mu}(y), \overline{\mu}(z)\}$ and $\omega(xyz) \leq \max\{\omega(x), \omega(y), \omega(z)\}$. Therefore $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic weak bi-ideal of R.

Conversly, assume that $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic weak bi-ideal of R. Let $x, y, z \in H$ be such that $\overline{\mu}(x) = \overline{\mu}(y) = \overline{\mu}(z) = [p_1, p_2]$ and $\omega(x) = \omega(y) = \omega(z) = 1 - p$. Since $\overline{\mu}$ is an interval-valued fuzzy weak bi-ideal of R, we have $\overline{\mu}(x - y) \geq \min\{\overline{\mu}(x), \overline{\mu}(y)\} = [p_1, p_2]$ and ω is a fuzzy weak bi-ideals of R, we have $\omega(x - y) \leq \max\{\omega(x), \omega(y)\} = 1 - p$.

Again, $\overline{\mu}(xyz) \geq \min\{\overline{\mu}(x), \overline{\mu}(y), \overline{\mu}(z)\} = [p_1, p_2]$ and $\omega(xyz) \leq \max\{\omega(x), \omega(y), \omega(z)\} = 1 - p$. So $x - y, xyz \in H$.

Hence H is a cubic weak bi-ideal of R. □

Theorem 3.12. *The direct product of cubic weak bi-ideals of near-rings is also a cubic weak bi-ideal.*

Proof. Let $\mathscr{A}_i = \langle \overline{\mu}_i, \omega_i \rangle$ be cubic weak bi-ideals of near-rings R_i for $i = 1, 2, 3, \ldots, n$. Let

$x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n) \in R_1 \times R_2 \times \cdots \times R_n$.

$$\begin{aligned}
\bar{\mu}_i(x-y) &= \bar{\mu}_i((x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n)) \\
&= \bar{\mu}_i(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n) \\
&= \min\{\bar{\mu}_1(x_1 - y_1), \bar{\mu}_2(x_2 - y_2), \ldots, \bar{\mu}_n(x_n - y_n)\} \\
&\geq \min\{\min\{\bar{\mu}_1(x_1), \bar{\mu}_1(y_1)\}, \min\{\bar{\mu}_2(x_2), \bar{\mu}_2(y_2)\}, \ldots, \min\{\bar{\mu}_n(x_n), \bar{\mu}_n(y_n)\}\} \\
&= \min\{\min\{\bar{\mu}_1(x_1), \bar{\mu}_2(x_2), \ldots, \bar{\mu}_n(x_n)\}, \min\{\bar{\mu}_1(y_1), \bar{\mu}_2(y_2), \ldots, \bar{\mu}_n(y_n)\}\} \\
&= \min\{(\bar{\mu}_1 \times \bar{\mu}_2 \times, \ldots, \times \bar{\mu}_n)(x_1, x_2, \ldots, x_n), (\bar{\mu}_1 \times \bar{\mu}_2 \times, \ldots, \times \bar{\mu}_n)(y_1, y_2, \ldots, y_n)\} \\
&= \min\{\bar{\mu}_i(x), \bar{\mu}_i(y)\}. \\
\omega_i(x-y) &= \omega_i((x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n)) \\
&= \omega_i(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n) \\
&= \max\{\omega_1(x_1 - y_1), \omega_2(x_2 - y_2), \ldots, \omega_n(x_n - y_n)\} \\
&\leq \max\{\max\{\omega_1(x_1), \omega_1(y_1)\}, \max\{\omega_2(x_2), \omega_2(y_2)\}, \ldots, \max\{\omega_n(x_n), \omega_n(y_n)\}\} \\
&= \max\{\max\{\omega_1(x_1), \omega_2(x_2), \ldots, \omega_n(x_n)\}, \max\{\omega_1(y_1), \omega_2(y_2), \ldots, \omega_n(y_n)\}\} \\
&= \max\{(\omega_1 \times \omega_2 \times, \ldots, \times \omega_n)(x_1, x_2, \ldots, x_n), (\omega_1 \times \omega_2 \times, \cdots, \times \omega_n)(y_1, y_2, \ldots, y_n)\} \\
&= \max\{\omega_i(x), \omega_i(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mu}_i(xyz) &= \bar{\mu}_i((x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n)(z_1, z_2, \ldots, z_n)) \\
&= \bar{\mu}_i(x_1 y_1 z_1, x_2 y_2 z_2, \ldots, x_n y_n z_n) \\
&= \min\{\bar{\mu}_1(x_1 y_1 z_1), \bar{\mu}_2(x_2 y_2 z_2), \ldots, \bar{\mu}_n(x_n y_n z_n)\} \\
&\geq \min\{\min\{\bar{\mu}_1(x_1), \bar{\mu}_1(y_1), \bar{\mu}_1(z_1)\}, \min\{\bar{\mu}_2(x_2), \bar{\mu}_2(y_2), \bar{\mu}_2(z_2)\}, \ldots, \\
&\qquad \min\{\bar{\mu}_n(x_n), \bar{\mu}_n(y_n), \bar{\mu}_n(z_n)\}\} \\
&= \min\{\min\{\bar{\mu}_1(x_1), \bar{\mu}_2(x_2), \ldots, \bar{\mu}_n(x_n)\}, \min\{\bar{\mu}_1(y_1), \bar{\mu}_2(y_2), \ldots, \bar{\mu}_n(y_n), \\
&\qquad \min\{\bar{\mu}_1(z_1), \bar{\mu}_2(z_2), \ldots, \bar{\mu}_n(z_n)\}\} \\
&= \min\{(\bar{\mu}_1 \times \bar{\mu}_2 \times, \cdots, \times \bar{\mu}_n)(x_1, x_2, \ldots, x_n), (\bar{\mu}_1 \times \bar{\mu}_2 \times, \ldots, \times \bar{\mu}_n)(y_1, y_2, \ldots, y_n), \\
&\qquad (\bar{\mu}_1 \times \bar{\mu}_2 \times, \ldots, \times \bar{\mu}_n)(z_1, z_2, \ldots, z_n)\} \\
&= \min\{\bar{\mu}_i(x), \bar{\mu}_i(y), \bar{\mu}_i(z)\}. \\
\omega_i(xyz) &= \omega_i((x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n)(z_1, z_2, \ldots, z_n)) \\
&= \omega_i(x_1 y_1 z_1, x_2 y_2 z_2, \ldots, x_n y_n z_n) \\
&= \max\{\omega_1(x_1 y_1 z_1), \omega_2(x_2 y_2 z_2), \ldots, \omega_n(x_n y_n z_n)\} \\
&\leq \max\{\max\{\omega_1(x_1), \omega_1(y_1), \omega_1(z_1)\}, \max\{\omega_2(x_2), \omega_2(y_2), \omega_2(z_2)\}, \ldots, \\
&\qquad \max\{\omega_n(x_n), \omega_n(y_n), \omega_n(z_n)\}\} \\
&= \max\{\max\{\omega_1(x_1), \omega_2(x_2), \ldots, \omega_n(x_n)\}, \max\{\omega_1(y_1), \omega_2(y_2), \ldots, \omega_n(y_n), \\
&\qquad \max\{\omega_1(z_1), \omega_2(z_2), \ldots, \omega_n(z_n)\}\} \\
&= \max\{(\omega_1 \times \omega_2 \times, \cdots, \times \omega_n)(x_1, x_2, \ldots, x_n), (\omega_1 \times \omega_2 \times, \cdots, \times \omega_n)(y_1, y_2, \ldots, y_n), \\
&\qquad (\omega_1 \times \omega_2 \times, \cdots, \times \omega_n)(z_1, z_2, \ldots, z_n)\} \\
&= \max\{\omega_i(x), \omega_i(y), \omega_i(z)\}.
\end{aligned}$$

□

4 Homomorphism of cubic weak bi-ideals of near-rings

Definition 4.1. [5] Let R and S be near rings. A map $\theta : R \to S$ is called a (near-ring) homomorphism if $\theta(x+y) = \theta(x) + \theta(y)$ and $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in R$.

Definition 4.2. [9] Let f be a mapping from a set X to Y and $\mathscr{A} = \langle \overline{\mu}, \lambda \rangle$ be a cubic set of X then the image of X $C_f(\mathscr{A}) = \langle C_f(\overline{\mu}), C_f(\lambda) \rangle$ is a cubic set of Y defined by

$$C_f(\mathscr{A})(y) = \begin{cases} C_f(\bar{\mu})(y) = \begin{cases} \sup\limits_{f(x)=y} \overline{\mu}(x) & \text{if } f^{-1}(y) \neq \emptyset \\ [0,0] & \text{otherwise} \end{cases} \\ C_f(\lambda)(y) = \begin{cases} \inf\limits_{f(x)=y} \lambda(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases} \end{cases}$$

and let f be a mapping from a set X to Y and $\mathscr{A} = \langle \overline{\mu}, \lambda \rangle$ is a cubic set of Y, then the pre image of Y $C_f{}^{-1}(\mathscr{A}) = \left\langle C_f{}^{-1}(\overline{\mu}), C_f{}^{-1}(\lambda) \right\rangle$ is a cubic set of X is defined by

$$C_f{}^{-1}(\mathscr{A})(x) = \begin{cases} C_f{}^{-1}(\bar{\mu})(x) = \overline{\mu}(f(x)) \\ C_f{}^{-1}(\lambda)(x) = \lambda(f(x)) \end{cases}$$

Theorem 4.3. *Let $f : R \to R_1$ be a homomorphism between two near-rings R and R_1. If $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic weak bi-ideal of R_1, then $C_f{}^{-1}(\mathscr{A}) = \left\langle C_f{}^{-1}(\overline{\mu}), C_f{}^{-1}(\omega) \right\rangle$ is a cubic weak bi-ideal of R.*

Proof. Let $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ be a cubic weak bi-ideal of R_1.

Let $x, y, z \in R$. Then $C_f(x), C_f(y), C_f(z) \in R_1$, we have $\overline{\mu}$ is an interval-valued fuzzy weak bi-ideal of R_1.

$$\begin{aligned} C_f^{-1}(\overline{\mu})(x-y) &= \overline{\mu}(f(x-y)) \\ &= \overline{\mu}(f(x) - f(y)) \\ &\geq \min\{\overline{\mu}(f(x)), \overline{\mu}(f(y))\} \\ &= \min\{C_f^{-1}(\overline{\mu})(x), C_f^{-1}(\overline{\mu})(y)\} \end{aligned}$$

and ω is a fuzzy weak bi-ideal of R_1

$$\begin{aligned} C_f^{-1}(\omega)(x-y) &= \omega(f(x-y)) \\ &= \omega(f(x) - f(y)) \\ &\leq \max\{\omega(f(x)), \omega(f(y))\} \\ &= \max\{C_f^{-1}(\omega)(x), C_f^{-1}(\omega)(y)\} \end{aligned}$$

$C_f{}^{-1}(\mathscr{A}) = \left\langle C_f{}^{-1}(\overline{\mu}), C_f{}^{-1}(\omega) \right\rangle$ is a cubic subgroup of R. Again

$$\begin{aligned} C_f^{-1}(\overline{\mu})(xyz) &= \overline{\mu}(f(xyz)) \\ &= \overline{\mu}(f(x)f(y)f(z)) \\ &\geq \min\{\overline{\mu}(f(x)), \overline{\mu}(f(y)), \overline{\mu}(f(z))\} \\ &= \min\{C_f^{-1}(\overline{\mu})(x), C_f^{-1}(\overline{\mu})(y), C_f^{-1}(\overline{\mu})(z)\} \end{aligned}$$

and

$$\begin{aligned} C_f^{-1}(\omega)(xyz) &= \omega(f(xyz)) \\ &= \omega(f(x)f(y)f(z)) \\ &\leq \max\{\omega(f(x)), \omega(f(y)), \omega(f(z))\} \\ &= \max\{C_f^{-1}(\omega)(x), C_f^{-1}(\omega)(y), C_f^{-1}(\omega)(z)\} \end{aligned}$$

Hence $C_f{}^{-1}(\mathscr{A}) = \left\langle C_f{}^{-1}(\overline{\mu}), C_f{}^{-1}(\omega) \right\rangle$ is a cubic weak bi-ideal of R. □

Remark 4.4. We can also state the converse of the theorem by strengthening the condition of f as follows.

Theorem 4.5. *Let $f : R \to R_1$ be a homomorphism between two near-rings R and R_1. Let $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic subset of R_1. If $C_f{}^{-1}(\mathscr{A}) = \left\langle C_f{}^{-1}(\overline{\mu}), C_f{}^{-1}(\omega) \right\rangle$ is a cubic weak bi-ideal of R, then $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic weak bi-ideal of R_1.*

Proof. Let $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ be a cubic subset of R_1 and $x, y, z \in R_1$. Then $f(a) = x, f(b) = y, f(c) = z$ for some $a, b, c \in R$, it follows that $\overline{\mu}$ is an interval-valued fuzzy weak bi-ideal of R_1

$$\begin{aligned}
\overline{\mu}(x-y) &= \overline{\mu}(f(a)-f(b)) \\
&= \overline{\mu}(f(a-b)) \\
&= (C_f{}^{-1}(\overline{\mu}))(a-b) \\
&\geq \min\{C_f{}^{-1}(\overline{\mu})(a), C_f{}^{-1}(\overline{\mu})(b)\} \\
&= \min\{(\overline{\mu})(f(a)), (\overline{\mu})(f(b))\} \\
&= \min\{\overline{\mu}(x), \overline{\mu}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\omega(x-y) &= \omega(f(a)-f(b)) \\
&= \omega(f(a-b)) \\
&= (C_f{}^{-1}(\omega))(a-b) \\
&\leq \max\{C_f{}^{-1}(\omega)(a), C_f{}^{-1}(\omega)(b)\} \\
&= \max\{(\omega)(f(a)), (\omega)(f(b))\} \\
&= \max\{\omega(x), \omega(f(y))\}
\end{aligned}$$

Again

$$\begin{aligned}
\overline{\mu}(xyz) &= \overline{\mu}(f(a)f(b)f(c)) \\
&= \overline{\mu}(f(abc)) \\
&= (C_f{}^{-1}(\overline{\mu}))(abc) \\
&\geq \min\{C_f{}^{-1}(\overline{\mu})(a), C_f{}^{-1}(\overline{\mu})(b), C_f{}^{-1}(\overline{\mu})(c)\} \\
&= \min\{\overline{\mu}(f(a)), \overline{\mu}(f(b)), \overline{\mu}(f(c))\} \\
&= \min\{\overline{\mu}(x), \overline{\mu}(y), \overline{\mu}(z)\}
\end{aligned}$$

and

$$\begin{aligned}
\omega(xyz) &= \omega(f(a)f(b)f(c)) \\
&= \omega(f(abc)) \\
&= (C_f{}^{-1}(\omega))(abc) \\
&\leq \max\{C_f{}^{-1}(\omega)(a), C_f{}^{-1}(\omega)(b), C_f{}^{-1}(\omega)(c)\} \\
&= \max\{\omega(f(a)), \omega(f(b)), \omega(f(c))\} \\
&= \max\{\omega(x), \omega(y), \omega(z)\}
\end{aligned}$$

Hence $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic weak bi-ideal of R_1. □

Theorem 4.6. *Let $f : R \to R_1$ be an onto near-ring homomorphism. If $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic weak bi-ideal of R then $C_f(\mathscr{A}) = \langle C_f(\overline{\mu}), C_f(\omega) \rangle$ is a cubic weak bi-ideal of R_1.*

Proof. Let $\mathscr{A} = \langle \overline{\mu}, \omega \rangle$ is a cubic weak bi-ideal of R. Since $C_f(\overline{\mu})(x') = \sup_{f(x)=x'} (\overline{\mu}(x))$ for $x' \in R_1$ and $C_f(\omega)(x') = \inf_{f(x)=x'} (\omega(x))$ for $x' \in R_1$.

So $C_f(\mathscr{A}) = \langle C_f(\overline{\mu}), C_f(\omega) \rangle$ is non-empty. Let $x', y', z' \in R_1$. Then we have

$$\begin{aligned} C_f(\overline{\mu})(x'-y') &= \sup_{f(p)=x'-y'} \overline{\mu}(p) \\ &\geq \sup_{f(x)=x', f(y)=y'} \overline{\mu}(x-y) \\ &\geq \sup_{f(x)=x', f(y)=y'} \min\{\overline{\mu}(x), \overline{\mu}(y)\} \\ &= \min\left\{ \sup_{f(x)=x'} \overline{\mu}(x), \sup_{f(y)=y'} \overline{\mu}(y) \right\} \\ &= \min\{C_f(\overline{\mu})(x'), C_f(\overline{\mu})(y')\} \\ C_f(\omega)(x'-y') &= \inf_{f(p)=x'-y'} \omega(p) \\ &\leq \inf_{f(x)=x', f(y)=y'} \omega(x-y) \\ &\leq \inf_{f(x)=x', f(y)=y'} \max\{\omega(x), \omega(y)\} \\ &= \max\left\{ \inf_{f(x)=x'} \omega(x), \inf_{f(y)=y'} \omega(y) \right\} \\ &= \max\{C_f(\omega)(x'), C_f(\omega)(y')\} \end{aligned}$$

and

$$\begin{aligned} C_f(\overline{\mu})(x'y'z') &= \sup_{f(p)=x'y'z'} \overline{\mu}(p) \\ &\geq \sup_{f(x)=x', f(y)=y', f(z)=z'} \overline{\mu}(xyz) \\ &\geq \sup_{f(x)=x', f(y)=y', f(z)=z'} \min\{\overline{\mu}(x), \overline{\mu}(y), \overline{\mu}(z)\} \\ &= \min\left\{ \sup_{f(x)=x'} \overline{\mu}(x), \sup_{f(y)=y'} \overline{\mu}(y), \sup_{f(z)=z'} \overline{\mu}(z) \right\} \\ &= \min\{C_f(\overline{\mu})(x'), C_f(\overline{\mu})(y'), C_f(\overline{\mu})(z')\} \\ C_f(\omega)(x'y'z') &= \inf_{f(p)=x'y'z'} \omega(p) \\ &\leq \inf_{f(x)=x', f(y)=y', f(z)=z'} \omega(xyz) \\ &\leq \inf_{f(x)=x', f(y)=y', f(z)=z'} \max\{\omega(x), \omega(y), \omega(z)\} \\ &= \max\left\{ \inf_{f(x)=x'} \omega(x), \inf_{f(y)=y'} \omega(y), \inf_{f(z)=z'} \omega(z) \right\} \\ &= \max\{C_f(\omega)(x'), C_f(\omega)(y'), C_f(\omega)(z')\} \end{aligned}$$

Hence $C_f(\mathscr{A}) = \langle C_f(\overline{\mu}), C_f(\omega) \rangle$ is a cubic weak bi-ideal of R_1. □

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Received: November 3, 2016.

Accepted: December 28, 2016.