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ADDITIVITY OF MULTIPLICATIVE GENERALIZED JORDAN DERIVATIONS ON RINGS

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Abstract. Let G be a mapping from ring R into itself such that

$$G(ab + ba) = G(a)b + ad(b) + G(b)a + bd(a),$$

for all $a, b \in R$, where d is a Jordan derivation on R. We show that G is the additive mapping, by taking some appropriate conditions on R.

1 Introduction

Let R be a ring and e a non trivial idempotent element of R. We denotes e, $1 - e$, by e_1, e_2 respectively, then we have Peirce decomposition [10] of R, as $R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$, where $R_{mn} = e_m R e_n$, $m, n = 1, 2$. An additive mapping $d : R \to R$ is called derivation (Jordan derivation) if $d(xy) = d(x)y + xd(y)$,$(d(xy + yx) = d(x)y + xd(y) + d(y)x + yd(x))$ for all x, $y \in R$. The definition of generalized derivation was first introduced by Brešar, in [1]. An additive mapping $G : R \to R$ is said to be generalized derivation (generalized Jordan derivation) associated with derivation (Jordan derivation) d if $G(xy) = G(x)y + xd(y)$, $(G(xy + yx) = G(x)y + xd(y) + G(y)x + yd(x))$ for all $x, y \in R$. If we remove additivity of d then d is said to be multiplicative derivation. In similar fashion without additivity G is said to be multiplicative generalized derivation (multiplicative generalized Jordan derivation).

Additive mappings are closely connected with the structure of rings. In this direction Posner [9] proved very striking results. Posner studied behaviour of rings with the help of derivation. Many author's studied of additivity of mappings, further information can be found [6, 7]. In general multiplicative maps are not additive. It is natural question that 'When are multiplicative maps addtive'? The answer of this question given by Martindale [8]. He proved the following result:

Theorem 1.1. *Let R be a ring containing a family $\{e_\alpha : \alpha \in \Lambda\}$ of idempotents which satisfies:*
(1) $xR = \{0\}$ *implies* $x = 0$;
(2)*If* $e_\alpha Rx = \{0\}$ *for each* $\alpha \in \Lambda$, *then* $x = 0$*(hence* $Rx = \{0\}$ *implies* $x = 0$);
(3) *For each* $\alpha \in \Lambda$, $e_\alpha x e_\alpha R(1 - e_\alpha) = \{0\}$ *implies* $e_\alpha x e_\alpha = 0$.
Then any multiplicative bijective map from R onto an arbitrary ring R' is additive.

In this line Daif [3] proved that multiplicative derivation is derivation by taking some suitable condition on R. Recently, Jing and Lu [5] proved the following results:

Theorem 1.2. *Let R be a ring with a nontrivial idempotent and satisfying:*
(1) *If* $a_{ij} x_{jk} = 0$ *for all* $x_{jk} \in R_{jk}$, *then* $a_{ij} = 0$;
(2) *If* $x_{ij} a_{jk} = 0$ *for all* $x_{ij} \in R_{ij}$, *then* $a_{jk} = 0$;
(3) *If* $a_{ii} x_{ii} + x_{ii} a_{ii} = 0$ *for all* $x_{ii} \in R_{ii}$, *then* $a_{ii} = 0$;
for $i, j, k \in \{1, 2\}$. *If a mapping* $\delta : R \to R$ *satisfies*

$$\delta(ab + ba) = \delta(a)b + a\delta(b) + \delta(b)a + b\delta(a)$$

for all $a, b \in R$, then δ is additive. Moreover if R is 2-torsion free, then δ is a Jordan derivation.

Motivated by the above results, it is natural to ask when are multiplicative generalized Jordan derivation is additive? In this paper we will provide answer of the above question.

2 Preliminaries

The following lemma will be used in our main result.

Lemma 2.1. [[5], Lemma 1.5] *Let R be a 2-torsion free semi prime ring with a nontrivial idempotent and $i, j, k \in \{1, 2\}$ and R satisfies following condition:If $a_{ii}x_{ij} = 0$ for all $x_{jk} \in R_{jk}(i \neq j)$, then $a_{ii} = 0$. Then R has following condition:*
(1) If $a_{ij}x_{jk} = 0$ for all $x_{jk} \in R_{jk}$, then $a_{ij} = 0$.

3 Main Results

Throughout this section, we will take R, a ring with a nontrivial idempotent e_1 and satisfies the following condition:

(C) If $a_{ij}x_{jk} = 0$, for all $x_{jk} \in R_{jk}$, then $a_{ij} = 0$.

We also assume that mapping $G : R \to R$ satisfies

$$G(ab + ba) = G(a)b + ad(b) + G(b)a + bd(a)$$

for all $a, b \in R$, where d is a Jordan derivation on R.

We start with the following lemma.

Lemma 3.1. $(1) G(a_{11} + b_{12}) = G(a_{11}) + G(b_{12})$
$(2) G(a_{11} + b_{21}) = G(a_{11}) + G(b_{21})$
$(3) G(a_{22} + b_{12}) = G(a_{22}) + G(b_{12})$
$(4) G(a_{22} + b_{21}) = G(a_{22}) + G(b_{21})$

Proof. For any $x_{22} \in R_{22}$, we compute

$$G[(a_{11} + b_{12})x_{22} + x_{22}(a_{11} + b_{12})]$$
$$= G(a_{11} + b_{12})x_{22} + (a_{11} + b_{12})d(x_{22}) + G(x_{22})(a_{11} + b_{12}) + x_{22}d(a_{11} + b_{12}).$$

On other hand we get

$$\begin{aligned} G[(a_{11} + b_{12})x_{22} + x_{22}(a_{11} + b_{12})] &= G(b_{12}x_{22}) \\ &= G(a_{11}x_{22} + x_{22}a_{11}) + G(b_{12}x_{22} + x_{22}b_{12}) \\ &= G(a_{11})x_{22} + a_{11}d(x_{22}) + G(x_{22})a_{11} + x_{22}d(a_{11}) \\ &\quad + G(b_{12})x_{22} + b_{12}d(x_{22}) + G(x_{22})b_{12} + x_{22}d(b_{12}) \end{aligned}$$

Combining both the equalities, we get

$$[G(a_{11} + b_{12}) - G(a_{11}) - G(b_{12})]x_{22} = 0$$

This gives us

$$[G(a_{11} + b_{12}) - G(a_{11}) - G(b_{12})]_{12}x_{22} = 0$$
$$[G(a_{11} + b_{12}) - G(a_{11}) - G(b_{12})]_{22}x_{22} = 0$$

Using hypothesis condition C, we obtain

$$[G(a_{11} + b_{12}) - G(a_{11}) - G(b_{12})]_{12} = 0$$
$$[G(a_{11} + b_{12}) - G(a_{11}) - G(b_{12})]_{22} = 0$$

For any $x_{12} \in R_{12}$, we have

$$\begin{aligned} G[(a_{11} + b_{12})x_{12} + x_{12}(a_{11} + b_{12})] &= G(a_{11}x_{12}) \\ &= G(a_{11}x_{12} + x_{12}a_{11}) + G(b_{12}x_{12} + x_{12}b_{12}) \end{aligned}$$

Consequently, we obtain

$[G(a_{11}+b_{12})-G(a_{11})-G(b_{12})]x_{12}=0$

Again using hypothesis condition C, we obtain
$[G(a_{11}+b_{12})-G(a_{11})-G(b_{12})]_{11}=0$
$[G(a_{11}+b_{12})-G(a_{11})-G(b_{12})]_{21}=0$

Therefore $G(a_{11}+b_{12})=G(a_{11})+G(b_{12})$. Similarly we can prove others. □

Lemma 3.2. $(1)G(a_{12}+b_{12}c_{22})=G(a_{12})+G(b_{12}c_{22})$
$(2)G(a_{21}+b_{22}c_{21})=G(a_{21})+G(b_{22}c_{21})$

Proof. We prove only (1) and the proof of (2) is similar. Using Lemma 3.1 , we obtain

$$\begin{aligned}
&G(a_{12}+b_{12}c_{22})\\
&\quad=G[(e_1+b_{12})(a_{12}+c_{22})+(a_{12}+c_{22})(e_1+b_{12})]\\
&\quad=G(e_1+b_{12})(a_{12}+c_{22})+(e_1+b_{12})d(a_{12}+c_{22})\\
&\quad+G(a_{12}+c_{22})(e_1+b_{12})+(a_{12}+c_{22})d(e_1+b_{12})\\
&\quad=[G(e_1)+G(b_{12})](a_{12}+c_{22})+(e_1+b_{12})[d(a_{12})+d(c_{22})]\\
&\quad+[G(a_{12})+G(c_{22})](e_1+b_{12})+(a_{12}+c_{22})[d(e_1)+d(b_{12})]\\
&\quad=G(a_{12})+G(b_{12}c_{22}).
\end{aligned}$$

□

Lemma 3.3. $(1)G(a_{12}+b_{12})=G(a_{12})+G(b_{12})$
$(2)G(a_{21}+b_{21})=G(a_{21})+G(b_{21})$

Proof. For any $x_{22}\in R_{22}$, we have

$$\begin{aligned}
&G[(a_{12}+b_{12})x_{22}+x_{22}(a_{12}+b_{12})]\\
&=G(a_{12}+b_{12})x_{22}+(a_{12}+b_{12})d(x_{22})+G(x_{22})(a_{12}+b_{12})+x_{22}d(a_{12}+b_{12}).
\end{aligned}$$

On other hand by Lemma 3.2, we have

$$\begin{aligned}
&G[(a_{12}+b_{12})x_{22}+x_{22}(a_{12}+b_{12})]\\
&\quad=G(a_{12}x_{22}+b_{12}x_{22})\\
&\quad=G(a_{12}x_{22})+G(b_{12}x_{22})\\
&\quad=G(a_{12}x_{22}+x_{22}a_{12})+G(b_{12}x_{22}+x_{22}b_{12})\\
&\quad=G(a_{12})x_{22}+a_{12}d(x_{22})+G(x_{22})a_{12}+x_{22}d(a_{12})\\
&\quad+G(b_{12})x_{22}+b_{12}d(x_{22})+G(x_{22})b_{12}+x_{22}d(b_{12})
\end{aligned}$$

Therefore we get,

$$G[(a_{12}+b_{12})-G(a_{12})-G(b_{12})]x_{22}=0$$

So, we have

$$G[(a_{12}+b_{12})-G(a_{12})-G(b_{12})]_{22}=0$$
$$G[(a_{12}+b_{12})-G(a_{12})-G(b_{12})]_{12}=0$$

For complete the proof take $x_{12}\in R_{12}$, we compute

$$\begin{aligned}
&G(a_{12}+b_{12})x_{12}+(a_{12}+b_{12})d(x_{12})+G(x_{12})(a_{12}+b_{12})+x_{12}d(a_{12}+b_{12})\\
&\quad=G[(a_{12}+b_{12})x_{12}+x_{12}(a_{12}+b_{12}]\\
&\quad=0\\
&\quad=G(a_{12}x_{12}+x_{12}a_{12})+G(b_{12}x_{12}+x_{12}b_{12})\\
&\quad=G(a_{12})x_{12}+a_{12}d(x_{12})+G(x_{12})a_{12}+x_{12}d(a_{12})\\
&\quad+G(b_{12})x_{12}+b_{12}d(x_{12})+G(x_{12})b_{12}+x_{12}d(b_{12})
\end{aligned}$$

This yields that
$[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]x_{12}=0$

Therefore we obtain
$[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]_{11}=0$

$[G(a_{12}+b_{12})-G(a_{12})-G(b_{12})]_{21}=0$

Hence we get $G(a_{12}+b_{12})=G(a_{12})+G(b_{12})$.
Similarly we can prove that $G(a_{21}+b_{21})=G(a_{21})+G(b_{21})$. □

Lemma 3.4. $(1)G(a_{11}+b_{11})=G(a_{11})+G(b_{11})$
$(2)G(a_{22}+b_{22})=G(a_{22})+G(b_{22})$

Proof. For any $x_{22}\in R_{22}$, we have

$$\begin{aligned}
&G(a_{11}+b_{11})x_{22}+(a_{11}+b_{11})d(x_{22})+G(x_{22})(a_{11}+b_{11})+x_{22}d(a_{11}+b_{11})\\
&\quad=G[(a_{11}+b_{11})x_{22}+x_{22}(a_{11}+b_{11}]\\
&\quad=0\\
&\quad=G(a_{11}x_{22}+x_{22}a_{11})+G(b_{11}x_{22}+x_{22}b_{11})\\
&\quad=G(a_{11})x_{22}+a_{11}d(x_{22})+G(x_{22})a_{11}+x_{22}d(a_{11})\\
&\quad+G(b_{11})x_{22}+b_{11}d(x_{22})+G(x_{22})b_{11}+x_{22}d(b_{11})
\end{aligned}$$

This yields that,
$[G(a_{11}+b_{11})-G(a_{11})-G(b_{11})]x_{22}=0$

Therefore, we obtain

$$[G(a_{11}+b_{11})-G(a_{11})-G(b_{11})]_{12}=0$$
$$[G(a_{11}+b_{11})-G(a_{11})-G(b_{11})]_{22}=0$$

Similarly, by considering $(a_{11}+b_{11})x_{12}+x_{12}(a_{11}+b_{11})$, we can show that

$$[G(a_{11}+b_{11})-G(a_{11})-G(b_{11})]_{11}=0$$
$$[G(a_{11}+b_{11})-G(a_{11})-G(b_{11})]_{21}=0$$

Hence, we obtain $G(a_{11}+b_{11})=G(a_{11})+G(b_{11})$.
Similarly we can prove that $G(a_{22}+b_{22})=G(a_{22})+G(b_{22})$. □

Lemma 3.5. $G(a_{12}+b_{21})=G(a_{12})+G(b_{21})$

Proof. For any $x_{12}\in R_{12}$, we have

$$\begin{aligned}
&G(a_{12}+b_{21})x_{12}+(a_{12}+b_{21})d(x_{12})+G(x_{12})(a_{12}+b_{21})+x_{12}d(a_{12}+b_{21})\\
&\quad=G(b_{21}x_{12}+x_{12}b_{21})\\
&\quad=G(b_{21}x_{12}+x_{12}b_{21})+G(a_{12}x_{12}+x_{12}a_{12})\\
&\quad=G(b_{21})x_{12}+b_{21}d(x_{12})+G(x_{12})b_{21}+x_{12}d(b_{21})\\
&\quad+G(a_{12})x_{12}+a_{12}d(x_{12})+G(x_{12})a_{12}+x_{12}d(a_{12})
\end{aligned}$$

Therefore, we get
$[G(a_{12}+b_{21})-G(a_{12})-G(b_{21})]x_{12}=0$

Using hypothesis condition C, we obtain

$$[G(a_{12}+b_{21})-G(a_{12})-G(b_{21})]_{11}=0$$
$$[G(a_{12}+b_{21})-G(a_{12})-G(b_{21})]_{21}=0$$

Similarly, by considering $(a_{12}+b_{21})x_{12}+x_{12}(a_{12}+b_{21})$, we can show that

$$[G(a_{12}+b_{21})-G(a_{12})-G(b_{21})]_{12}=0$$
$$[G(a_{12}+b_{21})-G(a_{12})-G(b_{21})]_{22}=0$$

Hence we get the required result. □

Lemma 3.6. $(1)G(a_{11}+b_{12}+c_{21})=G(a_{11})+G(b_{12})+G(c_{21})$
$(2)G(a_{12}+b_{21}+c_{22})=G(a_{12})+G(b_{21})+G(c_{22})$

Proof. For any $x_{22} \in R_{22}$, we have

$$\begin{aligned} &G[(a_{11}+b_{12}+c_{21})x_{22}+x_{22}(a_{11}+b_{12}+c_{21})] \\ &\qquad = G(a_{11}+b_{12}+c_{21})x_{22}+(a_{11}+b_{12}+c_{21})d(x_{22}) \\ &\qquad +G(x_{22})(a_{11}+b_{12}+c_{21})+x_{22}d(a_{11}+b_{12}+c_{21}) \end{aligned}$$

On other hand by Lemma 3.5, we also have

$$\begin{aligned} &G[(a_{11}+b_{12}+c_{21})x_{22}+x_{22}(a_{11}+b_{12}+c_{21})] \\ &\qquad = G(b_{12}x_{22}+x_{22}c_{21}) \\ &\qquad = G(b_{12}x_{22})+G(x_{22}c_{21}) \\ &\qquad = G(a_{11}x_{22}+x_{22}a_{11})+G(b_{12}x_{22}+x_{22}b_{12})+G(c_{21}x_{22}+x_{22}c_{21}) \\ &\qquad = G(a_{11})x_{22}+a_{11}d(x_{22})+G(x_{22})a_{11}+x_{22}d(a_{11}) \\ &\qquad +G(b_{12})x_{22}+b_{12}d(x_{22})+G(x_{22})b_{12}+x_{22}d(b_{12}) \\ &\qquad +G(c_{21})x_{22}+c_{21}d(x_{22})+G(x_{22})c_{21}+x_{22}d(c_{21}) \end{aligned}$$

It follows that
$G[(a_{11}+b_{12}+c_{21})-G(a_{11})-G(b_{12})-G(c_{21})]x_{22}=0$

Then we can obtain that

$$G[(a_{11}+b_{12}+c_{21})-G(a_{11})-G(b_{12})-G(c_{21})]_{12}=0$$
$$G[(a_{11}+b_{12}+c_{21})-G(a_{11})-G(b_{12})-G(c_{21})]_{22}=0$$

Similarly, we can show that,

$$G[(a_{11}+b_{12}+c_{21})-G(a_{11})-G(b_{12})-G(c_{21})]_{21}=0$$
$$G[(a_{11}+b_{12}+c_{21})-G(a_{11})-G(b_{12})-G(c_{21})]_{11}=0$$

which completes the proof. □

Lemma 3.7. $G(a_{11}+b_{12}+c_{21}+g_{22})=G(a_{11})+G(b_{12})+G(c_{21})+G(g_{22})$

Proof. For any $a_{11} \in R_{11}$,

$$\begin{aligned} &G(a_{11}+b_{12}+c_{21}+g_{22})x_{11}+(a_{11}+b_{12}+c_{21}+g_{22})d(x_{11}) \\ &\quad +G(x_{11})(a_{11}+b_{12}+c_{21}+g_{22})+x_{11}d(a_{11}+b_{12}+c_{21}+g_{22}) \\ &\quad = G[(a_{11}+b_{12}+c_{21}+g_{22})x_{11}+x_{11}(a_{11}+b_{12}+c_{21}+g_{22})] \\ &\quad = G[a_{11}x_{11}+c_{21}x_{11}+x_{11}a_{11}+x_{11}b_{12}] \\ &\quad = G(a_{11}x_{11}++x_{11}a_{11})+G(x_{11}b_{12})+G(c_{21}x_{11}) \\ &\quad = G(a_{11}x_{11}++x_{11}a_{11})+G(b_{12}x_{11}+x_{11}b_{12}) \\ &\quad +G(c_{21}x_{11}+x_{11}c_{21})+G(g_{22}x_{11}+x_{11}g_{22}) \end{aligned}$$

From above expression we find that
$[G(a_{11}+b_{12}+c_{21}+g_{22})-G(a_{11})-G(b_{12})-G(c_{21})-G(g_{22})]x_{11}=0$

we can infer that

$$[G(a_{11}+b_{12}+c_{21}+g_{22})-G(a_{11})-G(b_{12})-G(c_{21})-G(g_{22})]_{11}=0$$
$$[G(a_{11}+b_{12}+c_{21}+g_{22})-G(a_{11})-G(b_{12})-G(c_{21})-G(g_{22})]_{21}=0$$

Similarly we can prove that

$$[G(a_{11}+b_{12}+c_{21}+g_{22})-G(a_{11})-G(b_{12})-G(c_{21})-G(g_{22})]_{12}=0$$
$$[G(a_{11}+b_{12}+c_{21}+g_{22})-G(a_{11})-G(b_{12})-G(c_{21})-G(g_{22})]_{22}=0$$

This completes the proof. □

Theorem 3.8. *Let R be a ring with a nontrivial idempotent e_1 and satisfies:*

i) If $a_{ij}x_{jk} = 0$ for all $x_{jk} \in R_{jk}$, then $a_{ij} = 0$; for $i, j, k \in \{1, 2\}$. Suppose that the mapping $G : R \to R$ satisfies

$$G(ab + ba) = G(a)b + ad(b) + G(b)a + bd(a)$$

for all $a, b \in R$, where d is a Jordan derivation on R, then G is additive. Moreover if R is 2-torsion free, then G is a generalized Jordan derivation.

Proof. For any $a, b \in R$, we write $a = a_{11} + a_{12} + a_{21} + a_{22}$ and $b = b_{11} + b_{12} + b_{21} + b_{22}$. Applying Lemmas 3.1 – 3.7 we have

$$\begin{aligned} G(a+b) &= G(a_{11} + a_{12} + a_{21} + a_{22} + b_{11} + b_{12} + b_{21} + b_{22}) \\ &= G[(a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{21} + b_{21}) + (a_{22} + b_{22})] \\ &= G(a_{11} + b_{11}) + G(a_{12} + b_{12}) + G(a_{21} + b_{21}) + G(a_{22} + b_{22}) \\ &= G(a_{11}) + G(b_{11}) + G(a_{12}) + G(b_{12}) + G(a_{21}) + G(b_{21}) + G(a_{22}) + G(b_{22}) \\ &= G(a_{11} + a_{12} + a_{21} + a_{22}) + G(b_{11} + b_{12} + b_{21} + b_{22}) \\ &= G(a) + G(b) \end{aligned}$$

Hence G is the additive mapping. □

In [4], it has been proved that every Generalized Jordan derivation on a 2-torsion free semi prime ring is a generalized derivation. Using this result and applying Lemma 2.1 we get the following corollary.

Corollary 3.9. *Let R be a 2-torsion free semi prime ring with a nontrivial idempotent and satisfying:*

(P) If $a_{ii}x_{ij} = 0$ for all $x_{ij} \in R_{ij} (i \neq j)$, then $a_{ii} = 0$.

If mapping $G : R \to R$ satisfies

$$G(ab + ba) = G(a)b + ad(b) + G(b)a + bd(a)$$

for all $a, b \in R$, where d is a Jordan derivation on R, then G is additive. Moreover G, is a generalized derivation.

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