Palestine Journal of Mathematics

Vol. 6(Special Issue: II) (2017) , 204–214

A Note on Hermite-Based poly-Euler and Multi poly-Euler Polynomials

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Communicated by Jose Luis Lopez-Bonilla

MSC 2010 Classifications: Primary 33C45, 11B73, 11B68, 11A05.

Keywords and phrases: Hermite polynomials, poly-Euler polynomials, Hermite poly-Euler polynomials, multi poly-Euler Polynomials, multi Hermite poly-Euler polynomials, summation formulae, symmetric identities.

Abstract In this paper, we introduce a new class of generalized poly-Euler, Hermite poly-Euler, multi poly-Euler and multi Hermite poly-Euler polynomials. The concepts of poly-Euler numbers $E_n^{(k)}(a,b)$, generalized poly-Euler polynomials $E_n^{(k)}(x;a,b,c)$ of Jolany et al, Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ of Dattoli et al and ${}_HB_n^{(\alpha)}(x,y)$ of Pathan and Khan are generalized to the one ${}_HE_n^{(k)}(x,y;a,b,c)$. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions.

1 Introduction

Recently the generalized poly-Euler polynomials are defined by Jolany et al [4, 5, 6, 7] as follows

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}c^{xt}=\sum_{n=0}^{\infty}\frac{E_n^{(k)}(x;a,b,c)t^n}{n!},|t|<\frac{2\pi}{|\ln a+\ln b|} \tag{1.1}$$

Note that the poly-Euler polynomials of Sasaki and Bayad [1, 11] can be deduced from (1.1) by replacing t with 4t and taking $x=\frac{1}{2}$. when $x=0$, (1.1) gives

$$E_n^{(k)}(0;a,b,c)=E_n^{(k)}(a,b)$$

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}=\sum_{n=0}^{\infty}\frac{E_n^{(k)}(a,b)t^n}{n!},|t|<\frac{2\pi}{|\ln a+\ln b|} \tag{1.2}$$

and when $a=1$ and $b=c=e$, we get

$$E_n^{(k)}(x;1,e,e)=E_n^{(k)}(x)$$

where

$$\frac{2Li_k(1-e^{-t})}{1+e^t}e^{xt}=\sum_{n=0}^{\infty}\frac{E_n^{(k)}(x)t^n}{n!},|t|<\frac{2\pi}{|\ln a+\ln b|} \tag{1.3}$$

On the other hand in the same paper by Jolany et al [4, 5, 6, 7], they defined certain multi poly-Euler polynomials as follows

$$\frac{2Li_{k_1,\ldots,k_r}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}c^{rxt}=\sum_{n=0}^{\infty}\frac{E_n^{(k_1,\ldots,k_r)}(x;a,b,c)t^n}{n!},|t|<\frac{2\pi}{|\ln a+\ln b|} \tag{1.4}$$

where

$$Li_{(k_1,\ldots,k_r)}(z)=\sum_{r,k=1}^{\infty}\frac{z^{m_r}}{m_1^{k_1}\ldots m_r^{k_r}}$$

is the generalization of poly-logarithm.

In particular

$$E_n^{(k_1,\ldots,k_r)}(x;1,e,e) = E_n^{(k_1,\ldots,k_r)}(x)$$

$$E_n^{(k_1,\ldots,k_r)}(0;a,b,c) = E_n^{(k_1,\ldots,k_r)}(a,b)$$

Further by taking $r = 1$, (1.4) immediately yield (1.1).

The generalized Hermite-Bernoulli polynomials of two variables ${}_HB_n^{(\alpha)}(x,y)$ introduced by Pathan [12] and Pathan and Khan [13 to 18] in the form

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!} \tag{1.5}$$

which is essentially a generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ introduced by Dattoli et al [3, p.386(1.6)] in the form

$$\left(\frac{t}{e^t-1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x,y)\frac{t^n}{n!} \tag{1.6}$$

Definition 1.1. Let $c > 0$. The generalized 2-variable 1-parameter Hermite Kamp'e de Feriet polynomials $H_n(x,y,c)$ polynomials for nonnegative integer n are defined by

$$c^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y,c)\frac{t^n}{n!} \tag{1.7}$$

This is an extended 2-variable Hermite Kamp'e de Feriet polynomials $H_n(x,y)$ (see[2]) defined by

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} \tag{1.8}$$

Note that

$$H_n(x,y,e) = H_n(x,y)$$

In order to collect the powers of t we expand the left hand side of (1.7) to get

$$\left(\sum_{n=0}^{\infty}\frac{x^n(\ln c)^n t^n}{n!}\right)\left(\sum_{j=0}^{\infty}\frac{y^j(\ln c)^j t^{2j}}{j!}\right) = \sum_{n=0}^{\infty} H_n(x,y,c)\frac{t^n}{n!}$$

Thus we led to the representation

$$H_n(x,y,c) = \sum_{j=0}^{[\frac{n}{2}]}\left(\begin{array}{c} n \\ j \end{array}\right)(\ln c)^{n-j}x^{n-2j}y^j \tag{1.9}$$

In this note we first give definitions of the generalized poly-Euler polynomials $E_n^{(k)}(x;a,b.c)$ which generalize the concepts stated above and then research their basic properties and relationships with poly-Euler numbers $E_n^{(k)}(a,b)$, poly-Euler polynomials $E_n^{(k)}(x)$ and the generalized poly-Euler polynomials $E_n^{(k)}(x;a,b,c)$ of Jolany et al, Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ of Dattoli et al and ${}_HB_n^{(\alpha)}(x,y)$ of Pathan and Pathan and Khan. The remainder of this paper is organized as follows. We modify generating functions for the poly-Euler polynomials and derive some identities related to Hermite polynomials, poly-Euler polynomials and power sums. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These results extend some known summations and identities of generalized Hermite-Bernoulli polynomials studied by Dattoli et al, Zhang et al, Yang, Khan, Pathan and Pathan and Khan.

2 Definitions and Properties of the Generalized poly-Euler and Multi poly-Euler Polynomials

In this section, we are establish a definitions and properties of generalized poly-Euler polynomials $E_n^{(k)}(x, y; a, b, c)$ and multi poly-Euler polynomials $E_n^{(k_1,\ldots,k_r)}(x, y; a, b, c)$

Definition 2.1. Let $a, b, c > 0$ and $a \neq b$. The generalized Hermite poly-Euler polynomials $E_n^{(k)}(x, y; a, b, c)$ for nonnegative integer n are defined by

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE_n^{(k)}(x, y; a, b, c)\frac{t^n}{n!}, \mid t \mid < 2\pi/(\mid \ln a + \ln b \mid), x\epsilon\Re \quad (2.1)$$

whereas for $x = 0$ gives

$$E_n^{(k)}(0, y; a, b, c) = \sum_{m=0}^{[\frac{n}{2}]} \frac{n!}{m!(n-2m)!}(\ln c)^m E_{n-2m}^{(k)}(a, b)y^m \quad (2.2)$$

Another special case of (2.1) for $x = 0, y = 0$ leads to the extension of the generalized poly-Euler numbers $E_n^{(k)}(a, b)$ for nonnegative integer n defined by (1.2) in the form.

Further setting $c = e$ in (2.1), we get

Definition 2.2. Let $a, b > 0$ and $a \neq b$. The generalized Hermite poly-Euler polynomials ${}_HE_n^{(k)}(x, y; a, b, e)$ for nonnegative integer n are defined by

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE_n^{(k)}(x, y; a, b, e)\frac{t^n}{n!}, \quad \mid t \mid < 2\pi/(\mid \ln a + \ln b \mid), x\epsilon\Re \quad (2.3)$$

Definition 2.3. Let $a, b, c > 0$ and $a \neq b$. The generalized multi Hermite poly-Euler polynomials ${}_HE_n^{(k_1,\ldots,k_r)}(x, y; a, b, c)$ for nonnegative integer n are defined by

$$\frac{2Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}c^{r(xt+yt^2)} = \sum_{n=0}^{\infty} {}_HE_n^{(k_1,\ldots,k_r)}(x, y; a, b, c)\frac{t^n}{n!}, \quad \mid t \mid < 2\pi/(\mid \ln a + \ln b \mid), x\epsilon\Re \quad (2.4)$$

For $y = 0$ in (2.4), the result reduces to (1.4).

Further setting $c = e$ in (2.4), we get

Definition 2.4. Let $a, b > 0$ and $a \neq b$. The generalized multi Hermite poly-Euler polynomials ${}_HE_n^{(k_1,\ldots,k_r)}(x, y; a, b, e)$ for nonnegative integer n are defined by

$$\frac{2Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}e^{r(xt+yt^2)} = \sum_{n=0}^{\infty} {}_HE_n^{(k_1,\ldots,k_r)}(x, y; a, b, e)\frac{t^n}{n!}, \quad \mid t \mid < 2\pi/(\mid \ln a + \ln b \mid), x\epsilon\Re \quad (2.5)$$

The generalized poly-Euler polynomials $E_n^{(k)}(x, y; a, b, c)$ and generalized muti poly-Euler polynomials $E_n^{(k_1,\ldots,k_r)}(x, y; a, b, c)$ defined by (2.1) and (2.4) have the following properties which are stated as theorems below.

Theorem 2.1. Let $a, b, c > 0$ and $a \neq b$. For $x\epsilon R$ and $n \geq 0$. Then

$${}_HE_n^{(k)}(x, y, 1, e, e) = {}_HE_n^{(k)}(x, y), {}_HE_n^{(k)}(0, 0, a, b, 1) = E_n^{(k)}(a, b),$$

$${}_HE_n^{(k)}(0, 0, 1, e, 1) = E_n^{(k)}, {}_HE_n^{(k)}(x, y, a, b, e) = {}_HE_n^{(k)}(x, y; a, b) \quad (2.6)$$

$${}_HE_n^{(k)}(x+u, y+z; a, b, c) = \sum_{m=0}^{n} \binom{n}{m} H_m(z, u; c)\,{}_HE_{n-m}^{(k)}(x, y; a, b, c) \quad (2.7)$$

$$ {}_HE_n^{(k)}(x+z,y;a,b,c)=\sum_{m=0}^{n}\binom{n}{m}E_{n-m}^{(k)}(z;a,b,c)H_m(x,y;c) \tag{2.8}$$

Proof. The formula in (2.6) are obvious. Applying Definition (2.1), we have

$$\sum_{n=0}^{\infty}{}_HE_n^{(k)}(x+u,y+z;a,b,c)\frac{t^n}{n!}=\sum_{n=0}^{\infty}{}_HE_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!}\sum_{m=0}^{\infty}H_m(z,u;c)\frac{t^m}{m!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}H_m(z,u;c)\frac{t^m}{m!}{}_HE_{n-m}^{(k)}(x,y;a,b,c)\frac{t^n}{(n-m)!}$$

Now equating the coefficients of the like powers of t in the above equation, we get the result (2.7). Again by Definition (2.1) of generalized poly-Euler polynomials, we have

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}c^{(x+z)t+yt^2}=\sum_{n=0}^{\infty}{}_HE_n^{(k)}(x+z,y;a,b,c)\frac{t^n}{n!} \tag{2.9}$$

which can be written as

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}c^{zt}c^{xt+yt^2}=\sum_{n=0}^{\infty}E_n^{(k)}(z;a,b,c)\frac{t^n}{n!}\sum_{m=0}^{\infty}H_m(x,y,c)\frac{t^m}{m!} \tag{2.10}$$

Replacing n by n-m in (2.10), comparing with (2.9) and equating their coefficients of t^n leads to formula (2.8). □

Theorem 2.2. The generalized multi Hermite poly-Euler polynomials satisfy the following relation

$$ {}_HE_n^{(k_1,\ldots,k_r)}(x+y,z;a,b,c)=\sum_{m=0}^{n}\binom{n}{m}{}_HE_{n-m}^{(k_1,\ldots,k_r)}(x,z;a,b,c)y^m(r\ln c)^m \tag{2.11}$$

Proof. Using Definition (2.3)

$$\sum_{n=0}^{\infty}{}_HE_n^{(k_1,\ldots,k_r)}(x+y,z;a,b,c)\frac{t^n}{n!}=\frac{2Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}c^{r(x+y)t+rzt^2}$$

$$=\sum_{n=0}^{\infty}{}_HE_n^{(k_1,\ldots,k_r)}(x,z;a,b,c)\frac{t^n}{n!}\sum_{m=0}^{\infty}y^m(r\ln c)^m\frac{t^m}{m!}$$

Replacing n by n-m in above equation and equating their coefficients of t^n leads to formula (2.11).
□

Theorem 2.3. The generalized multi Hermite poly-Euler polynomials satisfy the following relation

$$ {}_HE_n^{(k_1,\ldots,k_r)}(x,y;a,b,c)=\sum_{m=0}^{[\frac{n}{2}]}\binom{n}{2m}E_{n-2m}^{(k_1,\ldots,k_r)}(x;a,b,c)y^m(r\ln c)^m \tag{2.12}$$

Proof. By the definitions of multi Hermite poly-Euler polynomials, we have

$$\sum_{n=0}^{\infty}{}_HE_n^{(k_1,\ldots,k_r)}(x,y;a,b,c)\frac{t^n}{n!}=\frac{2Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}c^{r(x+yt^2)}$$

$$=\sum_{n=0}^{\infty}E_n^{(k_1,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!}\sum_{m=0}^{\infty}y^m(r\ln c)^m\frac{t^{2m}}{m!}$$

Replacing n by n-2m in above equation and equating their coefficients of t^n leads to formula (2.12).
□

3 Implicit Summation Formulae Involving Generalized Hermite poly-Euler Polynomials

For the derivation of implicit formulae involving generalized Hermite poly-Euler polynomials ${}_HE_n^{(k)}(x,y;a,b,c)$ and generalized Hermite poly-Euler polynomials ${}_HE_n^{(k)}(x,y;a,b,e)$ the same considerations as developed for the ordinary Hermite and related polynomials in Khan et al [8] and Hermite-Bernoulli polynomials in Pathan [12] and Pathan et al [13 to 18] holds as well. First we prove the following results involving generalized Hermite-poly-Euler polynomials ${}_HE_n^{(k)}(x,y;a,b,c)$.

Theorem 3.1. Let $a,b,c>0$ and $a\neq b$. Then for $x,y\epsilon R$ and $n\geq 0$, The following implicit summation formulae for generalized Hermite poly-Euler polynomials ${}_HE_n^{(k)}(x,y;a,b,c)$ holds true:

$$ {}_HE_{m+l}^{(k)}(z,y;a,b,c) = \sum_{n,p=0}^{m,l} \binom{l}{p}\binom{m}{n}(z-x)^{n+p}\,{}_HE_{m+l-n-p}^{(k)}(x,y;a,b,c) \tag{3.1}$$

Proof. We replace t by $t+u$ and rewrite the generating function (2.1) as

$$\frac{2Li_k(1-(ab)^{-(t+u)})}{a^{-(t+u)}+b^{(t+u)}}c^{y(t+u)^2} = c^{-x(t+u)}\sum_{k,l=0}^{\infty}{}_HE_{m+l}^{(k)}(x,y;a,b,c)\frac{t^m}{m!}\frac{u^l}{l!} \tag{3.2}$$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$c^{(z-x)(t+u)}\sum_{m,l=0}^{\infty}{}_HE_{m+l}^{(k)}(x,y;a,b,c)\frac{t^m}{m!}\frac{u^l}{l!} = \sum_{m,l=0}^{\infty}{}_HE_{m+l}^{(k)}(z,y;a,b,c)\frac{t^m}{m!}\frac{u^l}{l!} \tag{3.3}$$

On expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty}\frac{[(z-x)(t+u)]^N}{N!}\sum_{m,l=0}^{\infty}{}_HE_{m+l}^{(k)}(x,y;a,b,c)\frac{t^m}{m!}\frac{u^l}{l!} = \sum_{m,l=0}^{\infty}{}_HE_{m+l}^{(k)}(z,y;a,b,c)\frac{t^m}{m!}\frac{u^l}{l!} \tag{3.4}$$

which on using formula [19,p.52(2)]

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty}f(n+m)\frac{x^n}{n!}\frac{y^m}{m!} \tag{3.5}$$

in the left hand side becomes

$$\sum_{n,p=0}^{\infty}\frac{(z-x)^{n+p}t^nu^p}{n!p!}\sum_{m,l=0}^{\infty}{}_HE_{m+l}^{(k)}(x,y;a,b,c)\frac{t^m}{m!}\frac{u^l}{l!} = \sum_{m,l=0}^{\infty}{}_HE_{m+l}^{(k)}(z,y;a,b,c)\frac{t^m}{m!}\frac{u^l}{l!} \tag{3.6}$$

Now replacing m by m-n, l by l-p and using the lemma [19,p.100(1)] in the left hand side of (3.6), we get

$$\sum_{n,p=0}^{\infty}\sum_{m,l=0}^{\infty}\frac{(z-x)^{n+p}}{n!p!}{}_HE_{m+l-n-p}^{(k)}(x,y;a,b,c)\frac{t^m}{(m-n)!}\frac{u^l}{(l-p)!} = \sum_{m,l=0}^{\infty}{}_HE_{m+l}^{(k)}(z,y;a,b,c)\frac{t^m}{m!}\frac{u^l}{l!} \tag{3.7}$$

Finally on equating the coefficients of the like powers of t and u in the above equation, we get the required result.

□

Remark 1. By taking $l = 0$ in equation (3.1), we immediately deduce the following result.

Corollary 3.1. The following implicit summation formula for Hermite poly-Euler polynomials ${}_HE_n^{(k)}(z,y;a,b,c)$ holds true:

$${}_HE_k^{(k)}(z,y;a,b,c) = \sum_{n=0}^{m}\binom{m}{n}(z-x)^n\,{}_HE_{m-n}^{(k)}(x,y,a,b,c) \tag{3.8}$$

Remark 2. On replacing z by z+x and setting $y = 0$ in Theorem (3.1), we get the following result involving generalized poly-Euler polynomials of one variable

$$E^{(k)}_{m+l}(z + x; a, b, c) = \sum_{n,p=0}^{m,l} \binom{l}{p} \binom{m}{n} (z)^{n+p} E^{(k)}_{m+l-p-n}(x; a, b, c) \tag{3.9}$$

whereas by setting z=0 in Theorem 3.1, we get another result involving generalized poly-Euler polynomials of one and two variables

$$E^{(k)}_{m+l}(y; a, b, c) = \sum_{n,p=0}^{m,l} \binom{m}{n} \binom{l}{p} (-x)^{n+p} {}_H E^{(k)}_{m+l-p-n}(x, y; a, b, c) \tag{3.10}$$

Remark 3. Along with the above results we will exploit extended forms of generalized poly-Euler polynomials $E^{(k)}_{m+l}(z; a, b, c)$ by setting y=0 in the Theorem (3.1) to get

$$E^{(k)}_{m+l}(z; a, b, c) = \sum_{n,p=0}^{m,l} \binom{l}{p} \binom{m}{n} (z - x)^{n+p} E^{(k)}_{m+l-p-n}(x; a, b, c) \tag{3.11}$$

Theorem 3.2. Let $a, b, c > 0$ and $a \neq b$. Then for $x \epsilon R$ and $n \geq 0$. Then

$$E^{(k)}_n(x + 1; a, b, c) = E^{(k)}_n(x; ac, \frac{b}{c}, c) \tag{3.12}$$

Proof. We start with the definition

$$\sum_{n=0}^{\infty} E^{(k)}_n(x + 1; a, b, c)\frac{t^n}{n!} = \frac{2Li_k(1 - (ab)^{-t})}{a^{-t} + b^t} c^{(x+1)t} = \frac{2Li_k(1 - (ab)^{-t})}{a^{-t} + b^t} c^{xt} c^t$$

$$\sum_{n=0}^{\infty} E^{(k)}_n(x + 1; a, b, c)\frac{t^n}{n!} = \frac{Li_k(1 - (ab)^{-t})}{(ac)^{-t} + (\frac{b}{c})^t} c^{xt} = \sum_{n=0}^{\infty} E^{(k)}_n(x; ac, \frac{b}{c}, c)\frac{t^n}{n!} \tag{3.13}$$

Equating the coefficients of t^n leads to formula (3.12). □

Theorem 3.3 Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \epsilon R$ and $n \geq 0$. Then

$${}_H E^{(k)}_n(x + 1, y; a, b, c) = \sum_{j=0}^{[\frac{n}{2}]} \binom{n}{2j} y^j (\ln c)^j E^{(k)}_{n-2j}(x; ac, \frac{b}{c}, c) \tag{3.14}$$

Proof. Since

$$\sum_{n=0}^{\infty} {}_H E^{(k)}_n(x + 1, y; a, b, c)\frac{t^n}{n!} = \frac{2Li_k(1 - (ab)^{-t})}{a^{-t} + b^t} c^{(x+1)t+yt^2} = \frac{2Li_k(1 - (ab)^{-t})}{(ac)^{-t} + (\frac{b}{c})^t} c^{xt} c^{yt^2}$$

$$= \left(\sum_{n=0}^{\infty} E^{(k)}_n(x; ac, \frac{b}{c}, c)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty} y^j (\ln c)^j \frac{t^2 j}{j!}\right)$$

Now replacing n by n-2j and comparing the coefficients of t^n, we get the result (3.14). □

Theorem 3.4. Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \epsilon R$ and $n \geq 0$. Then

$${}_H E^{(k)}_n(x, y; a, b, c) = \sum_{m=0}^{n} \binom{n}{m} E^{(k)}_{n-m}(a, b) H_m(x, y, c) \tag{3.15}$$

Proof. By the definition of generalized poly-Euler polynomials and the definition (1.1), we have

$$\frac{2Li_k(1 - (ab)^{-t})}{a^{-t} + b^t} c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H E^{(k)}_n(x, y; a, b, c)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} E^{(k)}_n(a, b)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} H_m(x, y; c)\frac{t^m}{m!}\right)$$

Now replacing n by n-m and comparing the coefficients of t^n, we get the result (3.15).
□

Remark. For $c = e$, (3.15) yields

$$_HE_n^{(k)}(x,y;a,b,e) = \sum_{m=0}^{n} \binom{n}{m} E_{n-m}^{k}(a,b)H_m(x,y)$$

Theorem 3.5 Let $a,b,c > 0$ and $a \neq b$. Then for $x, y\epsilon R$ and $n \geq 0$. Then

$$_HE_n^{(k)}(x,y;a,b,c) = \sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]} y^j x^{n-m-2j}(\ln c)^{n-m-j}E_m^{(k)}(a,b)\frac{n!}{m!j!(n-2j-m)!} \tag{3.16}$$

Proof. Applying the definition (2.1) to the term $\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}$ and expanding the exponential function c^{xt+yt^2} at $t = 0$ yields

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}c^{xt+yt^2} = \left(\sum_{m=0}^{\infty} E_m^{(k)}(a,b)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} x^n(\ln c)^n\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty} y^j(\ln c)^j\frac{t^{2j}}{j!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(\ln c)^{n-m}E_m^{(k)}(a,b)x^{n-m}\right)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty} y^j(\ln c)^j\frac{t^{2j}}{j!}\right)$$

Replacing n by n-2j, we have

$$\sum_{n=0}^{\infty} {}_HE_n^{(k)}(x,y;a,b)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}\binom{n-2j}{m}(\ln c)^{n-m-j}E_m^{(k)}(a,b)x^{n-m-2j}y^j\right)\frac{t^n}{(n-2j)!j!} \tag{3.17}$$

Combining (3.17) and (2.1) and equating their coefficients of t^n produce the formula (3.16).
□

Theorem 3.6. Let $a,b,c > 0$ and $a \neq b$. Then for $x, y\epsilon R$ and $n \geq 0$. Then

$$_HE_n^{(k)}(x+1,y;a,b,c) = \sum_{j=0}^{[\frac{n}{2}]}\sum_{m=0}^{n-2j}\binom{n-2j}{m} y^j(\ln c)^{n-m-j}E_m^{(k)}(x;a,b,c) \tag{3.18}$$

Proof. By the definition of generalized poly-Euler polynomials, we have

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}c^{(x+1)t+yt^2} = \sum_{n=0}^{\infty} {}_HE_n^{(k)}(x+1,y;a,b,c)\frac{t^n}{n!} \tag{3.19}$$

$$= \left(\sum_{m=0}^{\infty} E_m^{(k)}(x;a,b,c)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}(\ln c)^n\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty} y^j(\ln c)^j\frac{t^{2j}}{j!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(\ln c)^{n-m}E_m^{(k)}(x;a,b,c)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty} y^j(\ln c)^j\frac{t^{2j}}{j!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} y^j(\ln c)^{n-m+j}E_m^{(k)}(x;a,b,c)\frac{t^{n+2j}}{n!j!}$$

Replacing n by n-2j, we have

$$\sum_{n=0}^{\infty} {}_HE_n^{(k)}(x+1,y;a,b,c)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{[\frac{n}{2}]}\sum_{m=0}^{n-2j}\binom{n-2j}{m} y^j(\ln c)^{n-m-j}E_m^{(k)}(x;a,b,c)\right)\frac{t^n}{n!} \tag{3.20}$$

Combining (3.19) and (3.20) and equating their coefficients of t^n leads to formula (3.18).

□

Theorem 3.7. Let $a, b, c > 0$ and $a \neq b$. Then for $x, y \epsilon R$ and $n \geq 0$. The following implicit summation formula involving generalized Hermite poly-Euler polynomials ${}_HE_n^{(k)}(x, y; a, b, c)$ holds true:

$$ {}_HE_n^{(k)}(x+1, y; a, b, c) = \sum_{m=0}^{n} \binom{n}{m} (\ln c)^{n-m} {}_HE_m^{(k)}(x, y; a, b, c) \tag{3.21} $$

Proof. By the definition of generalized Hermite poly-Euler polynomials, we have

$$ \sum_{n=0}^{\infty} {}_HE_n^{(k)}(x+1, y; a, b, c)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HE_n^{(k)}(x, y; a, b, c)\frac{t^n}{n!} $$

$$ = \frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t} c^{xt+yt^2}(c^t - 1) $$

$$ = \left(\sum_{m=0}^{\infty} {}_HE_m^{(k)}(x, y; a, b, c)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} (\ln c)^n \frac{t^n}{n!}\right) - \sum_{n=0}^{\infty} {}_HE_n^{(k)}(x, y; a, b, c)\frac{t^n}{n!} $$

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{n} (\ln c)^{n-m} {}_HE_m^{(k)}(x, y; a, b, c)\frac{t^n}{(n-m)!} - \sum_{n=0}^{\infty} {}_HE_n^{(k)}(x, y; a, b, c)\frac{t^n}{n!} $$

Finally, equating the coefficients of the like powers of t^n, we get (3.21).

□

4 Symmetry Identities for the poly-Euler Polynomials

In this section, we give general symmetry identities for the generalized poly-Euler polynomials ${}_HE_n^{(k)}(x, y; a, b, c)$ and $E_n^{(k)}(x; a, b)$ by applying the generating function (1.1) and (2.1). The results extend some known identities of Zhang and Yang [21], Yang [20,Eqs.(9)], Khan [9, 10], Pathan [12] and Pathan et al [13 to 18].

Theorem 4.1. Let $a, b, c > 0$ and $a \neq b$. For $x, y \epsilon R$ and $n \geq 0$. Then the following identity holds true:

$$ \sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} {}_HE_{n-m}^{(k)}(bx, b^2y; b, c) {}_HE_m^{(k)}(ax, a^2y; a, c) $$

$$ = \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} {}_HE_{n-m}^{(k)}(ax, a^2y; a, c) {}_HE_m^{(k)}(bx, b^2y; b, c) \tag{4.1} $$

Proof. Start with

$$ g(t) = \left(\frac{(2Li_k(1-(ab)^{-t}))^2}{(a^{-at}+b^{at})(a^{-bt}+b^{bt})}\right) c^{abxt+a^2b^2yt^2} \tag{4.2} $$

Then the expression for g(t) is symmetric in a and b and we can expand g(t) into series in two ways to obtain

$$ g(t) = \sum_{n=0}^{\infty} {}_HE_n^{(k)}(bx, b^2y; b, c)\frac{(at)^n}{n!} \sum_{m=0}^{\infty} {}_HE_m^{(k)}(ax, a^2y; a, c)\frac{(bt)^m}{m!} $$

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_HE_{n-m}^{(k)}(bx, b^2y; b, c)\frac{a^{n-m}}{(n-m)!} {}_HE_m^{(k)}(ax, a^2y; a, c)\frac{b^m}{m!} t^n $$

On the similar lines we can show that

$$ g(t) = \sum_{n=0}^{\infty} {}_HE_n^{(k)}(ax, a^2y; a, c)\frac{(bt)^n}{n!} \sum_{m=0}^{\infty} {}_HE_m^{(k)}(bx, b^2y; b, c)\frac{(at)^m}{m!} $$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_HE_{n-m}^{(k)}(ax, a^2y; a, c)\frac{b^{n-m}}{(n-m)!}{}_HE_m^{(k)}(bx, b^2y; b, c)\frac{a^m}{m!}t^n$$

by comparing the coefficients of t^n on the right hand sides of the last two equations we arrive the desired result.
□

Remark 1. For $c = e$ in Theorem 4.1, we immediately deduce the following result involving generalized Hermite-poly-Euler polynomials ${}_HE_n^{(k)}(x, y; a, b, e)$ for nonnegative integer n

$$\sum_{m=0}^{n}\binom{n}{m} b^m a^{n-m}{}_HE_{n-m}^{(k)}(bx, b^2y; b, e){}_HE_m^{(k)}(ax, a^2y; a, e)$$

$$= \sum_{m=0}^{n}\binom{n}{m} a^m b^{n-m}{}_HE_{n-m}^{(k)}(ax, a^2y; a, e){}_HE_m^{(k)}(bx, b^2y; b, e) \tag{4.3}$$

Remark 2. By setting $b = 1$ in Theorem 4.1, we immediately following result

$$\sum_{m=0}^{n}\binom{n}{m} a^{n-m}{}_HE_{n-m}^{(k)}(x, y; 1, c){}_HE_m^{(k)}(ax, a^2y; a, c)$$

$$= \sum_{m=0}^{n}\binom{n}{m} a^m{}_HE_{n-m}^{(k)}(ax, a^2y; a, c){}_HE_m^{(k)}(x, y; 1, c) \tag{4.4}$$

Theorem 4.2. Let $a, b, c > 0$ and $a \neq b$. For $x, y\epsilon R$ and $n \geq 0$. Then the following identity holds true:

$$\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1} b^m a^{n-m}{}_HE_{n-m}^{(k)}\left(bx + \frac{b}{a}i + j, b^2z; A, B, c\right)E_m^{(k)}(ay; A, B, c)$$

$$= \sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1} a^m b^{n-m}{}_HE_{n-m}^{(k)}\left(ax + \frac{a}{b}i + j, a^2z; A, B, c\right)E_m^{(k)}(by; A, B, c) \tag{4.5}$$

Proof. Let

$$g(t) = \left(\frac{(2Li_k(1-(ab)^{-t}))^2}{(A^{-at}+B^{at})(A^{-bt}+B^{bt})}\right)\frac{(c^{abt}-1)^2 c^{ab(x+y)t+a^2b^2zt^2}}{(c^{at}-1)(c^{bt}-1)}$$

$$g(t) = \left(\frac{2Li_k(1-(ab)^{-t})}{(A^{-at}+B^{at}}\right)c^{abxt+a^2b^2zt^2}\left(\frac{c^{abt}-1}{c^{bt}-1}\right)\left(\frac{2Li_k(1-(ab)^{-t})}{A^{-bt}+B^{bt}}\right)c^{abyt}\left(\frac{c^{abt}-1}{c^{at}-1}\right)$$

$$= \left(\frac{2Li_k(1-(ab)^{-t})}{(A^{-at}+B^{at}}\right)c^{abxt+a^2b^2zt^2}\sum_{i=0}^{a-1}c^{bti}\left(\frac{2Li_k(1-(ab)^{-t})}{A^{-bt}+B^{bt}}\right)c^{abyt}\sum_{j=0}^{b-1}c^{atj} \tag{4.6}$$

$$= \left(\frac{2Li_k(1-(ab)^{-t})}{A^{-at}+B^{at}}\right)c^{a^2b^2zt^2}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}c^{(bx+\frac{b}{a}i+j)at}\sum_{m=0}^{\infty}E_m^{(k)}(ay; A, B, c)\frac{(bt)^m}{m!}$$

$$= \sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}{}_HE_n^{(k)}\left(bx + \frac{b}{a}i + j, b^2z; A, B, c\right)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}E_m^{(k)}(ay; A, B, c)\frac{(bt)^m}{(m)!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}{}_HE_{n-m}^{(k)}\left(bx + \frac{b}{a}i + j, b^2z; A, B, c\right)E_m^{(k)}(ay; A, B, c)b^m a^{n-m}t^n \tag{4.7}$$

On the other hand

$$g(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}{}_HE_{n-m}^{(k)}\left(ax+\frac{a}{b}i+j,a^2z;A,B,c\right)E_m^{(k)}(by;A,B,c)a^mb^{n-m}t^n \tag{4.8}$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.
□

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Received: May 11, 2015.

Accepted: November 21, 2015.