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# COEFFICIENT INEQUALITIES FOR TRANSFORMS OF ANALYTIC FUNCTIONS THROUGH GENERALIZED DIFFERENTIAL OPERATOR

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**Abstract**. We define two subclasses of analytic function viz $G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$ and $G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta,g}(\phi)$ and obtain the sharp upper bounds for the coefficient functional $|b_{2k+1} - \mu b^2_{k+1}|$ corresponding to the $k^{th}$ root transformation for the function $f$ in these classes. We also study certain applications of our results for the functions defined through convolution and fractional derivatives. We obtain the Fekete- Szegö inequality for the inverse function and for $\frac{z}{f(z)}$. The results of this paper generalize and unify the work of earlier researchers in this direction..

## 1 Introduction

Let $\mathcal{A}$ be the class of all functions $f$ analytic in the open unit disk $\Delta = [z \in C :| z |< 1]$ normalized by $f(0) = 0$ and $f'(0) = 1$. Let $f$ be a function in the class $\mathcal{A}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n; \quad \forall\, z \in \Delta. \tag{1}$$

Let $\mathbb{S}$ be the subclass of $\mathcal{A}$, consisting of univalent functions. For a univalent function $f$ of the form (1), the $k^{th}$ root transformation is defined by

$$F(z) = [f(z^k)]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{nk+1} z^{nk+1} (k \in N). \tag{2}$$

Let $B_o$ be the family of analytic functions $w$ in $\Delta$ with $w(0) = 0$ and $|w(z)| \leq 1$. The functions in the class $B_o$ are called as Schwartz functions. If $f$ is analytic in $\Delta$, $g$ is analytic and univalent in $\Delta$ and $f(0) = g(0)$ with $f(\Delta) \subset g(\Delta)$, then we say that $f$ is subordinate to $g$ and we write it as $f \prec g$. If $f \prec g$ then there exists a Schwartz function $w(z)$ in $B_o$ such that$f(z) = g(w(z))$.

**Definition 1.1.** Let $\phi(z)$ be a univalent, analytic function with positive real part on $\Delta$ with $\phi(0) = 1$, $\phi'(0) > 0$ where $\phi(z)$ maps $\Delta$ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Such a function $\phi$ has a series expansion of the form
$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ......$ with $B_1 > 0$, $B_2 \geq 0$ and $B_n s$ are real.

For a function $f \in \mathbb{S}$, Ramadan and Darus [13] introduced the generalized differential operator $D^n_{\alpha,\beta,\lambda,\delta}(\phi)$ as

$$D^n_{\alpha,\beta,\lambda,\delta} f(z) = z + \sum_{k=2}^{\infty} [(\lambda - \delta)(\beta - \alpha)(k - 1) + 1]^n a_k z^k, \tag{3}$$

$(\alpha,\ \beta,\ \lambda,\ \delta\ \geq 0;\ \beta > \alpha;\ \lambda > \delta;\ n \in N_0 = N \cup \{0\})$.
For $n = 0, D^0_{\alpha,\beta,\lambda,\delta} f(z) = f(z)$.
**Remarks:**

(i) Taking $\alpha = 0$, then operator $D^n_{0,\beta,\lambda,\delta} = D^n_{\beta,\lambda,\delta}$, was introduced and studied by Darus and Ibrahim [6].

(ii) Taking $\alpha = \delta = 0$ and $\beta = 1$, then operator $D^n_{0,1,\lambda,0} = D^n_{\lambda}$, was introduced and studied by Al-Oboundi [1].

(iii) Taking $\alpha = \delta = 0$ and $\lambda = \beta = 1$, then operator $D^n_{0,1,1,0} = D^n$, was introduced and studied by Salagean [15].

**Definition 1.2.** Let $G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$ be the class consisting of functions $f \in \mathcal{A}$ satisfying the subordination

$$1 + \frac{1}{b}\{\mu[\frac{z(D^n_{\alpha,\beta,\lambda,\delta} f(z))''}{(D^n_{\alpha,\beta,\lambda,\delta} f(z))'}] + (1-\mu)[\frac{z(D^n_{\alpha,\beta,\lambda,\delta} f(z))'}{(D^n_{\alpha,\beta,\lambda,\delta} f(z))} - 1]\} \prec [\phi(z)]^{\gamma},$$

where $\alpha, \beta, \lambda, \delta \geq 0;\ \ 0 \leq \mu \leq 1$ and $0 < \gamma \leq 1$.

For specific values of $\alpha, \beta, \lambda, \delta, n, b, \mu, \gamma$ and $\phi(z)$, we obtain the following subclasses studied by various researchers

(i) $G^{n,b,0,1}_{\alpha,\beta,\lambda,\delta}(\phi) = G^{n,b}_{\alpha,\beta,\lambda,\delta}(\phi)$, this class was introduced and studied by Aouf et.al [2].

(ii) $G^{n,1,0,1}_{\alpha,\beta,\lambda,\delta}(\phi) = M^n_{\alpha,\ \beta,\ \lambda,\ \delta}(\phi)$, this class was introduced and studied by Ramadan and Darus [13].

(iii) $G^{n,b,0,1}_{0,1,1,0}(\phi) = \ H_{n,b}(\phi)$, this class was introduced and studied by Aouf et.al [4].

(iv) $G^{0,b,0,1}_{0,1,1,0}(\phi) = S^{\star}_b(\phi)$ and $G^{1,b,0,1}_{0,1,1,0}(\phi) = C_b(\phi)$, these classes were introduced and studied by Ravichandran et.al [14].

(v) $G^{n,b,0,1}_{0,1,1,0}(\frac{1+z}{1-z}) = S^n(b)$, this class was introduced and studied by Aouf et.al [2].

(vi) $G^{0,b,0,1}_{0,1,1,0}(\frac{1+z}{1-z}) = S(b)$, this class was introduced and studied by Nasr and Aouf [11, 12] and also improved by Aouf et.al [3].

(vii) $G^{1,b,0,1}_{0,1,1,0}(\frac{1+z}{1-z}) = C(b)$, this class was introduced and studied by Nasr and Aouf [11, 12] and also improved by Aouf et.al [4].

(viii) $G^{0,(1-\rho)\cos\eta e^{-i\eta},0,1}_{0,1,1,0}(\frac{1+z}{1-z}) = S^{\eta}(\rho);\ |\eta| < \frac{\pi}{2},\ 0 \leq \rho < 1$, this class was introduced by Libera [9] and also improved by Keogh and Merkes [8].

(ix) $G^{1,(1-\rho)\cos\eta e^{-i\eta},0,\ 1}_{0,1,1,0}(\frac{1+z}{1-z}) = C^{\eta}(\rho);\quad |\eta| < \frac{\pi}{2},\ 0 \leq \rho < 1$, this class was introduced by Chichra [5].

**Definition 1.3.** Let $G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta,g}(\phi)$ be the class of functions $f \in \mathcal{A}$ satisfying the subordination

$$1 + \frac{1}{b}\{\mu\{\frac{z(D^n_{\alpha,\beta,\lambda,\delta}(f*g)(z)''}{(D^n_{\alpha,\beta,\lambda,\delta}(f*g)(z)'}\} + (1-\mu)\{\frac{z(D^n_{\alpha,\beta,\lambda,\delta}(f*g)(z)'}{(D^n_{\alpha,\beta,\lambda,\delta} f(f*g)(z)}\} - 1\} \prec [\phi(z)]^{\gamma},$$

where $\alpha, \beta, \lambda, \delta\ \geq 0;\ 0 \leq \mu \leq 1$ and $0 < \gamma \leq 1$.

**Remarks:** If $g(z) = \frac{z}{(1-z)}$, then $G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta,g}(\phi) = G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$.

## 2 Preliminaries

To prove our result, we require the following two Lemmas

**Lemma 2.1.** *[14] If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + .......$ is an analytic function with positive real part in $\Delta$ then for any complex number $\nu$*

$$| c_2 - \nu c_1^2 | \leq 2 \max\{1, \ | 2\nu - 1 |\}.$$

*The result is sharp for the functions defined by $p(z) = \frac{1+z^2}{1-z^2}$ or $p(z) = \frac{1+z}{1-z}$.*

**Lemma 2.2.** *[10] If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + .......$ is an analytic function with positive real part in $\Delta$ then for any real number $\nu$, we have*

$$| c_2 - \nu c_1^2 | \leq \begin{cases} -4\nu + 2, & \text{if } \nu \leq 0; \\ 2, & \text{if } 0 \leq \nu \leq 1; \\ 4\nu - 2, & \text{if } \nu \geq 1, \end{cases}$$

*when $\nu < 0$ or $\nu > 0$ the equality holds if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$ then the equality holds if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$ then the equality holds if and only if $p(z) = [\frac{1+\lambda}{2}][\frac{1+z}{1-z}] + [\frac{1-\lambda}{2}][\frac{1+z}{1-z}](0 \leq \lambda \leq 1)$ or one of its rotations. If $\nu = 1$ the equality holds only for the reciprocal of $p(z)$ for the case $\nu = 0$. Also the above upper bound is sharp and it can be further improved as follows when $0 < \nu < 1$.*

$$| c_2 - \nu c_1^2 | + \nu | c_1 |^2 \leq 2 \ (0 \leq \nu \leq \frac{1}{2}),$$

$$| c_2 - \nu c_1^2 | + (1 - \nu) | c_1 |^2 \leq 2 \ (\frac{1}{2} \leq \nu \leq 1).$$

## 3 Main Results

We now derive our main results for the function $f$ in the class $G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$.

**Theorem 3.1.** *If $f \in G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$ and $F$ is the $k^{th}$ root transformation of $f$ given by (2) then for any complex number $\tau$,*

$$|b_{2k+1} - \tau b_{k+1}^2| \leq \frac{|b|\gamma \ B_1}{2k(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n} \times \max\{1, \ |\omega_1[(k-1)+2\tau] - \frac{B_2}{B_1} - \frac{(\gamma-1)}{2}B_1 - \frac{b\gamma B_1(3\mu+1)}{(\mu+1)^2}|\}, \tag{4}$$

*where*

$$\omega_1 = \frac{b \ \gamma \ B_1(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}{k[(\lambda-\delta)(\beta-\alpha)+1]^{2n}(\mu+1)^2}. \tag{5}$$

*The result is sharp.*

**Proof.** If $f \in G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$, then there exists a Schwartz function $w(z)$ in $B_0$ with $w(0) = 0$ and $|w(z)| \leq 1$ such that

$$1 + \frac{1}{b}\{\mu\{\frac{z(D^n_{\alpha,\beta,\lambda,\delta}f(z))''}{(D^n_{\alpha,\beta,\lambda,\delta}f(z))'}\} + (1-\mu)\{\frac{z(D^n_{\alpha,\beta,\lambda,\delta}f(z))'}{(D^n_{\alpha,\beta,\lambda,\delta}f(z))}\} - 1\} = [\phi(w(z))]^\gamma. \tag{6}$$

Consider

$$1 + \frac{1}{b}\{\mu\{\frac{z(D^n_{\alpha,\beta,\lambda,\delta}f(z))''}{(D^n_{\alpha,\beta,\lambda,\delta}f(z))'}\} + (1-\mu)\{\frac{z(D^n_{\alpha,\beta,\lambda,\delta}f(z))'}{(D^n_{\alpha,\beta,\lambda,\delta}f(z))}\} - 1\} = 1 + \frac{[(\lambda-\delta)(\beta-\alpha)+1]^n(\mu+1)}{b}a_2 z + \frac{1}{b}\{(4\mu+2)[2(\lambda-\delta)(\beta-\alpha)+1]^n a_3 - (3\mu+1)[(\lambda-\delta)(\beta-\alpha)+1]^{2n}a_2^2\}z^2 + ......... \tag{7}$$

Define a function $p(z)$ and by substituting $w(z)$ in $\phi(z)$ and by increasing the power to $\gamma,$ we have

$$[\phi(w(z))]^{\gamma}=1+\{\frac{\gamma B_1 w_1}{2}\}z+\{\frac{\gamma B_1}{2}[w_2-\frac{w_1^2}{2}]+[\frac{\gamma B_2 w_1^2}{4}]+[\frac{\gamma(\gamma-1)w_1^2}{8}B_1^2]\}z^2+..... \quad (8)$$

From equations (7), (8) and (9), we have

$$a_2=\frac{b\,\gamma\,B_1 w_1}{2[(\lambda-\delta)(\beta-\alpha)+1]^n(\mu+1)}, \quad (9)$$

$$a_3=\frac{b\,\gamma\,B_1}{4(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}\{w_2-\frac{w_1^2}{2}(1-\frac{B_2}{B_1}-\{\frac{(\gamma-1)}{2}\}B_1$$
$$-\frac{b\,\gamma\,B_1(3\mu+1)}{(\mu+1)^2})\}. \quad (10)$$

If $F$ is the $k^{th}$ root transformation of $f$ then

$$F(z)=\{f(z^k)\}^{\frac{1}{k}}=z+(\frac{a_2}{k})z^{k+1}+[\frac{a_3}{k}-\frac{(k-1)}{2k^2}a_2^2]z^{2k+1}+....=z+\sum_{n=1}^{\infty}b_{nk+1}z^{nk+1}. \quad (11)$$

Upon equating the coefficients of $z^{k+1},\ z^{2k+1}$ and from equations (9), (10) and (11), we have

$$b_{k+1}=\frac{b\gamma B_1 w_1}{2k[(\lambda-\delta)(\beta-\alpha)+1]^n(\mu+1)}, \quad (12)$$

$$b_{2k+1}=\frac{b\gamma B_1}{4k(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}\times\{w_2-\frac{w_1^2}{2}\{1-\frac{B_2}{B_1}-\{\frac{\gamma-1}{2}\}B_1$$
$$-\frac{b\gamma B_1(3\mu+1)}{(\mu+1)^2}+\omega_1(k-1)\}\}, \quad (13)$$

where $\omega_1$ is given by (5). For any complex number $\mu,$ we have

$$[b_{2k+1}-\tau b_{k+1}^2]=\frac{b\gamma B_1}{4k(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}\{w_2-tw_1^2\}, \quad (14)$$

where $t=\frac{1}{2}\{1-\frac{B_2}{B_1}-(\frac{\gamma-1}{2})B_1-\frac{b\gamma\,B_1(3\mu+1)}{(\mu+1)^2}+\omega_1[(k-1)+2\tau]\}$ .
Taking modulus on both sides of the equation (14) and applying Lemma 2.1 on right hand side we get the result as in (4). This proves the result of the Theorem 3.1 and the result is sharp for $p(z)=\{\frac{1+z^2}{1-z^2}\}^{\gamma}$ and $p(z)=\{\frac{1+z}{1-z}\}^{\gamma}$

**Remark 3.2.** (i) Taking $\mu=0,\ \gamma=k=1$ in Theorem 3.1, we improve the result obtained by Aouf et.al [3, Theorem 1];

(ii) Taking $n=\mu=0$ and $\gamma=k=1$ in Theorem 3.1, we improve the result obtained by Ravichandran et.al [13, Theorem 4.1];

(iii) Taking $\alpha=\delta=\mu=0,\ \beta=\lambda=k=\gamma=1,\ b=(1-\rho)\cos\eta e^{i\eta}\{|\eta|<\frac{\pi}{2}\leq\rho<1\}$ and $\phi(z)=\frac{1+z}{1-z}$ in Theorem 3.1, we obtain the result obtained by Goyal and Kumar [7, Corollary 2.10];

(iv) Taking $\alpha=\delta=\mu=0$ and $\beta=\lambda=\gamma=k=1$ in Theorem 3.1, we obtain the result obtained by Aouf et.al [4, Theorem 1].

For specific values of the parameter in Theorem 3.1, we obtain the following new sharp results.
Taking $b=1,\mu=0,\gamma=1$ & $k=1$ in Theorem 3.1 we obtain the following Corollary:

**Corollary 3.3.** *If $f(z)\in M^n_{\alpha,\beta,\lambda,\delta}(\phi)$, then for any complex number $\tau$, we have*

$$|b_3-\tau\,b_2^2|\leq\frac{B_1}{2[2(\lambda-\delta)(\beta-\alpha)+1]^n}\max\{1,|\frac{2[2(\lambda-\delta)(\beta-\alpha)+1]^n\tau B_1}{[(\lambda-\delta)(\beta-\alpha)+]^{2n}}-\frac{B_2}{B_1}-B_1|\}.$$

*The result is sharp.*

Taking $\phi(z) = \frac{1+Az}{1+Bz}(-1 \leq B < A \leq 1)$ or equivalently $B_1 = A - B$ and $B_2 = -B(A - B)$, $\mu = 0, \gamma = 1$ and $k = 1$ in Theorem 3.1, we obtain the following Corollary:

**Corollary 3.4.** *If $f(z) \in S^{n,b}_{\alpha,\beta,\lambda,\delta}(A,B)$, then for any complex number $\tau$ we have*

$$|b_3 - \tau\, b_2^2| \leq \frac{|b|(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^n}\max\{1, |b(A-B)\{\frac{2[2(\lambda-\delta)(\beta-\alpha)+1]^n\tau}{[(\lambda-\delta)(\beta-\alpha)+1]^{2n}} - 1\} - B|\}.$$

*The result is sharp.*

Let $b = 1$ and taking $\tau$ to be real we now obtain the coefficient inequality for the function $f \in G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$

**Theorem 3.5.** *If $f \in G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$ and $F$ is the $k^{th}$ root transformation of $f$ given by (2) then for any complex number $\tau$ and*

$$\sigma_1 = \frac{1}{2\omega_2}\{-1 + \frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 + \frac{\gamma\, B_1(3\mu+1)}{(\mu+1)^2} - \omega_2(k-1)\},$$
$$\sigma_2 = \frac{1}{2\omega_2}\{1 + \frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 + \frac{\gamma\, B_1(3\mu+1)}{(\mu+1)^2} - \omega_2(k-1)\},$$
$$\sigma_3 = \frac{1}{2\omega_2}\{\frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 + \frac{\gamma\, B_1(3\mu+1)}{(\mu+1)^2} - \omega_2(k-1)\},$$

*where*

$$\omega_2 = \frac{\gamma\, B_1(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}{k[(\lambda-\delta)(\beta-\alpha)+1]^{2n}(\mu+1)^2}. \tag{15}$$

*We have*

$$| b_{2k+1} - \tau b_{k+1}^2 | \leq \begin{cases} \frac{\gamma\, B_1}{2k(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}\{\frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 + \frac{\gamma\, B_1(3\mu+1)}{(\mu+1)^2} & \\ -\omega_2[(k-1)+2\tau]\}, & \text{if } \tau \leq \sigma_1; \\ \frac{\gamma\, B_1}{2k(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{\gamma\, B_1}{2k(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}\{\omega_2[(k-1)+2\tau] - \frac{B_2}{B_1} & \\ -(\frac{\gamma-1}{2})B_1 - \frac{\gamma\, B_1(3\mu+1)}{(\mu+1)^2}\}, & \text{if } \tau \geq \sigma_2, \end{cases} \tag{16}$$

*and the result is sharp.*
*Furthermore if $\sigma_1 \leq \tau \leq \sigma_3$, then*

$$| b_{2k+1} - \tau b_{k+1}^2 | + \frac{k[(\lambda-\delta)(\beta-\alpha)+1]^{2n}(\mu+1)^2}{2\gamma\, B_1(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}\{1 - \frac{B_2}{B_1} - (\frac{\gamma-1}{2})B_1$$
$$- \frac{\gamma\, B_1(3\mu+1)}{(\mu+1)^2} + \omega_2[(k-1)+2\tau]\}|b_{k+1}^2| \leq \frac{\gamma\, B_1}{4k(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}, \tag{17}$$

*and if $\sigma_3 \leq \tau \leq \sigma_2$, then*

$$| b_{2k+1} - \tau b_{k+1}^2 | + \frac{k[(\lambda-\delta)(\beta-\alpha)+1]^{2n}(\mu+1)^2}{2\gamma\, B_1(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}\{1 + \frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1$$
$$+ \frac{\gamma\, B_1(3\mu+1)}{(\mu+1)^2} - \omega_2[(k-1)+2\tau]\}|b_{k+1}^2| \leq \frac{\gamma\, B_1}{4k(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}. \tag{18}$$

**Proof.** Since $f \in G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$ then for $b = 1$ in equations (12), (13) and (14) and for any real number $\tau$, we have

$$[b_{2k+1} - \tau b_{k+1}^2] = \frac{\gamma B_1}{4k(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}\{w_2 - t w_1^2\}, \tag{19}$$

where $t = \frac{1}{2}\{1 - \frac{B_2}{B_1} - (\frac{\gamma-1}{2})B_1 - \frac{\gamma\ B_1(3\mu+1)}{(\mu+1)^2} + \omega_2\{(k-1)+2\tau\}\}$ and $\omega_2$, is defined as in (15).
By applying Lemma 2.2 on the right hand side of (19), we have the following cases
**Case(i)**: If $\tau \leq \sigma_1$, then

$$\tau \leq \frac{1}{2\omega_2}\{-1 + \frac{B_2}{B_1} + \{\frac{\gamma-1}{2}\}B_1 + \frac{\gamma\ B_1(3\mu+1)}{(\mu+1)^2} - \omega_2(k-1)\}.$$

Upon simplifying we get

$$\Rightarrow |w_2 - tw_1^2| \leq \{\frac{2B_2}{B_1} + (\gamma-1)B_1 + \frac{2\gamma\ B_1(3\mu+1)}{(\mu+1)^2} - 2\omega_2[(k-1)+2\tau]\}. \tag{20}$$

**Case(ii)**: If $\sigma_1 \leq \tau \leq \sigma_2$, then

$$\frac{1}{2\omega_2}\{-1 + \frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 + \frac{\gamma\ B_1(3\mu+1)}{(\mu+1)^2} - \omega_2(k-1)\} \leq \tau$$
$$\leq \frac{1}{2\omega_2}\{1 + \frac{B_2}{B_1} + \{\frac{\gamma-1}{2}\}B_1 + \frac{\gamma B_1(3\mu+1)}{(\mu+1)^2} - \omega_2(k-1)\}.$$

Upon simplifying we get

$$\Rightarrow |w_2 - tw_1^2| \leq 2. \tag{21}$$

**Case(iii)**: If $\tau \geq \sigma_2$, then

$$\tau \geq \frac{1}{2\omega_2}\{1 + \frac{B_2}{B_1} + \{\frac{\gamma-1}{2}\}B_1 + \frac{\gamma\ B_1(3\mu+1)}{(\mu+1)^2} - \omega_2(k-1)\}.$$

Upon simplifying we get

$$\Rightarrow |w_2 - tw_1^2| \leq \{2\omega_2[(k-1)+2\tau] - \frac{2B_2}{B_1} - (\gamma-1)B_1 - \frac{2\gamma B_1(3\mu+1)}{(\mu+1)^2}\}. \tag{22}$$

From equations (19), (20), (21) and (22), we get the result as in (16).
**Case(iv)**: Furthermore, if $\sigma_1 \leq \tau \leq \sigma_3$, then

$$\frac{1}{2\omega_2}\{-1 + \frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 + \frac{\gamma\ B_1(3\mu+1)}{(\mu+1)^2} - \omega_2(k-1)\} \leq \tau$$
$$\leq \frac{1}{2\omega_2}\{\frac{B_2}{B_1} + \{\frac{\gamma-1}{2}\}B_1 + \frac{\gamma\ B_1(3\mu+1)}{(\mu+1)^2} - \omega_2(k-1)\}.$$

Upon simplifying we get

$$\Rightarrow |w_2 - tw_1^2| + t\ |\ w_1\ |^2 \leq 2. \tag{23}$$

From equations (19) and (23), we get the result as in (17).
**Case(v)**: Furthermore, if $\sigma_3 \leq \tau \leq \sigma_2$, then

$$\frac{1}{2\omega_2}\{\frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 + \frac{\gamma\ B_1(3\mu+1)}{(\mu+1)^2} - \omega_2(k-1)\} \leq \tau$$
$$\leq \frac{1}{2\omega_2}\{1 + \frac{B_2}{B_1} + \{\frac{\gamma-1}{2}\}B_1 + \frac{\gamma\ B_1(3\mu+1)}{(\mu+1)^2} - \omega_2(k-1)\}.$$

Upon simplifying we get

$$\Rightarrow |w_2 - tw_1^2| + (1-t)\ |\ w_1\ |^2 \leq 2. \tag{24}$$

From equations (19) and (24), we get the result as in (18). This completes the proof of the Theorem and the sharpness of the result is verified from Lemma 2.2.

**Remark 3.6.** (i) Taking $\mu = 0$ and $\gamma = k = 1$ in Theorem 3.4 we improve the result obtained by Aouf et.al [3, Theorem 2 and Theorem 3];

(ii) Taking $\mu = 0$ and $\gamma = k = 1$ in Theorem 3.4 we improve the result obtained by Ramadan and Darus [13, Theorem 1];

(iii) Taking $\alpha = \delta = \mu = 0$ and $\beta = \lambda = k = \gamma = 1$ in Theorem 3.4, we obtain the result obtained by Goyal and Kumar [7, Corollary 2.7] and Aouf et.al [4, Theorem 2].

## 4 Coefficient inequalities for the functions defined through convolution

**Theorem 4.1.** *If $f \in G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta,g}(\phi)$ and $F$ is the $k^{th}$ root transformation of $f$ given by (2)then for any complex number $\tau$,*

$$|b_{2k+1} - \tau b_{k+1}^2| \leq \frac{|b|\gamma B_1}{2k(2\mu+1)g_3[2(\lambda-\delta)(\beta-\alpha)+1]^n} \times max\{1, |\omega_3[(k-1)+2\tau] - \frac{B_2}{B_1} - [\frac{\gamma-1}{2}]B_1 - \frac{b\gamma B_1(3\mu+1)}{(\mu+1)^2}\}, \quad (25)$$

*where*

$$\omega_3 = \frac{b\gamma B_1(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n g_3}{k[(\lambda-\delta)(\beta-\alpha)+1]^{2n} g_2^2}. \quad (26)$$

*The result is sharp.*

Proceeding in a way similar to Theorem 3.1 for the function $(f*g)$ one can obtain this result. Let $b = 1$ and taking $\tau$ to be real we now obtain the coefficient inequality for the function $f \in G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta,g}(\phi)$.

**Theorem 4.2.** *If $f \in G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta,g}(\phi)$ and $F$ is the $k^{th}$ root transformation of $f$ given by (2), then for any real number $\tau$,*

$$\sigma_1 = \frac{1}{2\omega_4}\{-1 + \frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 + \frac{\gamma\, B_1(3\mu+1)}{(\mu+1)^2} - \omega_4(k-1)\},$$
$$\sigma_2 = \frac{1}{2\omega_4}\{1 + \frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 + \frac{\gamma\, B_1(3\mu+1)}{(\mu+1)^2} - \omega_4(k-1)\},$$
$$\sigma_3 = \frac{1}{2\omega_4}\{\frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 + \frac{\gamma\, B_1(3\mu+1)}{(\mu+1)^2} - \omega_4(k-1)\},$$

*where*

$$\omega_4 = \frac{\gamma\, B_1(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n g_3}{k[(\lambda-\delta)(\beta-\alpha)+1]^{2n}(\mu+1)^2 g_2^2}. \quad (27)$$

*We have*

$$| b_{2k+1} - \tau b_{k+1}^2 | \leq \begin{cases} \frac{\gamma\, B_1}{2kg_3(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}\{\frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1, & \\ + \frac{\gamma\, B_1(3\mu+1)}{(\mu+1)^2} - \omega_4[(k-1)+2\tau]\}, & if\, \tau \leq \sigma_1; \\ \frac{\gamma\, B_1}{2kg_3(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}, & if\, \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{\gamma\, B_1}{2kg_3(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}\{\omega_4[(k-1)+2\tau] - \frac{B_2}{B_1} & \\ -(\frac{\gamma-1}{2})B_1 - \frac{\gamma\, B_1(3\mu+1)}{(\mu+1)^2}\}, & if\, \tau \geq \sigma_2, \end{cases} \quad (28)$$

*and the result is sharp.*

Proceeding in a way similar to Theorem 3.1 for the function $(f*g)$ one can obtain this result.

## 5 Functions defined through fractional derivatives

We now obtain our result for the function in the class $G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$

For fixed $g \in A$, let $G^{n,b,\mu,\gamma,g}_{\alpha,\beta,\lambda,\delta}(\phi)$ be the class of functions $f \in A$ for which $(f*g) \in G^{n,b,\mu,\gamma,g}_{\alpha,\beta,\lambda,\delta}(\phi)$.

**Definition 5.1.** Let $f$ be analytic in a simply connected region of the $z - plane$ containing the origin. The fractional derivative of $f$ of order $\rho$ is defined by

$$D_z^\rho f(z) = \frac{1}{\Gamma(1-\rho)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^\rho}d\zeta (0 \leq \rho < 1), \quad (29)$$

where the multiplicity of $(z-\zeta)^\rho$ is removed by requiring that $log(z-\zeta)$ is real for $(z-\zeta) > 0$.

Using Definition 5.1 and its known extensions involving the fractional derivatives and fractional integral, Owa and Srivastava introduced the operator

$$(\Omega^{\rho} f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\rho)}{\Gamma(n-\rho+1)} a_n z^n. \tag{30}$$

This operator is known as the Owa-Srivastava operator. In terms of the Owa-operator ($\Omega^{\rho}$ defined by (30),we now introduce the class $G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta,g}(\phi)$ in the following way :
$G^{n,b,\mu,\gamma,\rho}_{\alpha,\beta,\lambda,\delta,g}(\phi)$ is a special case of the class $G^{n,b,\mu,\ \gamma}_{\alpha,\beta,\lambda,\delta,g}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\rho)}{\Gamma(n-\rho+1)} z^n. \tag{31}$$

For $g_n > 0$ we can obtain the coefficient estimates for function in the class $G^{n,b,\mu,\gamma,\rho}_{\alpha,\beta,\lambda,\delta,g}(\phi)$ from the corresponding estimates for functions in the class $G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta,g}(\phi)$ when $g$ corresponds to Owa-Operator given in (31), we obtain

$$g_2 = \frac{\Gamma(3)\Gamma(2-\rho)}{\Gamma(3-\rho)} = \frac{2}{(2-\rho)}, \tag{32}$$

$$g_3 = \frac{\Gamma(4)\Gamma(2-\rho)}{\Gamma(4-\rho)} = \frac{6}{(2-\rho)(3-\rho)}. \tag{33}$$

For $g_2$ and $g_3$ given by (32) and (33), respectively Theorems 4.1 and 4.2 reduces to the following results.

**Theorem 5.2.** *If $f \in G^{n,b,\mu,\gamma,\rho}_{\alpha,\beta,\lambda,\delta,g}(\phi)$; $\alpha \geq 0$ and $g_n > 0$ and $F$ is the $k^{th}$ root transformation of $f$ given by (2), then for any complex number $\tau$, we have*

$$|b_{2k+1} - \tau b_{k+1}^2| \leq \frac{|b|\gamma B_1(2-\rho)(3-\rho)}{12k(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}$$
$$\times \max\{1, |\omega_5[(k-1)+2\tau] - \frac{B_2}{B_1} - [\frac{\gamma-1}{2}]B_1 - \frac{b\gamma B_1(3\mu+1)}{(\mu+1)^2}\}, \tag{34}$$

*where*

$$\omega_5 = \frac{3b\gamma B_1(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n(2-\rho)}{k[(\lambda-\delta)(\beta-\alpha)+1]^{2n}(\mu+1)^2(3-\rho)}, \tag{35}$$

*and the result is sharp.*

**Theorem 5.3.** *If $f \in G^{n,b,\mu,\gamma,\rho}_{\alpha,\beta,\lambda,\delta,g}(\phi)$; $\alpha \geq 0$ and $g_n > 0$ and $F$ is the $k^{th}$ root transformation of $f$ given by (2), then for any real number $\tau$ and for*

$$\sigma_1 = \frac{1}{2\omega_6}\{-1 + \frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 + \frac{\gamma\ B_1(3\mu+1)}{(\mu+1)^2} - \omega_6(k-1)\},$$

$$\sigma_2 = \frac{1}{2\omega_6}\{1 + \frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 + \frac{\gamma\ B_1(3\mu+1)}{(\mu+1)^2} - \omega_6(k-1)\},$$

$$\sigma_3 = \frac{1}{2\omega_6}\{\frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 + \frac{\gamma\ B_1(3\mu+1)}{(\mu+1)^2} - \omega_6(k-1)\},$$

*where*

$$\omega_6 = \frac{3\gamma B_1(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n(2-\rho)}{2k[(\lambda-\delta)(\beta-\alpha)+1]^{2n}(\mu+1)^2(3-\rho)}. \tag{36}$$

*We have*

$$| b_{2k+1} - \tau b_{k+1}^2 | \leq \begin{cases} \frac{\gamma\ B_1(2-\rho)(3-\rho)}{12k(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}\{\frac{B_2}{B_1} + (\frac{\gamma-1}{2})B_1 \\ +\frac{\gamma\ B_1(3\mu+1)}{(\mu+1)^2} - \omega_6[(k-1)+2\tau]\}, & \text{if } \tau \leq \sigma_1; \\ \frac{\gamma\ B_1(2-\rho)(3-\rho)}{12k(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{\gamma\ B_1(2-\rho)(3-\rho)}{12k(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}\{\omega_6[(k-1)+2\tau] - \frac{B_2}{B_1} \\ -(\frac{\gamma-1}{2})B_1 - \frac{\gamma\ B_1(3\mu+1)}{(\mu+1)^2}\}, & \text{if } \tau \geq \sigma_2, \end{cases} \tag{37}$$

*and the result is sharp.*

## 6 Coefficient inequality for the function $\frac{z}{f(z)}$

**Theorem 6.1.** *If $f \in G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$ and $G(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} d_n z^n$ then for any complex number $\tau$, we have*

$$| d_2 - \tau d_1^2 | \leq \frac{|b|\gamma\, B_1\mu(\mu+1)}{4(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n} \times \max\{1,\ |\omega_7(1-\tau) - \frac{B_2}{B_1} - (\frac{\gamma-1}{2})B_1 - \frac{b\gamma B_1(3\mu+1)}{(\mu+1)^2}\}, \tag{38}$$

*where*

$$\omega_7 = \frac{2b\gamma B_1(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}{[(\lambda-\delta)(\beta-\alpha)+1]^{2n}(\mu+1)^2}, \tag{39}$$

*and the result is sharp.*

Proceeding in a way similar to Theorem 3.1 for the function $\frac{z}{f(z)}$ one can obtain this result.

**Theorem 6.2.** *If $f \in G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta,g}(\phi)$ and $G(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} d_n z^n$ then for any complex number $\tau$, we have*

$$| d_2 - \tau d_1^2 | \leq \frac{|b|\gamma\, B_1\mu(\mu+1)}{4g_3(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n} \times \max\{1, |\omega_8(1-\tau) - \frac{B_2}{B_1} - (\frac{\gamma-1}{2})B_1 - \frac{b\gamma B_1(3\mu+1)}{(\mu+1)^2}\}, \tag{40}$$

*where*

$$\omega_8 = \frac{2b\gamma B_1 g_3(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n}{[(\lambda-\delta)(\beta-\alpha)+1]^{2n} g_2^2(\mu+1)^2}, \tag{41}$$

*and the result is sharp.*

Proceeding in a way similar to Theorem 3.1 for the function $\frac{z}{f(z)}$ through convolution one can obtain this result.

## 7 Coefficient inequality for $f^{-1}$

**Theorem 7.1.** *If $f \in G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$ and $f^{-1}(w) = w + \sum_{n=1}^{\infty} q_n z^n$ is the inverse function of $f$ with $|w| < r_0$ where $r_0$ is greater than the radius of the Koebe domain of the class $f \in G^{n,b,\mu,\gamma}_{\alpha,\beta,\lambda,\delta}(\phi)$, then for any complex number $\tau$, we have*

$$| q_2 - \tau q_1^2 | \leq \frac{|b|\gamma\, B_1\mu(\mu+1)}{2(2\mu+1)[2(\lambda-\delta)(\beta-\alpha)+1]^n} \times \max\{1, |\omega_7(2-\tau) - \frac{B_2}{B_1} - (\frac{\gamma-1}{2})B_1 - \frac{b\gamma B_1(3\mu+1)}{(\mu+1)^2}\}, \tag{42}$$

*where $\omega_7$ is defined as in (39) and the result is sharp.*

Proceeding in a way similar to Theorem 3.1 for the inverse of the function $f$ one can obtain this result.

## References


[1] F.M.Al-Oboundi , On Univalent functions defined by a generalised Salagean Operator,*Int.J.Math.Sci.* **27**, 1429–1436 (2004).

[2] M.K.Aouf, R.M.El-Ashwah, A.M.Hassan and A.H.Hassan , Fekete-Szegöproblem for a new class of analytic functions defined by using a generalized differential operator, *Acta Univ. Palacki, Olonuc., Fac.rer.nat., Mathematica.***52**(1),21–34 (2013).

[3] M.K.Aouf, S.Owa and M.Obradovic , Certain classes of analytic functions of complex order and type beta,*Rend.Mat.Appl.* **7**(4),691–714 (1991).

[4] M.K.Aouf and H.Silverman , Fekete-Szegö inequality for n-starlike functions of complex order, *Advances in Mathematics* 1Ű12(2008).

[5] P.N.Chichra , Regular functions $f(z)$ for which $zf'(z)$ is $\alpha$ -spirallike functions, *Proc. American Math.Soc.* **49**(1), 151-160 (1975).

[6] M.Darus and R.W.Ibrahim , On subclasses for generalized operators of complex order, *Far East Jour.Math.Sci.* **33**(3), 299-308(2009).

[7] S.P.Goyal and R.Kumar , Fekete-Szegö problem for a class of complex order related to salagean operator , *Bull.Math.Anal.Appl.* **3**(4), 240-246 (2011).

[8] F.R.Keogh and E.P.Merkes , A coefficient inequality for certain classes of analytic functions, *Proc. American Math. Soc.* **20**(1),8-12 (1969).

[9] R.J.Libera and M.R.Ziegler , Regular function $f(z)$ for which $zf'(z)$ is $\alpha$-spiral, *Trans.Amer.Soc.* **166**, 361-370 (1972).

[10] W.C.Ma and D.Minda , A unified treatment of some special classes of univalent functions, *Proc. Conference on Complex Analysis Tianjin* Internat. Press, Cambridge, 157-169 (1992).

[11] M.A.Nasr and M.K.Aouf , Bounded starlike functions of complex order, *Proc.Indian Aca.Sci.* **92**, 97-102 (1983).

[12] M.A.Nasr and M.K.Aouf , Starlike function of complex order, *J.Natur.Sci.Math.* **25**, 1-12 (1985).

[13] S.F.Ramadan and M.Darus , On the Fekete-Szegö inequality for class of analytic functions defined by using generalized, *Act. Univ.Apulensis* **26**, 167-178 (2011).

[14] V.Ravichandran, Y.Polatoglu, M.Bolcal and A.Sen , Certain subclasses of starlike and convex functions of complex order, *Haccetepe Jour. Math. and Statis.***34**, 9-15 (2005).

[15] G.S.Salagean , Subclasses of Univalent functions, II Section Ů Function Theory Of One Complex Variable Complex Analysis Ů *Fifth Romanian-Finnish Seminar, Lecture Notes in Mathematics* Volume **1013** of the series Lecture Notes in Mathematics, 362-372 (1983).


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