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COEFFICIENT ESTIMATES FOR SOME SUBCLASSES OF ANALYTIC AND Bi-UNIVALENT FUNCTIONS

Pravati Sahoo and Saumya Singh

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Abstract. In the present paper, we introduce and investigate two new subclasses $\mathcal{B}_\Sigma(\alpha,\lambda,\mu)$ and $\mathcal{M}_\Sigma(\beta,\lambda,\mu)$ of bi-valent functions in the unit disk $\mathbb{U}$. For functions belonging to the classes $\mathcal{B}_\Sigma(\alpha,\lambda,\mu)$ and $\mathcal{M}_\Sigma(\beta,\lambda,\mu)$, we obtain bounds of the first two Taylor-Maclaurin coefficients of $f(z)$.

1 Introduction and Preliminaries

Let $\mathcal{A}$ be the class of analytic functions defined on the unit disc $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ with the normalized conditions $f(0)=0=f'(0)-1$. Let $\mathcal{S}$ be the class of all functions $f\in\mathcal{A}$ which are univalent in $\mathbb{U}$. So $f(z)\in\mathcal{S}$ has the form

$$f(z)=z+\sum_{n=1}^{\infty}a_nz^n,\quad z\in\mathbb{U}. \tag{1.1}$$

Let $f^{-1}(z)$ be inverse of the function $f(z)$ and it is well known that every function $f\in\mathcal{S}$ has an inverse $f^{-1}(z)$, defined by

$$f^{-1}(f(z))=z,\quad z\in\mathbb{U}$$

and

$$f(f^{-1}(w))=w,\quad for\quad |w|<r_0(f);\ r_0(f)\geq\frac{1}{4},$$

where

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{1.2}$$

A function $f\in\mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(w)$ are univalent in $\mathbb{U}$.
Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).
Many interesting examples of the functions of the class Σ, together with various other properties and characteristics associated with bi-univalent functions (including also several open problems and conjectures involving bounds of the coefficients of the functions in Σ), can be found in the earlier work studied by Lewin[7], Brannan and Clunie [5], Netanyahu[8] and others. They introduced subclasses of Σ, like class of bi-starlike and bi-convex functions, bi-strongly starlike and bi-convex functions similar to the well-known subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}^*(\alpha)$ of starlike and convex functions of order $\alpha(0<\alpha<1)$, respectively (see [2]) and obtained non-sharp estimates on the initial coefficients in the Taylor-Maclaurin series exapansion (1.1) see [4, 5, 13]. More recently, Srivastava et.al [12, 14, 15], Frasin and Aouf [6], R.M. Ali et.al [1] introduced some new subclasses of Σ and obtained bounds for the initial coefficients of the function given by (1.1).
Motivated by the work of [12, 14, 15] and Sahoo et.al [11], we introduce and study some new subclasses $\mathcal{B}_\Sigma(\alpha,\lambda,\mu)$ and $\mathcal{M}_\Sigma(\alpha,\lambda,\mu)$.

Definition 1.1. A function f given by (1.1) is said to be in the class $\mathcal{B}_\Sigma(\alpha,\lambda,\mu)$ if the following conditions are satisfied:
$f\in\Sigma,\quad 0<\alpha\leq 1,\ 0<\mu<1,\ \lambda>\mu$

$$\left|\arg\left((1-\lambda)\left(\frac{z}{f(z)}\right)^{\mu}+\lambda\left(\frac{z}{f(z)}\right)^{\mu+1}\right)\right|<\frac{\alpha\pi}{2}\quad z\in\mathbb{U}, \tag{1.3}$$

and

$$\left|\arg\left((1-\lambda)\left(\frac{w}{g(w)}\right)^{\mu}+\lambda\left(\frac{w}{g(w)}\right)^{\mu+1}\right)\right|<\frac{\alpha\pi}{2}\quad w\in\mathbb{U}, \tag{1.4}$$

where

$$g(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots.$$

$\mathcal{B}_\Sigma(\alpha,\lambda,-1)$ was introduced and studied in [6] and $\mathcal{B}_\Sigma(\alpha,1,-1)$ was introduced and studied in [12]. In this paper, we found the estimates for the initial coefficients a_2 and a_3 of bi-univalent functions belonging to the class $\mathcal{B}_\Sigma(\alpha,\lambda,\mu)$. Our results generalizes several well-known results in [6, 12, 15].
In order to prove our main result we need the following lemma:

Lemma 1.2. *[9] If $p\in\mathcal{P}$, then $|c_k|\le 2$ for each k, where $\mathcal{P}$is the family of all functions $p(z)$ analytic in $\mathbb{U}$ for which* $\operatorname{Re} p(z)>0$, *$p(z)=1+c_1z+c_2z^2+\cdots$ for $z\in\mathbb{U}$.*

2 Coefficient bounds for the function belonging to the class $\mathcal{B}_\Sigma(\alpha,\lambda,\mu)$

Theorem 2.1. *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\alpha,\lambda,\mu)$, $0<\mu<\alpha\le 1$, $\lambda>\mu$. Then*

$$|a_2|\le\frac{2\alpha}{\sqrt{(\lambda-\mu)^2+\alpha(2\lambda-\lambda^2-\mu)}} \tag{2.1}$$

and

$$|a_3|\le\frac{4\alpha^2}{(\lambda-\mu)^2}+\frac{2\alpha}{2\lambda-\mu}. \tag{2.2}$$

Proof. It follows from (1.3) and (1.4) that

$$(1-\lambda)\left(\frac{z}{f(z)}\right)^{\mu}+\lambda\left(\frac{z}{f(z)}\right)^{\mu+1}=(p(z))^{\alpha} \tag{2.3}$$

$$(1-\lambda)\left(\frac{w}{g(w)}\right)^{\mu}+\lambda\left(\frac{w}{g(w)}\right)^{\mu+1}=(q(w))^{\alpha}, \tag{2.4}$$

where $p(z)=1+p_1z+p_2z^2+\cdots$ and $q(w)=1+q_1w+q_2w^2+\cdots$ in $\mathcal{P}$. Now on equating the coefficients in (2.3) and (2.4), we have

$$(\lambda-\mu)a_2=\alpha p_1 \tag{2.5}$$

$$(2\lambda-\mu)a_3+\frac{(\mu-2\lambda)(\mu+1)}{2}a_2^2=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2 \tag{2.6}$$

$$-(\lambda-\mu)a_2=\alpha q_1 \tag{2.7}$$

and

$$-(2\lambda-\mu)a_3+\frac{(3-\mu)(2\lambda-\mu)}{2}a_2^2=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.8}$$

From (2.5) and (2.7) we get

$$p_1=-q_1 \tag{2.9}$$

and

$$2(\lambda-\mu)^2a_2^2=\alpha^2(p_1^2+q_1^2). \tag{2.10}$$

From (2.6), (2.8) and (2.10), we get

$$\begin{aligned}[(\mu-1)(\mu-2\lambda)]a_2^2 &= (p_2+q_2)\alpha+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)\\ &= (p_2+q_2)\alpha+\frac{\alpha-1}{\alpha}(\lambda-\mu)^2a_2^2.\end{aligned}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{(\lambda-\mu)^2+\alpha(2\lambda-\mu-\lambda^2)}. \tag{2.11}$$

Applying Lemma 1.2 for (2.11), we get

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda-\mu)^2+\alpha(2\lambda-\mu-\lambda^2)}},$$

which gives us desired estimate on $|a_2|$ as asserted in (2.1).
Next in order to find the bound on $|a_3|$, by subtracting (2.8) from (2.6), we get

$$2(2\lambda-\mu)a_3 - 2(2\lambda-\mu)a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2 - \left(\alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2\right). \tag{2.12}$$

It follows from (2.9), (2.10) and (2.12)

$$a_3 = \frac{\alpha^2(p_1^2+q_1^2)}{2(\lambda-\mu)^2} + \frac{\alpha(p_2-q_2)}{2(2\lambda-\mu)}. \tag{2.13}$$

Applying Lemma 1.2 for (2.13), we get

$$|a_3| \leq \frac{4\alpha^2}{(\lambda-\mu)^2} + \frac{2\alpha}{2\lambda-\mu}.$$

This completes the proof of Theorem 2.1.

If we take $\mu = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\alpha,\lambda,1)$, $0<\alpha\leq 1$, $\lambda>1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda-1)^2+\alpha(2\lambda-\lambda^2-1)}} \tag{2.14}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\lambda-1)^2} + \frac{2\alpha}{2\lambda-1}. \tag{2.15}$$

3 Coefficient bounds for the function belonging to the class $\mathcal{M}_\Sigma(\beta,\lambda,\mu)$

Definition 3.1. A function f given by (1.1) is said to be in the class $\mathcal{M}_\Sigma(\beta,\lambda,\mu)$ if the following conditions are satisfied:
$f\in\Sigma,\ 0\leq\beta<1,\ 0<\mu<1,\ \lambda>\mu$

$$\operatorname{Re}\left((1-\lambda)\left(\frac{z}{f(z)}\right)^{\mu} + \lambda\left(\frac{z}{f(z)}\right)^{\mu+1}\right) > \beta \quad z\in\mathbb{U}, \tag{3.1}$$

and

$$\operatorname{Re}\left((1-\lambda)\left(\frac{w}{g(w)}\right)^{\mu} + \lambda\left(\frac{w}{g(w)}\right)^{\mu+1}\right) > \beta \quad w\in\mathbb{U}, \tag{3.2}$$

where

$$g(w) = w - a_2w^2 + (2a_2^2-a_3)w^3 - (5a_2^3-5a_2a_3+a_4)w^4 + \cdots .$$

Theorem 3.2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{M}_\Sigma(\beta,\lambda,\mu)$, $0<\beta<1,\ 0<\mu<1,\ \lambda>\mu$. Then*

$$|a_2| \leq \min\left\{\frac{2(1-\beta)}{\lambda-\mu},\ 2\sqrt{\frac{1-\beta}{(1-\mu)(2\lambda-\mu)}}\right\} \tag{3.3}$$

and

$$|a_3| \leq \min\left\{\frac{4(1-\beta)^2}{(\lambda-\mu)^2} + \frac{2(1-\beta)}{2\lambda-\mu},\ \frac{4(1-\beta)}{(2\lambda-\mu)(1-\mu)}\right\}. \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that

$$(1-\lambda)\left(\frac{z}{f(z)}\right)^{\mu}+\lambda\left(\frac{z}{f(z)}\right)^{\mu+1}=\beta+(1-\beta)p(z) \tag{3.5}$$

$$(1-\lambda)\left(\frac{w}{g(w)}\right)^{\mu}+\lambda\left(\frac{w}{g(w)}\right)^{\mu+1}=\beta+(1-\beta)q(w), \tag{3.6}$$

where $p(z)=1+p_1z+p_2z^2+\cdots$ and $q(w)=1+q_1w+q_2w^2+\cdots$ in $\mathcal{P}$. Now on equating the coefficients in (3.5) and (3.6), we have

$$(\lambda-\mu)a_2=(1-\beta)p_1 \tag{3.7}$$

$$(2\lambda-\mu)a_3-\frac{(2\lambda-\mu)(\mu+1)}{2}a_2^2=(1-\beta)p_2 \tag{3.8}$$

$$-(\lambda-\mu)a_2=(1-\beta)q_1 \tag{3.9}$$

and

$$-(2\lambda-\mu)a_3+\frac{(3-\mu)(2\lambda-\mu)}{2}a_2^2=(1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1=-q_1 \tag{3.11}$$

and

$$2(\lambda-\mu)^2a_2^2=(1-\beta)^2(p_1^2+q_1^2). \tag{3.12}$$

From (3.8) and (3.10), we get

$$[(1-\mu)(2\lambda-\mu)]a_2^2=(p_2+q_2)(1-\beta). \tag{3.13}$$

From (3.12) and (3.13), we get

$$|a_2|^2\le\frac{(1-\beta)^2(|p_2|^2+|q_2|^2)}{2(\lambda-\mu)^2} \tag{3.14}$$

and

$$|a_2|^2\le\frac{(1-\beta)(|p_2|+|q_2|)}{(1-\mu)(2\lambda-\mu)}. \tag{3.15}$$

Applying Lemma 1.2 for (3.14) and (3.15) , we get

$$|a_2|\le\frac{2(1-\beta)}{\lambda-\mu},$$

and

$$|a_2|\le 2\sqrt{\frac{1-\beta}{(1-\mu)(2\lambda-\mu)}},$$

which gives us desired estimate on $|a_2|$ as asserted in (3.3).
Next in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$2(2\lambda-\mu)a_3-2(2\lambda-\mu)a_2^2=(1-\beta)(p_2-q_2). \tag{3.16}$$

It follows from (3.12) and (3.16)

$$a_3=\frac{(1-\beta)^2(p_1^2+q_1^2)}{2(\lambda-\mu)^2}+\frac{(1-\beta)(p_2-q_2)}{2(2\lambda-\mu)}. \tag{3.17}$$

Applying Lemma 1.2 for (3.17), we get

$$|a_3|\le\frac{4(1-\beta)^2}{(\lambda-\mu)^2}+\frac{2(1-\beta)}{2\lambda-\mu}.$$

On the other hand, by using (3.13) and (3.16), we obtain

$$a_3 = \frac{1-\beta}{2(2\lambda-\mu)}\left[\frac{3-\mu}{1-\mu}p_2 + \frac{1+\mu}{1-\mu}q_2\right], \tag{3.18}$$

which gives

$$|a_3| = \frac{4(1-\beta)}{(2\lambda-\mu)(1-\mu)}. \tag{3.19}$$

This completes the proof of Theorem 3.2.

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Author information

Pravati Sahoo and Saumya Singh, Department of Mathematics, Banaras Hindu University, Banaras, INDIA and
Department of Mathematics, O. P. Jindal University, Raigarh, INDIA.
E-mail: `pravatis@yahoo.co.in`

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