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Infinitesimal CL-transformations on $(LCS)_n$-manifolds

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Abstract. In this paper, we study infinitesimal CL-transformations on $(LCS)_n$-manifolds, whose metric tensors are Ricci solitons. We find conditions for Ricci solitons to be expanding, steady and shrinking on CL-flat and conformally flat $(LCS)_n$-manifolds.

1 Introduction

In 2003, Shaikh [19] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [16]. Then Shaikh and Baishya ([22], [23]) investigated some applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. Thereafter $(LCS)_n$-manifolds are studied by several authors.

In 1982, Hamilton [11] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for the metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generelization of Einstein metric such that [12]

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is the Ricci tensor and $\pounds_V$ is the Lie derivative along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive, respectively.

In last two decades, Ricci solitons have been studied by many mathematicians. In particular, the study of Ricci solitons becomes more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [25], Sharma studied the Ricci solitons in contact geometry. Later on, Ricci solitons in contact geometry have been studied by various authors such as Bagewadi et al. ([4], [5], [6], [15]), Bejan and Crasmareanu [7], Chen and Deshmukh [9], Deshmukh et al. [10], Nagaraja and Premalatha [17], Tripathi [26] and many others. Recently Hui et al. studied Ricci solitons on $(LCS)_n$-manifolds ([8], [13]) and on generalized Sasakian-space-forms [14], respectively.

Moreover, n-dimensional compact Ricci soliton with closed vector field and scalar curvature satisfying certain conditions has been studied by Sharfuddin, Ahsan and Deshmukh [24]; while for general theory of relativity, the role of Ricci Solitons has been explored by Ali and Ahsan (see [1], [2], [3]). Motivated by the above studies in the present paper, we study $(LCS)_n$-manifolds whose metrics are Ricci solitons. The paper is organized as follows: Section 2 is concerned with preliminaries. Section 3 is devoted to the study of infinitesimal CL-transformations on $(LCS)_n$-manifolds whose metric tensors are Ricci solitons. In section 4, we study conformally flat $(LCS)_n$-manifolds whose metric are also Ricci solitons.

2 Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of the type (0, 2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \cdots, +)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the set of real numbers. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp, ≤ 0, = 0, > 0) [18].

Definition 2.1. [27] In a Lorentzian manifold (M, g), a vector field P defined by

$$g(X, P) = A(X),$$

for any $X \in \Gamma(TM)$, the section of all smooth tangent vector fields on M, is said to be a *concircular vector field* if

$$(\nabla_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\}$$

where α is a non-zero scalar and ω is a closed 1-form and ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let M be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of M. Then, we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that

$$g(X, \xi) = \eta(X), \tag{2.2}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0 \tag{2.3}$$

that is,

$$\nabla_X \xi = \alpha[X + \eta(X)\xi] \tag{2.4}$$

for all vector fields X, Y, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.5}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.6}$$

then from (2.4) and (2.6), we have

$$\phi X = X + \eta(X)\xi, \tag{2.7}$$

from which it follows that ϕ is a symmetric (1, 1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, it is associated with 1-form η and a (1, 1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, an $(LCS)_n$-manifold) [20]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure introduced by Matsumoto [16]. In an $(LCS)_n$-manifold $(n > 2)$, the following relations hold ([20], [22], [23]):

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.8}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.9}$$

$$S(X,\xi) = (n-1)(\alpha^2-\rho)\eta(X), \tag{2.10}$$

$$R(X,Y)\xi = (\alpha^2-\rho)[\eta(Y)X-\eta(X)Y], \tag{2.11}$$

$$R(\xi,Y)Z = (\alpha^2-\rho)[g(Y,Z)\xi-\eta(Z)Y], \tag{2.12}$$

$$(\nabla_X\phi)Y = \alpha\{g(X,Y)\xi+2\eta(X)\eta(Y)\xi+\eta(Y)X\}, \tag{2.13}$$

$$(X\rho) = d\rho(X) = \beta\eta(X) \tag{2.14}$$

for any vector fields $X,\ Y,\ Z$ on M and $\beta = -(\xi\rho)$ is a scalar function, where R is the curvature tensor and S is the Ricci tensor of the manifold. From (2.4), we have

$$(\pounds_\xi g)(X,Y) = 2\alpha\{g(X,Y)+\eta(X)\eta(Y)\}. \tag{2.15}$$

3 Infinitesimal CL-transformations and Ricci solitons

This section deals with the infinitesimal CL-trasformations on $(LCS)_n$-manifolds and CL-flat $(LCS)_n$-manifolds whose metric tensors are Ricci solitons.

Definition 3.1. A vector field V in an $(LCS)_n$-manifold M is said to be an infinitesimal CL-transformation [21] if it satisfies

$$\pounds_V\{^{\,h}_{ij}\} = \mu_j\delta^h_i+\mu_i\delta^h_j+a(\eta_j\phi^h_i+\eta_i\phi^h_j)+b\phi_{ji}\xi^h,\ \ \phi_{ji}=\phi^l_j g_{li} \tag{3.1}$$

for certain constants a and b, where μ_i are the components of the 1-form μ, $\pounds_V$ denotes the Lie derivative with respect to V and $\{^{\,h}_{ij}\}$ is the Christoffel symbol of the Lorentzian metric g.

In [21], Shaikh and Ahmad studied infinitesimal CL-transformations on an $(LCS)_n$-manifold M and obtained the following useful result.

Theorem 3.2. *If V is an infinitesimal CL-transformation on an $(LCS)_n$-manifold M, then the relation*

$$\begin{aligned}(\alpha^2-\rho)(\pounds_V g)(Y,Z) &= -(\nabla_Y\mu)(Z)+\{\alpha(a+b)-(2\alpha\rho-\beta)\eta(V)\}g(Y,Z) \\ &+ \alpha(3a+b)\eta(Y)\eta(Z)\end{aligned} \tag{3.2}$$

holds for any vector fields Y and Z on M.

From (1.1) and (3.2), we obtain

$$\begin{aligned}2(\alpha^2-\rho)S(Y,Z) &= (\nabla_Y\mu)(Z)-\{\alpha(a+b)-(2\alpha\rho-\beta)\eta(V) \\ &+ 2\lambda(\alpha^2-\rho)\}g(Y,Z)-\alpha(3a+b)\eta(Y)\eta(Z).\end{aligned} \tag{3.3}$$

Thus, we can state the following:

Theorem 3.3. *Let (g,V,λ) be a Ricci soliton on a $(LCS)_n$-manifold M. If V is an infinitesimal CL-transformation on M, then the Ricci tensor S is given by the relation (3.3).*

Also, Shaikh and Ahmad [21] proved the tensor field

$$\begin{aligned}A(X,Y)Z &= R(X,Y)Z-\frac{1}{n-2}[\{S(Y,Z)X-S(X,Z)Y\} \\ &+ \{S(Y,Z)\eta(X)-S(X,Z)\eta(Y)\}\xi] \\ &+ \frac{\alpha^2-\rho}{n-2}[\{g(Y,Z)X-g(X,Z)Y\} \\ &+ (n-1)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi]\end{aligned}$$

is invariant on an $(LCS)_n$-manifold M under a CL-transformation, and it is called the CL-curvature tensor field on M.

Definition 3.4. An $(LCS)_n$-manifold M is said to be CL-flat if the CL-curvature tensor field A of the type $(1,3)$ vanishes identically on M.

It is known that [21] a CL-flat $(LCS)_n$-manifold M is an η-Einstein manifold, and its Ricci tensor is of the form

$$S(X,Y) = \Big\{\frac{r}{n-1} - (\alpha^2-\rho)\Big\}g(X,Y) + \Big\{\frac{r}{n-1} - n(\alpha^2-\rho)\Big\}\eta(X)\eta(Y), \tag{3.4}$$

where $r\big(\neq n(n-1)(\alpha^2-\rho)\big)$ is the scalar curvature of the manifold.
Again in [8], Chandra, Hui and Shaikh obtained the following:

Theorem 3.5. [8] *A second order parallel symmetric tensor on an $(LCS)_n$-manifold with $(\alpha^2 - \rho) \neq 0$, is a constant multiple of the metric tensor.*

Theorem 3.6. [8] *If the tensor field $\pounds_V g + 2S$ on an $(LCS)_n$-manifold with $\alpha^2 - \rho \neq 0$ is parallel for any vector field V, then (g, V, λ) is a Ricci soliton.*

Suppose that h is a $(0,2)$ symmetric parallel tensor field on an $(LCS)_n$-manifold such that

$$h(X,Y) = (\pounds_\xi g)(X,Y) + 2S(X,Y) \tag{3.5}$$

Using (2.15) and (3.4) in (3.5), we get

$$\begin{aligned} h(X,Y) &= 2[\frac{r}{n-1} - (\alpha^2-\rho) + \alpha]g(X,Y) \\ &+ 2[\frac{r}{n-1} - n(\alpha^2-\rho) + \alpha]\eta(X)\eta(Y). \end{aligned} \tag{3.6}$$

Putting $X = Y = \xi$ in (3.6), we obtain

$$h(\xi,\xi) = -2(n-1)(\alpha^2-\rho) \tag{3.7}$$

If (g, ξ, λ) is a Ricci soliton on an $(LCS)_n$-manifold M, then from (1.1), we have

$$h(X,Y) = -2\lambda g(X,Y) \tag{3.8}$$

and hence

$$h(\xi,\xi) = 2\lambda. \tag{3.9}$$

From (3.7) and (3.9) we get $\lambda = -(n-1)(\alpha^2-\rho)$. Since $n > 1$ and $(\alpha^2-\rho) \neq 0$, we have $\lambda > 0$ or < 0 according as $(\alpha^2-\rho) < 0$ or $(\alpha^2-\rho) > 0$, respectively. Thus we can state the following:

Theorem 3.7. *If the tensor field $\pounds_\xi g + 2S$ on a CL-flat $(LCS)_n$-manifold with $(\alpha^2 - \rho) \neq 0$ is parallel, then the Ricci soliton (g, ξ, λ) is shrinking if $(\alpha^2-\rho) > 0$ and expanding if $(\alpha^2-\rho) < 0$.*

4 Conformally flat $(LCS)_n$-manifolds and Ricci solitons

In this section we study conformally flat $(LCS)_n$-manifolds whose metric tensors are Ricci solitons with a conformal Killing vector field.

Definition 4.1. A vector field V in an $(LCS)_n$-manifold M is said to be a *conformal Killing vector field* if it satisfies

$$(\pounds_V g)(Y,Z) = 2\sigma g(Y,Z) \tag{4.1}$$

for any vector fields Y and Z on M, where σ is a smooth function on M and $\pounds$ is the operator of Lie differentiation.

In particular, if σ is constant, then V is called homothetic and if $\sigma = 0$, then V is called isometric as well as Killing vector field.

Now, we consider (g, V, λ) is a Ricci soliton on an $(LCS)_n$-manifold M. If V is a conformal Killing vector field, then by virtue of (4.1) and (1.1), we obtain

$$S(X,Y) = -(\lambda + \sigma)g(X,Y), \tag{4.2}$$

$$QX = -(\lambda + \sigma)X, \tag{4.3}$$

$$r = -n(\lambda + \sigma). \tag{4.4}$$

Let us consider that M be a conformally flat manifold. Then, we have

$$\begin{aligned} R(X,Y)Z &= \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{4.5}$$

Using (4.2)-(4.4) in (4.5), we obtain

$$R(X,Y)Z = -\frac{(\lambda + \sigma)}{n-1}[g(Y,Z)X - g(X,Z)Y]. \tag{4.6}$$

Putting $Z = \xi$ in (4.6), we derive

$$R(X,Y)\xi = -\frac{(\lambda + \sigma)}{n-1}[\eta(Y)X - \eta(X)Y]. \tag{4.7}$$

From (3.2) and (4.7), we find

$$\lambda = -[\sigma + (n-1)(\alpha^2 - \rho)].$$

This leads to the following:

Theorem 4.2. *Let (g, V, λ) be a Ricci soliton on a conformally flat $(LCS)_n$-manifold M. If V is conformal Killing, then (g, V, λ) is*

(i) shrinking for $\sigma + (n-1)(\alpha^2 - \rho) > 0$,

(ii) steady for $\sigma + (n-1)(\alpha^2 - \rho) = 0$,

(iii) expanding for $\sigma + (n-1)(\alpha^2 - \rho) < 0$.

Corollary 4.3. *Let (g, V, λ) be a Ricci soliton on an $(LCS)_3$-manifold M. If V is conformal Killing, then (g, V, λ) is shrinking, steady and expanding according as $\sigma + 2(\alpha^2 - \rho) > 0$, $= 0$ and < 0, respectively.*

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