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# A REMARK ABOUT THE COMPOSITION OPERATORS IN THE SPACE OF BOUNDED $\Lambda$–VARIATION FUNCTIONS IN WATERMAN SENSE

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**Abstract**. In this paper, we demonstrate the generalization of uniform continuous of the composition operators in the space of the bounded $\Lambda$-variation functions [2, 3]. In this paper we extend the result obtained recently in [2, 3] and [11] the space of bounded $\Lambda$-variation in the sense Waterman [21]. Also we give some results about locally defined operators.

## 1 Introduction

Let $I$ be an interval of $\mathbb{R}$, $X$ a real normed space, $C$ a closed convex subset of $X$, $Y$ a real Banach space and $h : I \times C \to Y$. Denote by $X^I$ the algebra of all functions $f : I \longrightarrow X$ and by $H : X^I \longrightarrow Y^I$ the Nemytskii composition operator generated by the function $h$ defined by

$$(Hf)(t) = h(t, f(t)), \quad t \in I, \;\; f \in X^I. \tag{1.1}$$

Let $(\Lambda BV(I, X), \|\cdot\|_\Lambda)$ be the Banach space of functions $f : I \to X$ which are of bounded $\Lambda$ -variation in the sense of Waterman, where the norm $\|\cdot\|_\Lambda$ is defined with the aid of Luxemburg-Nakano-Orlicz seminorm [16, 10, 18].

Assume that $H$ maps the set of functions $f \in \Lambda BV(I, X)$ such that $f(I) \subset C$ into $\Lambda BV(I, Y)$. In the present paper, we prove that, if $H$ is uniformly continuous, then the left and right regularization of its generator $h$ with respect for the first variable are affine functions in the second variable. This extends the main result of paper [2, 3].

## 2 Preliminaries

In this section we recall some facts which will be needed further on.

Denote by $\mathbb{R}$ the set of all real numbers and put $\mathbb{R}_+ = [0, \infty)$.

Next, let $\Lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive real numbers such that $\sum \frac{1}{\lambda_n}$ diverges.

If $\{I_n\}$ denote a sequence of non-overlapping intervals $I_n = [a_n, b_n] \subset I$ and we write $f(I_n) = f(b_n) - f(a_n)$. Throughout this paper, when we consider a collection of intervals, they will be assumed to be non-decreasing without further reference to that fact.

Let $I \subset \mathbb{R}$ be an interval. Then, for a set $X$ we denote by $X^I$ the set of all mappings $f : I \longrightarrow X$ acting from $I$ into $X$.

**Definition 2.1** ([21]). A function $f \in X^I$ is said to be of $\wedge$-bounded variation ($\Lambda BV$), in the sense of Waterman in $I$, if for every $\{I_n\}$, we have

$$v_\wedge(f) = v_\wedge(f, I) := \sup \sum_n \frac{\|f(I_n)\|}{\lambda_n} < \infty, \tag{2.1}$$

the supremum being taken over all $\{I_n\}$, $\{I_n\} \subseteq I$.

Various spaces of the functions of generalized bounded variation which have been considered can be obtained by making special choices of the functions $\lambda_n$, $n = 1, 2, \cdots$. If we take $\Lambda = \{n\}$ rise to be class of functions of harmonic bounded variation $HBV$. The definition (2.1) coincides with the classical concept of variation in the sense of Jordan. For $\lambda_n = \varphi$, the condition (2.1) coincides with the classical concept of variation in the sense of Wiener [22], where $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ denote a continuous, convex and non–decreasing function, with $\varphi(0) = 0$, $\varphi(x) > 0$ for $x > 0$.

It is easily seen that $\Lambda BV = BV$, the space of functions of ordinary Jordan bounded variation on $I$, if and only if $\Lambda$ is a bounded sequence. Consequently, if we suppose that $\sup_{i \in \mathbb{N}} \lambda_i = \infty$, then $BV$ is a proper subspace of $\Lambda BV$.

It is known that for all $a, b, c \in I$, such that $a \le c \le b$, we have $v_\wedge(f, [a, c]) \le v_\wedge(f, [a, b])$ (that is, $v_\wedge$ is increasing with respect to the interval) and $v_\wedge(f, [a, c]) + v_\wedge(f, [c, b]) \le v_\wedge(f, [a, b])$.

In what follows we denote by $V_\Lambda(I, X)$ the set of all bounded $\Lambda$-variation functions $f \in X^I$ in the Waterman sense. This is a symmetric and convex set; but it is not necessarily a linear space. In fact, Musielak-Orlicz proved the following statement: this class of functions ($V_\varphi(I, X) \supset V_\Lambda(I, X)$ ) is a linear space if, and only if, $\varphi$ satisfies the $\delta_2$ condition [15] (there exist $a > 0$ and $k > 0$ such that $\varphi(2u) \le k\varphi(u)$ for $0 < u \le a$). We denote by $\Lambda BV(I, X)$ the linear space of all functions $f \in X^I$ such that $v_\wedge(\lambda f) < \infty$ for some constant $\lambda > 0$.

In the linear space $\Lambda BV(I, X)$, the function $\|\cdot\|_\Lambda$ defined by

$$\|f\|_\Lambda := |f(a)| + p_\wedge(f), \quad f \in \Lambda BV(I, X),$$

where

$$p_\wedge(f) := p_\wedge(f, I) = \inf \Big\{ \epsilon > 0 : v_\wedge\big(f/\epsilon\big) \le 1 \Big\}, \quad f \in \Lambda BV(I, X), \tag{2.2}$$

is a norm (see for instance [15, 6, 20]).

For $X = \mathbb{R}$, the linear normed space $(BV_\Lambda(I, \mathbb{R}), \|\cdot\|_\Lambda)$ was studied by Daniel Waterman ([21]). Also he joint with Perlman shows that the space $(\Lambda BV(I, \mathbb{R}), \|\cdot\|_\Lambda)$ is a Banach algebra ([14, 19]). The functional $p_\wedge(\cdot)$ defined by (2.2) is called *the Luxemburg-Nakano-Orlicz seminorm* [16, 10, 18].

In the sequel, the symbol $\Lambda BV(I, C)$ stands for the set of all functions $f \in \Lambda BV(I, X)$ such that $f : I \longrightarrow C$ and $C$ is a subset of $X$.

**Lemma 2.2.** *For* $f \in \Lambda BV(I, X)$, *we have:*

- **(a)** *if* $t, t' \in I$, *then* $\|f(t) - f(t')\| \le \lambda_1 p_\wedge(f)$;
- **(b)** *if* $p_\wedge(f) > 0$ *then* $v_\wedge\big(f/p_\wedge(f)\big) \le 1$;
- **(c)** *for* $\epsilon > 0$,
  - **(c1)** $p_\wedge(f) \le \epsilon$ *if and only if* $v_\wedge\big(f/\epsilon\big) \le 1$;
  - **(c2)** *if* $v_\wedge\big(f/\epsilon\big) = 1$ *then* $p_\wedge(f) = \epsilon$.

*Proof.* (a) Take $\epsilon > p_\wedge(f)$; then for any $t, s \in I$ and for any finite collection $\{I_n\}$, by virtue (2.1) and (2.2), we have

$$\left(\frac{\|f(t) - f(s)\|}{\lambda_1 \epsilon}\right) \le \sum_n \left(\frac{\|f(I_n)\|}{\lambda_n \epsilon}\right) \le v_\wedge\left(\frac{f}{\epsilon}\right) \le 1$$

whence, taking the function $\lambda_1$ we obtain (a). Property **(a)** in Lemma 2.2 implies that any function $f \in \Lambda BV(I,X)$ is bounded. Indeed, we have $\|f\| \leq \|f(a)\| + \|f(t) - f(a)\|$, whence

$$\|f\|_\infty \leq \|f(a)\| + \lambda_n^{-1}(1)p_\wedge(f) < \infty.$$

(b) Suppose that sequence of the numbers $\lambda_n > \lambda = p_\wedge(f)$ converges a $\lambda$ as $n \longrightarrow \infty$. If follows from the definition of the number $\lambda$ that $v_\wedge(f) \leq 1$ for all positive integers $n$. Since $f/\lambda_n$ pointwise converges to $f/\lambda$ on $I$ as $n \longrightarrow \infty$, by the lower semicontinuity of the functional $v_\wedge(\cdot)$, we obtain that $v_\wedge(f/\lambda) \leq \lim_{n \longrightarrow \infty} v_\wedge(f/\lambda_n) \leq 1$.

(c) To prove (c.1), it suffices to show that if $0 < p_\wedge(f) < \epsilon$, then $v_\wedge(f/\epsilon) < 1$, and this is directly implied by the convexity of $v_\wedge(\cdot)$ and of the part (b), that is,

$$v_\wedge(f/\epsilon) \leq \frac{p_\wedge(f)}{\epsilon} v_\wedge\left(\frac{f}{p_\wedge(f)}\right) \leq \frac{p_\wedge(f)}{\epsilon} \leq 1.$$

To prove the second assertion (c.2), it suffices to observe that the cases where $p_\wedge(f) > \epsilon$ and $p_\wedge(f) < \epsilon$ are impossible. □

We consider the following notation of interval $I^-$ by formula $I^- := I \backslash \{\inf I\}$. If $(X, |\cdot|)$ is a Banach space and $f \in \Lambda BV(I,X)$, then $f^-(t) := \lim_{s \uparrow t} f(s)$, $t \in I^-$, exists and is called the *left regularization* of $f$ it was proved in ([6]).

Let $\Lambda BV^-(I,X)$ denote the subset in $\Lambda BV(I,X)$ that consists of those functions that are left continuous on $I^-$.

**Lemma 2.3.** *If $X$ is a Banach space and $f \in \Lambda BV(I,X)$, then $f^- \in \Lambda BV^-(I,X)$. The prove is similar to the since by Chistyakov [4, Lemma 6].*

Thus, if a function has a bounded $\Lambda$-variation, then its left regularization is a left continuous function.

**Lemma 2.4.** *[6] If $f : I \to X$ is monotone, then $v_\wedge(f) = \frac{|f(b) - f(a)|}{\lambda_1}$.*

## 3 The Composition Operator

Our main result reads as follows:

**Theorem 3.1.** *Let $(X, |\cdot|_X)$ be a real normed space, $(Y, |\cdot|_Y)$ a real Banach space, $C \subset X$ a closed convex set. Suppose that $\Lambda_1 = \{\lambda_n\}$, $\Lambda_2 = \{\varphi_n\}$ two sequence in sense Waterman and $h : I \times C \longrightarrow Y$. If a composition operator $H : C^I \longrightarrow Y^I$ generated by $h$, maps $\Lambda_1 BV(I,C)$ into $\Lambda_2 BV(I,Y)$ and is uniformly continuous, then the left regularization of $h$, i.e. the function $h^- : I^- \times X \longrightarrow Y$, defined by*

$$h^-(t,y) \quad := \quad \lim_{s \uparrow t} h(s,y), \quad t \in I^-; \ \ y \in C,$$

*exists and*

$$h^-(t,y) = A(t)y + B(t), \quad t \in I^-, \ \ y \in C,$$

*for some $A : I^- \longrightarrow \mathcal{L}(X,Y)$[1] and $B \in \Lambda_2 BV(I^-, Y)$. Moreover the functions $A$ and $B$ are left-continuous in $I^-$.*

*Proof.* For every $y \in C$, the constant function $f(t) = y$ $(t \in I)$ belongs to $\Lambda_1 BV(I,C)$. Since $H$ maps $\Lambda_1 BV(I,C)$ into $\Lambda_2 BV(I,Y)$, it follows that the function $t \mapsto h(t,y)$ $(t \in I)$ belongs to $\Lambda_2 BV(I,Y)$. Now, by Lemma 2.3, the completeness of $(Y, |\cdot|_Y)$ implies the existence of the left regularization $h^-$ of $h$.

By assumption, $H$ is uniformly continuous on $\Lambda_1 BV(I,C)$. Let $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be the *modulus continuity* of $H$ that is

$$\omega(\rho) := \sup\left\{ \Big\| H(f_1) - H(f_2) \Big\|_{\Lambda_2 BV(I,Y)} : \|f_1 - f_2\|_{\Lambda_1 BV(I,C)} \leq \rho \right\},$$

[1] $\mathcal{L}(X,Y)$ denote the space of all linear mappings $A : X \longrightarrow Y$

for $f_1,\ f_2 \in \Lambda_1 BV(I,C)$ and $\rho > 0$.

Hence we get

$$\|H(f_1) - H(f_2)\|_{\Lambda_2 BV(I,Y)} \le \omega\big(\|f_1 - f_2\|_{\Lambda_1 BV(I,C)}\big), \quad \text{for} \quad f_1, f_2 \in \Lambda_1 BV(I,C). \tag{3.1}$$

From the definition of the norm $\|\cdot\|_\Lambda$, we obtain

$$p_\wedge\big(H(f_1) - H(f_2)\big) \le \|H(f_1) - H(f_2)\|_{\Lambda_2 BV(I,Y)}, \quad \text{for} \quad f_1, f_2 \in \Lambda_1 BV(I,C). \tag{3.2}$$

From (3.1), (3.2) and Lemma 2.2 (c1), if $\omega\big(\|f_1 - f_2\|_{\Lambda_1 BV(I,C)}\big) > 0$, then

$$v_\wedge\left(\frac{H(f_1) - H(f_2)}{\omega\big(\|f_1 - f_2\|_{\Lambda_1 BV(I,C)}\big)}\right) \le 1. \tag{3.3}$$

Therefore, for any $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m,\ \alpha_i, \beta_i \in I$, $i \in \{1, 2, \cdots, m\},\ m \in \mathbb{N}$, the definitions of the operator $H$ and the functional $v_\wedge(\cdot)$ imply

$$\sum_{n=1}^{m}\left(\frac{|h(\beta_i, f_1(\beta_i)) - h(\beta_i, f_2(\beta_i)) - h(\alpha_i, f_1(\alpha_i)) + h(\alpha_i, f_2(\alpha_i))|}{\lambda_n \omega(\|f_1 - f_2\|_{\Lambda_1 BV(I,C)})}\right) \le 1. \tag{3.4}$$

For $\alpha, \beta \in \mathbb{R},\ \alpha < \beta$, we define auxiliary Lipschitz functions $\eta_{\alpha,\beta} : \mathbb{R} \longrightarrow [0,1]$ by

$$\eta_{\alpha,\beta}(t) := \begin{cases} 0 & \text{if} \quad t \le \alpha \\ \dfrac{t-\alpha}{\beta-\alpha} & \text{if} \quad \alpha \le t \le \beta \\ 1 & \text{if} \quad \beta \le t\,. \end{cases} \tag{3.5}$$

Let us fix $t \in I^-$. For arbitrary finite sequence $\inf I < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < t$ and $y_1, y_2 \in C,\ y_1 \ne y_2$, the functions $f_1, f_2 : I \longrightarrow X$ defined by

$$f_\ell(\tau) := \frac{1}{2}\big(\eta_{\alpha_i,\beta_i}(\tau)(y_1 - y_2) + y_\ell + y_2\big), \quad \tau \in I, \quad \ell = 1, 2, \tag{3.6}$$

belong to the space $\Lambda_1 BV(I,C)$. From (3.6), we have

$$f_1(\cdot) - f_2(\cdot) = \frac{y_1 - y_2}{2},$$

therefore

$$\|f_1 - f_2\|_{\Lambda_1 BV(I,C)} = \left|\frac{y_1 - y_2}{2}\right|;$$

moreover

$$f_1(\beta_i) = y_1; \quad f_2(\beta_i) = \frac{y_1 + y_2}{2}; \quad f_1(\alpha_i) = \frac{y_1 + y_2}{2}; \quad f_2(\alpha_i) = y_2.$$

Using (3.4), we hence get

$$\begin{aligned} &\sum_{i=1}^{m}\left(\frac{\left|h(\beta_i, y_1) - h\left(\beta_i, \frac{y_1+y_2}{2}\right) - h\left(\alpha_i, \frac{y_1+y_2}{2}\right) + h(\alpha_i, y_2)\right|}{\lambda_i \omega\left(\left|\frac{y_1-y_2}{2}\right|\right)}\right) \\ &\le \sum_{i\ge 1}\left(\frac{\left|h(\beta_i, y_1) - h\left(\beta_i, \frac{y_1+y_2}{2}\right) - h\left(\alpha_i, \frac{y_1+y_2}{2}\right) + h(\alpha_i, y_2)\right|}{\lambda_i \omega\left(\left|\frac{y_1-y_2}{2}\right|\right)}\right) \le 1. \end{aligned} \tag{3.7}$$

It is of great importance remarks that the constants functions defined on the interval $I$ belong to the space $\Lambda_1 BV(I,C)$ since the composition operator H generate by h acts from $\Lambda_1 BV(I,C)$ into $\Lambda_2 BV(I,Y)$, it follows that the function $t \mapsto h(t,y)\ (t \in I)$ belong to $\Lambda_2 BV(I,Y)$ for all $y \in C$. From the continuity of $\Lambda_2$ and the definition of $h^-$, passing to the limit in (3.7) when $\alpha_1 \uparrow t$, we obtain that

$$\sum_{i=1}^{m}\left(\frac{\left|h^-(t, y_1) - h^-\left(t, \frac{y_1+y_2}{2}\right) - h^-\left(t, \frac{y_1+y_2}{2}\right) + h^-(t, y_2)\right|}{\lambda_i \omega\left(\left|\frac{y_1-y_2}{2}\right|\right)}\right) \le 1,$$

The sum of the left hand side suppose without lost generality fix $i = n$ for $n = 1, 2, ..., m$, such that

$$m \cdot \left( \frac{\left| h^-(t, y_1) - 2h^-\left(t, \frac{y_1 + y_2}{2}\right) + h^-(t, y_2) \right|}{\lambda_n \omega\left(\left|\frac{y_1 - y_2}{2}\right|\right)} \right) \leq 1 \,.$$

we get

$$\left( \frac{\left| h^-(t, y_1) - 2h^-\left(t, \frac{y_1 + y_2}{2}\right) + h^-(t, y_2) \right|}{\omega\left(\left|\frac{y_1 - y_2}{2}\right|\right)} \right) \leq \frac{1}{m}$$

and since that $m \in \mathbb{N}$ is arbitraries we derive

$$\left( \frac{\left| h^-(t, y_1) - 2h^-\left(t, \frac{y_1 + y_2}{2}\right) + h^-(t, y_2) \right|}{\omega\left(\left|\frac{y_1 - y_2}{2}\right|\right)} \right) = 0,$$

then

$$\left| h^-(t, y_1) - 2h^-\left(t, \frac{y_1 + y_2}{2}\right) + h^-(t, y_2) \right| = 0.$$

Or equivalently

$$h^-\left(t, \frac{y_1 + y_2}{2}\right) = \frac{h^-(t, y_1) + h^-(t, y_2)}{2}$$

for all $t \in I^-$ and all $y_1,\ y_2 \in C$.

Thus, for each $t \in I^-$, the function $h^-(t, \cdot)$ satisfies the Jensen functional equation in $C$. Modifying a little the standard argument (cf. Kuczma [9]), we conclude that, for each $t \in I^-$, there exist $A(t) : C \longrightarrow \mathcal{L}(X, Y)$ and $B(t) \in Y$ such that $h^-(t, y) = A(t)y + B(t)$.

The uniform continuity of the operator $H : \Lambda_1 BV(I, C) \longrightarrow \Lambda_2 BV(I, Y)$ implies the continuity of the additive function $A(t)$. Consequently $A(t) \in \mathcal{L}(X, Y)$, for each $t \in I^-$. □

**Remark 3.2.** Obviously, the counterpart of Theorem 3.1 for the right regularization $h^+$ of $h$ defined by

$$h^+(t, y) := \lim_{s \downarrow t} h(s, y); \quad t \in I^+ := I \backslash \{\sup I\},$$

is also true.

**Remark 3.3.** Taking $X = Z = \mathbb{R}$, $\Lambda = \varphi := id\Big|_{[0,+\infty)}$ in Theorem 3.1 and $C := J$ where $J \subset \mathbb{R}$ is an interval we obtain the main result from [11].

**Remark 3.4.** Theorem 3.1 extends also the result of Guerrero ([2, 3]).

**Remark 3.5.** In the proof of Theorem 3.1 we apply the uniform continuity of the operator $H$ only on the set of functions $U \subset \Lambda_1 BV(I, C)$ such that $f \in U$ if, and only if, there are $\alpha,\ \beta \in I,\ \alpha < \beta$, such that

$$f(t) = \frac{1}{2}\Big[\eta_{\alpha,\beta}(t)(y_1 - y_2) + y + y_2\Big], \quad t \in I,$$

where $\eta_{\alpha,\beta}$ is defined by (3.5), $y_1, y_2 \in C$ and $y = y_1$ or $y = y_2$.

Thus the assumption of the uniform continuity of $H$ on $\Lambda_1 BV(I, C)$ in Theorem 3.1 can be replaced by a weaker condition of the uniform continuity of $H$ on $U$.

## 4 Locally Defined Operators

It is well know that every Nemytskii composition operator is locally defined (cf. [1], also [13, 23, 24]). To recall the definition of a local operator assume that $\mathcal{G} = \mathcal{G}(I, \mathbb{R})$ and $\mathcal{H} = \mathcal{H}(I, \mathbb{R})$ are two classes of functions $\varphi : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval. A mapping $K : \mathcal{G} \to \mathcal{H}$ is said to be a locally defined operator or $(\mathcal{G}, \mathcal{H})$–local operator if for any open interval $J \subset \mathbb{R}$ and for any functions $\varphi, \psi \in \mathcal{G}$,

$$\varphi\Big|_{J\cap I} = \psi\Big|_{J\cap I} \Rightarrow K(\varphi)\Big|_{J\cap I} = K(\psi)\Big|_{J\cap I},$$

where $\varphi\Big|_{J\cap I}$ denotes the restriction of $\varphi$ to $J \cap I$.

The form of the locally defined operator strongly depends on the nature of the function spaces $\mathcal{G}$ and $\mathcal{H}$ which are its domains and ranges, respectively.

Let $C(I)$ be a family of real continuous functions defined on $I$ and $CM_+(I)$ and $CM_-(I)$ denote, respectively, a family of continuous nondecreasing and continuous nonincreasing functions $f : I \to \mathbb{R}$.

We write $CBV(I)$ for $C(I) \bigcap BV(I, \mathbb{R})$.

**Proposition 4.1.** *If a locally defined operator $K$ maps $CBV(I)$ into $CM_+(I)$, then it is constant, that is, a function $b \in CM_+(I)$ such that*

$$K(\varphi) = b, \quad \varphi \in CBV(I).$$

*Proof.* Let $K : CBV(I) \to CM_+(I)$ be a local operator. Since $CM_+(I) \subset CBV(I)$ and $CM_-(I) \subset CBV(I)$, an operator $K$ is $(CM_+, CM_+)-$ and $(CM_-, CM_+)-$ locally defined. Hence, $K$ is the Nemytskii composition operator and by Theorem 1 and Theorem 4 from [24], we get our claim. □

Similarly, by [24, Remark 4], we can get the following

**Proposition 4.2.** *If a locally defined operator $K$ maps $CBV(I)$ into $CM_-(I)$, then it is constant, that is there is a function $b \in CM_-(I)$ such that*

$$K(\varphi) = b, \quad \varphi \in CBV(I).$$

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