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Solving magneto-hydrodynamic squeezing flow between two parallel disks with suction or injection using three classes of polynomials

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Abstract In this paper, three numerical methods based on the Spectral methods to consider magneto-hydrodynamic (MHD) squeezing flow of a viscous incompressible fluid between two parallel disks with suction or injection are introduced. It is assumed that upper disk is movable in upward and downward directions while the lower disk is fixed but permeable. First, the governing partial differential equations by using viable similarity transforms convert to a system of nonlinear ordinary differential equations. Then, the system is solved by collocation method by using polynomials of shifted Chebyshev, Euler, and Bessel. Influence of flow parameters is discussed and to show the efficiency and capability of these methods, our results are compared with each other and with other researchers. Numerical solutions are obtained by using few numbers of collocation points.

1 Introduction

The magneto-hydrodynamic (MHD) squeeze flow between two parallel disks with suction or injection at the porous disk has application in liquid metal lubricants in high-temperature bearings. Several studies have been conducted in this regard after the pioneer works done by Stefan (1874) [1] and Kuzma et al. (1964) [2]. Hamza (1989) [3] has studied a similar flow between two disks in the presence of a magnetic field by using the homotopy perturbation method (HPM). Siddiqui et al. (2008) [4] have studied the two-dimensional MHD squeezing flow between parallel plates by using the HPM. Sajid et al. (2008) [5] have calculated a series solution for unsteady axisymmetric flow and heat transfer over a radially stretching sheet by using the Homotopy analysis method (HAM). Domairy and Aziz (2009) [6] have studied this problem for parallel disk similar by using the HPM. Khuri and Sayfy (2009) [7] have presented a perturbation analysis study of the flow of an electrically conducting power-law fluid in the presence of a uniform transverse magnetic field over a stretching sheet. Joneidi et al. (2011) [8] have studied the mass transfer effect on squeezing flow between parallel disks by using the HAM. Hayat et al. (2012) [9] have studied the influence of heat transfer in the MHD squeezing Flow between parallel disks by using the HAM. Shaban et al. (2013) [10] have studied analyzing magneto-hydrodynamic squeezing flow between two parallel disks with suction or injection by using a hybrid method based on the Tau method and the HAM. Ganji et al. (2014) [11] have studied the MHD squeeze flow between two parallel disks with suction or injection by using the HAM and HPM. Khan et al. (2015) [12] have studied heat transfer analysis for squeezing flow between parallel disks by using the variational iteration method (VIM).

Now, we try to use Spectral methods based on three classes of basic functions (the polynomials of the shifted Chebyshev, Euler, and Bessel) to solve this equation and show the power of Spectral methods for solving this class of equations, and also comparing the results to calculate an appropriate solution of the equation.

The organization of the paper is expressed as follows: The remainder of this section, mathematical formulation of the problem and some basic definitions and theorems are presented. In section 2, the definitions and properties of the polynomials of the shifted Chebyshev, Euler, and

Bessel are expressed. In Section 3, the work methods are explained. Results and discussions of the proposed methods are shown in section 4. Finally, a conclusion is provided.

1.1 Mathematical formulation

Magneto-hydrodynamic squeezing flow between two parallel disks with suction or injection has been mathematically formulated by some researchers, that the totality of this formulation is as follows [5, 12]:

The magneto-hydrodynamic flow of a viscous incompressible fluid is considered in a system consisting:

(i) Two parallel infinite disks distance $h(t) = H(1-at)^{1/2}$ apart.

(ii) Magnetic field is applied normal to the disks to $B_0(1-at)^{1/2}$, and is assumed that there is no induced magnetic field.

(iii) The constant temperatures at $z = 0$ and $z = h(t)$ are T_w and T_h, respectively.

(iv) Upper disk at $z = h(t)$ is moving with velocity $\frac{aH(1-at)^{-1/2}}{2}$ toward or away from the static lower but permeable disk at $z = 0$.

(v) The cylindrical coordinate system (r, ϕ, z) is chosen.

(vi) Rotational symmetry of the flow ($\frac{\partial}{\partial\phi} = 0$) allows to take azimuthal component v of the velocity $V = (u, v, w)$ equal to zero.

Figure 1 shows the geometry of the problem. The governing equation for unsteady two-dimensional flow and heat transfer of a viscous fluid can be written as [5]

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{1.1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial \widehat{p}}{\partial r} + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) - \frac{\sigma}{\rho}B^2(t)u, \tag{1.2}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial \widehat{p}}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right), \tag{1.3}$$

$$\begin{aligned} C_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z}\right) &= \frac{K_0}{\rho}\left(\frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial z^2} + \frac{1}{r}\frac{\partial T}{\partial r} - \frac{u}{r^2}\right) \\ &+ v\left\{2\frac{u^2}{r^2} + \left(\frac{\partial u}{\partial z}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2\right. \\ &+ \left. 2\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2 + \frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\right\}, \end{aligned} \tag{1.4}$$

Auxiliary conditions are [9, 12]

$$\begin{aligned} &u = 0, \quad w = \frac{dh}{dt} \quad at\ z = h(t) \\ &u = 0, \quad w = -w_0 \quad at\ z = 0 \\ &T = T_w \quad at\ z = 0 \\ &u = T_h \quad at\ z = h(t). \end{aligned}$$

u and w are the velocity components in r and z directions respectively, μ is dynamic viscosity, $\widehat{p}$ is the pressure and ρ is density. Further T denotes temperature, K_0 is thermal conductivity, C_p is specific heat, v is kinematic viscosity and w_0 is suction/injection velocity.

Using the following transformations [9, 12]

$$u=\frac{ar}{2(1-at)}f'(\eta),\quad w=-\frac{aH}{\sqrt{1-at}}f'(\eta),$$
$$B(t)=\frac{B_0}{\sqrt{1-at}},\quad \eta=\frac{z}{H\sqrt{1-at}},\quad \theta=\frac{T-T_h}{T_w-T_h}, \tag{1.5}$$

into Eqs. (1.1)-(1.4) and eliminating pressure terms of the resulting equations, we obtain

$$f''''-S\left(\eta f'''+3f''-2ff''\right)-M^2f''=0, \tag{1.6}$$
$$\theta''+S\,Pr\left(2f\theta'-\eta\theta'\right)+Pr\,Ec(f''^2+12\delta^2f'^2)=0, \tag{1.7}$$

with the boundary conditions

$$f(0)=A,\quad f'(0)=0,\quad \theta(0)=1,$$
$$f(1)=\frac{1}{2},\quad f'(1)=0,\quad \theta(1)=0, \tag{1.8}$$

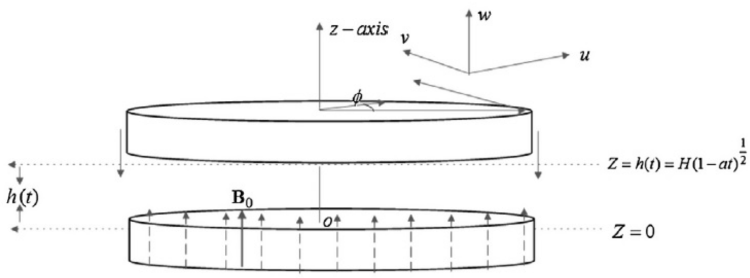
where S denotes the squeeze number, A is suction/injection parameter, M is Hartman number, Pr is Prandtl number, Ec is modified Eckert number, and δ denotes the dimensionless length defined as

$$S=\frac{aH^2}{2v},\quad M^2=\frac{aB_0^2H^2}{v},\quad Pr=\frac{\mu C_p}{K_0},$$
$$Ec=\frac{1}{C_p(T_w-T_h)}\left(\frac{ar}{2(1-at)}\right)^2,\quad \delta^2=\frac{H^2(1-at)}{r^2}, \tag{1.9}$$

Skin friction coefficient and the Nusselt number are defined in terms of variables (1.5) as

$$\frac{H^2}{r^2}Re_rC_{fr}=f''(1),\qquad (1-at)^{1/2}Nu=-\theta'(1), \tag{1.10}$$
$$Re_r=\frac{raH(1-at)^{1/2}}{2v}. \tag{1.11}$$

Figure 1. Geometry of the problem.



1.2 Basical definitions

In this section, some basic definitions and theorems which are useful for our methods are introduced [13].

Definition 1. For any real function $f(x)$, $x>0$, if there exists a real number $p>\mu$, such that $f(x)=x^pf_1(x)$, where $f_1(x)\in C(0,\infty)$, is said to be in space C_μ, $\mu\in\Re$, and it is in the space C_μ^n if and only if $f^n\in C_\mu,\ n\in N$.

Definition 2. Suppose that $f(x),\ g(x) \in C(0,1)$ and $w(x)$ is a weight function, then

$$\begin{aligned} \| f(x) \|_w^2 &= \int_0^1 f^2(x)w(x)dx, \\ \langle f(x), g(x)\rangle_w &= \int_0^1 f(x)g(x)w(x)dx. \end{aligned}$$

Theorem 1. (*Taylor's formula*) Suppose that $f(x) \in C[0,1]$ and $D^k f(x) \in C[0,1]$, where $k = 0, 1, ..., m$. Then we have

$$f(x) = \sum_{i=0}^{m-1} \frac{x^i}{i!} D^i f(0^+) + \frac{x^m}{m!} D^m f(\xi), \tag{1.12}$$

with $0 < \xi \leqslant x,\ \forall x \in [0,1]$. And thus

$$|f(x) - \sum_{i=0}^{m-1} \frac{x^i}{i!} D^i f(0^+)| \leqslant M \frac{x^m}{m!}, \tag{1.13}$$

where $M \geqslant |D^m f(\xi)|$.
Proof: See Ref. [14]. ■

2 Polynomials of shifted Chebyshev, Euler and Bessel

In this section, polynomials of shifted Chebyshev, Euler, and Bessel are defined.

2.1 Definition of Shifted Chebyshev Polynomials (SCP)

The Chebyshev polynomials are frequently used in the polynomial approximation, Gauss-quadrature integration, integral and differential equations and Spectral methods. For these reasons, many researchers have used these polynomials in their research [15, 16, 17, 18].

By transformation $t = 2x - 1$ on the Chebyshev polynomials of the first kind, the shifted Chebyshev orthogonal polynomials in interval $[0,1]$ are introduced, that be denoted by $T_n^*(x) = T_n(2x-1)$ and they used to solve many differential equations [19, 20, 21, 22].

The $T_n^*(x)$ can be obtained using the recursive relation as follows:

$$\begin{aligned} &T_0^*(x) = 1 \quad , \quad T_1^*(x) = 2x - 1, \\ &T_{n+1}^*(x) = (4x-2)\, T_n^*(x) - \, T_{n-1}^*(x), \qquad n = 1, 2, \cdots . \end{aligned}$$

The analytical form of $T_n^*(x)$ of degree n is given by

$$T_n^*(x) = \sum_{k=0}^{n} (-1)^k \frac{n 2^{2k} (n+k-1)!}{(n-k)!(2k)!} (1-x)^k \tag{2.1}$$

The weight function for the SCPs is $w(x) = \frac{1}{\sqrt{x-x^2}}$, and are orthogonal in the interval $(0,1)$:

$$\int_0^1 T_n^*(x)\, T_m^*(x) w(x)dx = \frac{\pi}{2} c_n \delta_{mn}, \tag{2.2}$$

where δ_{mn} is Kronecker delta, $c_0 = 2$, and $c_n = 1$ for $n \geq 1$.

2.2 Definition of Euler Polynomials (EP)

Euler polynomials are a class of special functions which are applied to solve the number of problems in physics, engineering, mathematics, and etc. [23, 24, 25]. The analytical form and recurrence relation of Euler polynomials of the first kind are defined as follows [26]:

$$E_n(x) = \frac{1}{n+1} \sum_{k=1}^{n+1} (2 - 2^{k+1}) \binom{n+1}{k} B_k x^{n+1-k}, \quad n = 0, 1, ... \tag{2.3}$$

$$\sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x) = 2x^n, \quad n = 1, 2, ... \tag{2.4}$$

where $x \in [0, 1]$, $E_0(x) = 1$, and $B_k = B_k(0)$ are the Bernoulli numbers for $k = 0, 1, ..., n$.

Euler polynomials can be expressed by the Bernoulli polynomials as[27]

$$E_n(x) = \sum_{k=0}^{n} \frac{-2}{k+1} \binom{n}{k} E_{k+1}(0) B_{n-k}(x).$$

where $E_n = 2^n E_n(\frac{1}{2}),\ n \in \mathbb{N}$ call Euler numbers. These two classes of the polynomials have many similar properties [28, 29].

2.3 Definition of Bessel Polynomials (BP)

Bessel functions are first defined by the Daniel Bernoulli on heavy chains (1738) and then generalized by Friedrich Bessel. More general Bessel functions were studied by Leonhard Euler in (1781) and in his study of the vibrating membrane in (1764) [30, 31].

Bessel differential equation of order $n \in \mathbb{R}$ is:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0, \ \ x \in (-\infty, \infty). \tag{2.5}$$

One of the solutions of equation (2.5) by applying frobenius' method as follows [32]:

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!} (\frac{x}{2})^{2r+n}, \tag{2.6}$$

where series (2.6) is convergent for all $x \in (-\infty, \infty)$.

Bessel functions and polynomials are used to solve the number of problems in physics, engineering, mathematics, and etc. [33, 34, 35, 36].

Bessel polynomials have been introduced as follows [37]:

$$B_n(x) = \sum_{r=0}^{[\frac{m-n}{2}]} \frac{(-1)^r}{r!(n+r)!} (\frac{x}{2})^{2r+n}, \quad x \in [0, 1]. \tag{2.7}$$

where $n \in \mathbb{N}$, and m is the number of the basis of Bessel polynomials.

2.4 Approximation of functions

Suppose that $\phi_n(x)$ is one of the polynomials of $T^*_n(x),\ E_n(x)$, or $B_n(x)$.

Any function $y(x) \in C[0, 1]$ can be expanded as follows:

$$y(x) = \sum_{n=0}^{\infty} a_n \ \phi_n(x).$$

But in the numerical methods, we have to use first m-terms of polynomials and approximate $y(x)$:

$$y(x) \approx y_m(x) = \sum_{n=0}^{m-1} a_n \ \phi_n(x) = F^T \Phi(x), \tag{2.8}$$

with

$$F = [a_0, a_1, ..., a_{m-1}]^T, \tag{2.9}$$

$$\Phi(x) = [\phi_0(x),\ \phi_1(x), ...,\ \phi_{m-1}(x)]^T. \tag{2.10}$$

The coefficients F can obtain by the inner product:

$$\langle y_m(x), \Phi^T(x)\rangle_w = \langle F^T \Phi(x), \Phi^T(x)\rangle_w,$$

thus

$$F^T = \langle y_m(x), \Phi^T(x)\rangle_w \ \langle \Phi(x), \Phi^T(x)\rangle_w^{-1}.$$

2.5 Convergence analysis

The following theorem shows that by increasing m, the approximation solution $f_m(x)$ is convergent to $f(x)$ exponentially.

Theorem 2. Suppose that $D^k f(x) \in C[0,1]$ for $k = 0, 1, ..., m$, and E_m is the subspace generated by $\{\phi_0(x),\ \phi_1(x), ...,\ \phi_{m-1}(x)\}$. If $f_m(x) = F^T\Phi(x)$ is the best approximation to $f(x)$ from E_m, then the error bound is presented as follows

$$\| f(x) - f_m(x) \|_w \leqslant \begin{cases} \frac{\sqrt{\pi}M}{2^m m! \sqrt{m!}} & for\ SCPs \\ \\ \frac{M}{m!\sqrt{2m+1}} & for\ EPs\ and\ BPs \end{cases} \tag{2.11}$$

where $M \geqslant |D^m f(x)|,\ \ x \in [0,1]$.
Proof. By theorom 1, we have $y = \sum_{i=0}^{m-1} \frac{x^i}{i!} D^i f(0^+)$ and

$$|f(x) - y(x)| \leqslant M\frac{x^m}{m!}.$$

Since the best approximation to $f(x)$ from E_m is $F^T\Phi(x)$, and $y(x) \in E_m$, thus

$$\begin{aligned} \| f(x) - f_m(x) \|_w^2 &\leqslant \| f(x) - y(x) \|_w^2 \\ &\leqslant \frac{M^2}{m!^2} \int_0^1 x^{2m}\ w(x)\ dx \end{aligned}$$

where the weight function for SCPs is $w(x) = \frac{1}{\sqrt{x - x^2}}$ and for EPs and BPs is $w(x) = 1$, thus by integration of the above equation, the Eq. (2.11) can be proved. ■

3 Application of the Methods

Different numerical methods have been introduced for solving problems on various domains such as Finite difference method [38], Finite element method [38], Meshfree methods [39, 40], and Spectral methods [41, 42, 43].

In this section, the SCPs, EPs, and BPs collocation methods are applied to solve a system of equations (1.6) and (1.7) and the boundary conditions (1.8).

Suppose that $\phi_n(x)$ is one of the polynomials of $T_n^*(x),\ E_n(x)$, or $B_n(x)$.

By the Eq. (2.8), we suppose that

$$\widehat{f_m}(\eta) = \sum_{n=0}^{m-1} a_n\ \phi_n(\eta) = F^T\Phi(\eta), \tag{3.1}$$

$$\widehat{\theta_m}(\eta) = \sum_{n=0}^{m-1} b_n\ \phi_n(\eta) = C^T\Phi(\eta). \tag{3.2}$$

For satisfying the boundary conditions (1.8):

$$\begin{aligned} f_m(\eta) &= A + 0.5(3-6A)\eta^2 + (-1+2A)\eta^3 + \eta^2(\eta-1)^2\ \widehat{f_m}(\eta), \\ \theta_m(\eta) &= 1 - \eta + \eta(\eta-1)\ \widehat{\theta_m}(\eta). \end{aligned}$$

To apply the collocation method, the residual functions are constructed by substituting $f_m(\eta)$ and $\theta_m(\eta)$ in Eqs. (1.6) and (1.7):

$$\begin{aligned} Res1(\eta) &= f_m'''' - S\,(\eta f_m''' + 3f_m'' - 2f_m f_m'') - M^2 f_m'', \\ Res2(\eta) &= \theta_m'' + S\ Pr\,(2f_m\theta_m' - \eta\theta_m') + Pr\ Ec(f_m''^2 + 12\delta^2 f_m'^2). \end{aligned}$$

The equations for obtaining the coefficient $\{a_i\}_{i=0}^{m-1}$ and $\{b_i\}_{i=0}^{m-1}$ in the Eqs. (3.1) and (3.2) arise from equalizing $Res1(\eta)$ and $Res2(\eta)$ to zero on m collocation points:

$$Res1(\eta_i) = 0, \quad i = 0, 1, ..., m-1,$$
$$Res2(\eta_i) = 0, \quad i = 0, 1, ..., m-1,$$

In this study, the roots of the SCPs are used in the interval $[0, 1]$ as collocation points (i.e. $\eta_i = (1 + \cos(\frac{(2i-1)\pi}{2m}))/2$, $i = 0, 1, ..., m-1$). By solving the obtained set of equations, the approximating functions $f_m(\eta)$ and $\theta_m(\eta)$ are obtained.

4 Results and Discussions

In this study, the problem of magneto-hydrodynamic squeezing flow of a viscous incompressible fluid between two parallel disks with suction or injection by the SCPs, EPs, and BPs collocation methods is considered. In this section, the effects of different flow parameters are discussed such as the squeeze number, Hartmann number, Prandtl and Eckert numbers on the velocity and temperature distributions in both cases of the suction and injection.

4.1 Suction case $A > 0$

The influence of porosity parameter A on the radial and axial velocity are displayed in Fig. 4(a). By the Fig. 4(a) can see that the axial velocity increases with increasing values of A, and Boundary layer thickness is a decreasing function of A. Due to the permeability of upper disk when suction plays a dominant role it allows the fluid to flow near the walls which result in a thinner boundary layer. Effects of deformation parameter S are displayed in Fig. 4(c). $S > 0$ corresponds to the movement of the upper disk away from the lower static disk while $S < 0$ stands for its fall toward the lower disk. By the Fig. 4(c) can see that the absolute of $f'(\eta)$ in interval of $0 < \eta \leq 0.4$ increases for increasing S, and decreases in interval of $0.4 < \eta \leq 1$. Effects of Hartman parameter M are displayed in Fig. 4(e). By the Fig. 4(e) can see that the absolute of $f'(\eta)$ in intervals of $0 < \eta \leq 0.275$ and $0.725 < \eta \leq 1$ increases for increasing M, and decreases in interval of $0.275 < \eta \leq 0.725$.

Figs. 5(a),5(c),5(e),6(a),6(c) have shown the effects of flow parameters (in suction case) on the temperature profile. Fig. 5(a) shows $\theta(\eta)$ to be an increasing function of A, on the other hand, the thermal boundary layer becomes thinner with raising A. Consequences of increasing S are presented in Fig. 5(c) according to which temperature profile falls with surging S. The influence of the Pr number of the temperature distribution is displayed in Fig. 5(e) which declares $\theta(\eta)$ to be directly variant with Pr. On the other hand, the thermal boundary layer is inversely proportional to Pr, it is due to the fact that for higher Pr low thermal conductivity is observed which results in the narrow thermal boundary layer. Higher values of Prandtl number are associated with large viscosity oils while lowers Pr corresponds to low viscosity fluids. Figs. 6(a) and 6(c) indicate that Eckert number Ec and dimensionless length δ have similar effects on the temperature profile as Pr.

4.2 Injection case $A < 0$

Figs. 4(b),4(d),4(f),5(b),5(d),5(f),6(b),6(d) have shown the effects of physical parameters on velocity and temperature distributions in the case of injection. It is observed that on velocity profiles, effects of involved parameters are opposite to the ones discussed earlier in suction case. Behavior of temperature distribution, however, remains invariant in both cases of the suction and injection.

To demonstrate the effectiveness of the SCPs, EPs, and BPs collocation methods, the results are compared with each other and the variational iteration method (VIM) [12].

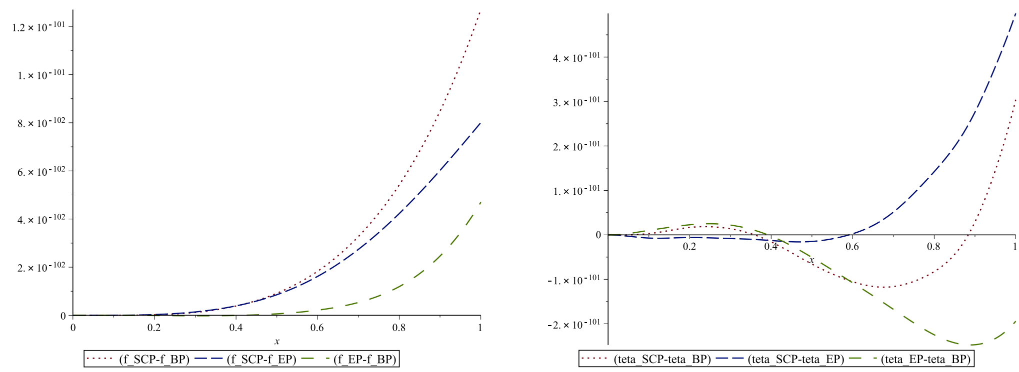
Tables 1-4 shows the obtained values for $f(\eta)$, $f'(\eta)$, $\theta(\eta)$, and $\theta'(\eta)$ by the SCPs, EPs, and BPs collocation methods, and the comparison of them with VIM [12] for $m = 15$ and $S = 0.1$, $M = 0.2$, $A = 0.1$, $Pr = 0.3$, $Ec = 0.2$, and $\delta = 0.1$.

Figures 2 shows the absolute errors between approximation solutions by the SCPs, EPs, and BPs collocation methods for $m = 15$ and $S = 0.1$, $M = 0.2$, $A = 0.1$, $Pr = 0.3$, $Ec =$

0.2, and $\delta = 0.1$ with $Digits = 100$, i.e $(f_{SCP} - f_{BP})$, $(f_{SCP} - f_{EP})$, $(f_{EP} - f_{BP})$ and $(\theta_{SCP} - \theta_{BP})$, $(\theta_{SCP} - \theta_{EP})$, $(\theta_{EP} - \theta_{BP})$. As can be seen, the results are exactly the same in three methods.

Figures 3 shows the residual errors for approximation solutions by the SCPs, EPs, and BPs collocation methods with $m = 15$ and $S = 0.1,\ M = 0.2,\ A = 0.1,\ Pr = 0.3,\ Ec = 0.2$, and $\delta = 0.1$. As can be seen, the results are exactly the same in three methods.

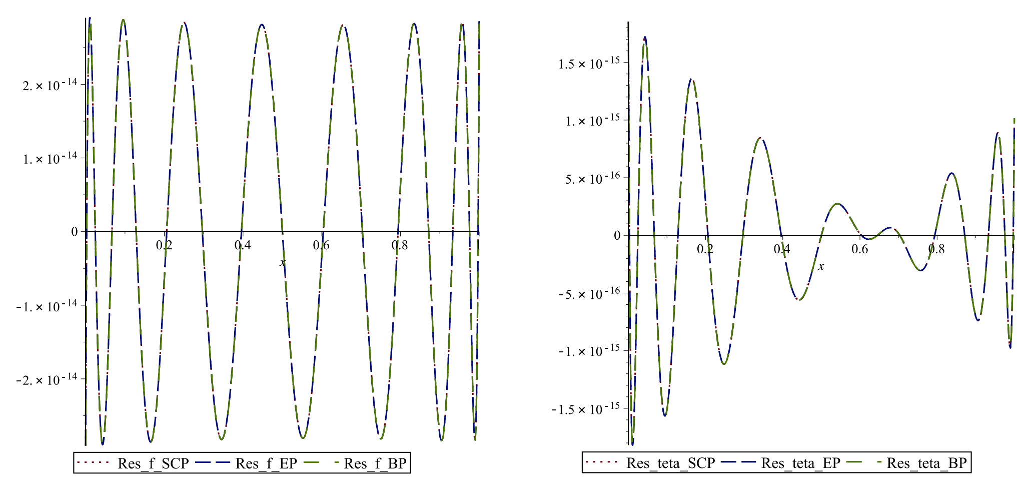
Tables 5 and 6 display the numerical values of Nusselt number and coefficient of skin friction for different values of flow parameters. Absolute of skin friction is found to be a decreasing function of permeability parameter A. This observation is important due to its industrial implication since the amount of energy required to squeeze the disk can be reduced by increasing values of A. As discussed earlier the suction parameter A decreases the thermal boundary layer thickness hence at the plates we have a higher rate of heat transfer.



(a) The absolute errors for $f(\eta)$

(b) The absolute errors for $\theta(\eta)$

Figure 2. Absolute errors between approximation solutions by the SCPs, EPs, and BPs collocation methods with $m = 15, S = 0.1, M = 0.2, a = 0.1, Pr = 0.3, Ec = 0.2$, and $\delta = 0.1$.



(a) The residual errors for $f(\eta)$

(b) The residual errors for $\theta(\eta)$

Figure 3. The residual errors for approximation solutions by the SCPs, EPs, and BPs collocation methods with $m = 15$ and $S = 0.1,\ M = 0.2,\ a = 0.1,\ Pr = 0.3,\ Ec = 0.2$, and $\delta = 0.1$.

5 Conclusion

In this paper, we have considered the collocation method by using the shifted Chebyshev, Euler and Bessel polynomials to study a magneto-hydrodynamic squeezing flow between two parallel

Table 1. Obtained values of $f(\eta)$ by the SCPs, EPs, and BPs collocation methods with $m = 15$

x	SCP Method	BP Method	EP Method	Res(t)
0.1	0.11125941997424494438017399909951	0.11125941997424494438017399909951	0.11125941997424494438017399909951	2.8e-14
0.2	0.14175294253230344145947124748 25	0.14175294253230344145947124748 25	0.14175294253230344145947124748 25	6.4e-15
0.3	0.18660427988206843778199765582 21	0.18660427988206843778199765582 21	0.18660427988206843778199765582 21	3.1e-15
0.4	0.24099026100735903704201032228 66	0.24099026100735903704201032228 66	0.24099026100735903704201032228 66	3.4e-15
0.5	0.30012381254370284921421242417 21	0.30012381254370284921421242417 21	0.30012381254370284921421242417 21	2.8e-40
0.6	0.35923799843807410124424910231 44	0.35923799843807410124424910231 44	0.35923799843807410124424910231 44	3.9e-15
0.7	0.41357059263043973508679055564 75	0.41357059263043973508679055564 75	0.41357059263043973508679055564 75	3.0e-15
0.8	0.45834867996684510759528479915 79	0.45834867996684510759528479915 79	0.45834867996684510759528479915 79	6.3e-15
0.9	0.48877278336314792953396729088 67	0.48877278336314792953396729088 67	0.48877278336314792953396729088 67	2.7e-14

Table 2. Obtained values of $f'(\eta)$ by the SCPs, EPs, and BPs collocation methods with $m = 15$ and the comparison of them with the variational iteration method (VIM) [12]

x	SCP Method	BP Method	EP Method	VIM[12]
0.1	0.216945117087951199014817966776	0.216945117087951199014817966776	0.216945117087951199014817966776	———
0.2	0.384801280290557318104914501993	0.384801280290557318104914501993	0.384801280290557318104914501993	0.384801
0.3	0.504189671461316244328348817498	0.504189671461316244328348817498	0.504189671461316244328348817498	———
0.4	0.575554143054331140256127880760	0.575554143054331140256127880760	0.575554143054331140256127880760	0.575554
0.5	0.599174553522456296304445305255	0.599174553522456296304445305255	0.599174553522456296304445305255	———
0.6	0.575174675564395436411845094392	0.575174675564395436411845094392	0.575174675564395436411845094392	0.575174
0.7	0.503524982928277574212379692958	0.503524982928277574212379692958	0.503524982928277574212379692958	———
0.8	0.384040432902588459379779642740	0.384040432902588459379779642740	0.384040432902588459379779642740	0.384040
0.9	0.216373183738477512380691010520	0.216373183738477512380691010520	0.216373183738477512380691010520	———

Table 3. Obtained values of $\theta(\eta)$ by the SCPs, EPs, and BPs collocation methods with $m = 15$ and the comparison of them with the variational iteration method (VIM) [12]

x	SCP Method	BP Method	EP Method	VIM[12]	Res(t)
0.1	0.904177630060793299491172164	0.904177630060793299491172164	0.904177630060793299491172164	———	7.2e-16
0.2	0.806144282850332131149518438	0.806144282850332131149518438	0.806144282850332131149518438	0.806144	1.3e-16
0.3	0.706865303890395656919987319	0.706865303890395656919987319	0.706865303890395656919987319	———	5.1e-17
0.4	0.607022034290870021379068128	0.607022034290870021379068128	0.607022034290870021379068128	0.607022	4.0e-17
0.5	0.507022003350412439865868591	0.507022003350412439865868591	0.507022003350412439865868591	———	4.4e-40
0.6	0.407003862839112678343074305	0.407003862839112678343074305	0.407003862839112678343074305	0.407004	8.7e-18
0.7	0.306837646669244272048476401	0.306837646669244272048476401	0.306837646669244272048476401	———	6.6e-18
0.8	0.206120555470330403373570615	0.206120555470330403373570615	0.206120555470330403373570615	0.206121	4.3e-17
0.9	0.104168093326752655615459603	0.104168093326752655615459603	0.104168093326752655615459603	———	3.1e-16

Table 4. Obtained values of $\theta'(\eta)$ by the SCPs, EPs, and BPs collocation methods with $m = 15$

x	SCP Method	BP Method	EP Method
0.1	-0.97113122635535035841604504001861 5	-0.97113122635535035841604504001861 5	-0.97113122635535035841604504001861 5
0.2	-0.98792951618139099473865712136847 8	-0.98792951618139099473865712136847 8	-0.98792951618139099473865712136847 8
0.3	-0.99651616959526725661070098039834 7	-0.99651616959526725661070098039834 7	-0.99651616959526725661070098039834 7
0.4	-0.99967081552497800055685864313611 5	-0.99967081552497800055685864313611 5	-0.99967081552497800055685864313611 5
0.5	-1.00009875557472173402900709308513 9	-1.00009875557472173402900709308513 9	-1.00009875557472173402900709308513 9
0.6	-1.00047996629431986105601142541783 2	-1.00047996629431986105601142541783 2	-1.00047996629431986105601142541783 2
0.7	-1.00351422528851105955359027720378 9	-1.00351422528851105955359027720378 9	-1.00351422528851105955359027720378 9
0.8	-1.01196610359883664633853431652934 5	-1.01196610359883664633853431652934 5	-1.01196610359883664633853431652934 5
0.9	-1.02871356585601513266548338947893 0	-1.02871356585601513266548338947893 0	-1.02871356585601513266548338947893 0

disks with suction or injection and the velocity and temperature distributions. In order to test the applicability, accuracy, and efficiency of these methods, we have compared our results with each other and the obtained values by Khan et al. [12], as shown, the accuracy is very good. The following key points are useful:

(i) Numerical solutions are exactly the same in three methods, and them computing power is greater than other methods, such as VIM, for this problem.

(ii) To calculate the approximation solutions of suitable from the velocity and temperature distributions, we do not need to use of many collocation points.

(iii) The behavior of all physical parameters is opposite on velocity profile in the cases of suction and injection. On the other hand, the effect of parameters remains similar on temperature profile in both cases of suction and injection.

(iv) Temperature $\theta(\eta)$ is directly proportional to the Prandtl number.

(v) These methods are the good experience and method for the other sciences.

Table 5. Values of skin friction coefficient $\frac{H^2}{r^2} Re_r C_{fr}$ and the Nusselt number $(1-at)^{1/2} Nu$ for different values of A, S, and M by the SCPs, EPs, and BPs collocation methods, $m = 15$, and the comparison of them with VIM.

A	S	M	$\frac{H^2}{r^2} Re_r C_{fr}$	VIM	Pr	Ec	δ	$(1-at)^{1/2} Nu$	VIM [12]
-0.1	0.1	0.2	-3.62305946165224495208	-3.62306	0.3	0.2	0.1	1.13169733124754295412	1.1317
0.0			-3.01553080658883492711	-3.01553				1.09056149128240299921	1.0906
0.1			-2.40947876984344076848	-2.40948				1.05680415677248143845	1.0568
0.2			-1.80490151977390173962	-1.80490				1.03039740554173366714	1.0304
0.1	0.01		-2.40238773483842290128	-2.40239				1.05814239736334546725	1.0581
	0.2		-2.41735178309209312967	-2.41735				1.05531828028348476386	1.0553
	0.3		-2.42521822337204163843	-2.42522				1.05383355199884105578	1.0538
	0.1	0.01	-2.40788673402023191920	-2.40789				1.05680386400966663099	1.0568
		0.1	-2.40828183504393844683	-2.40828				1.05680393466980411984	1.0568
		0.2	-2.40947876984344076848	-2.40948				1.05680415677248143845	1.0568

Table 6. Values of Nusselt number $(1-at)^{1/2} Nu$ for different values of Pr, Ec, and δ when $A = 0.1,\ S = 0.1,\ M = 0.2$ by the SCPs, EPs, and BPs collocation methods with $m = 15$ and the comparison of them with the VIM.

Pr	Ec	δ	$(1-at)^{1/2} Nu$	VIM [12]
0.0	0.2	0.1	1.00000000000000000000	1.00000
0.1			1.01893685003010788664	1.01894
0.2			1.03787156909761297343	1.03787
0.3	0.0		0.99859957004178043188	0.99860
	0.2		1.05680415677248143845	1.05680
	0.4		1.11500874350318244502	1.11501
	0.3	0.0	1.08487154864359339485	1.08487
		0.5	1.11074408599955706710	1.11074
		1.0	1.18836169806744808387	1.18836

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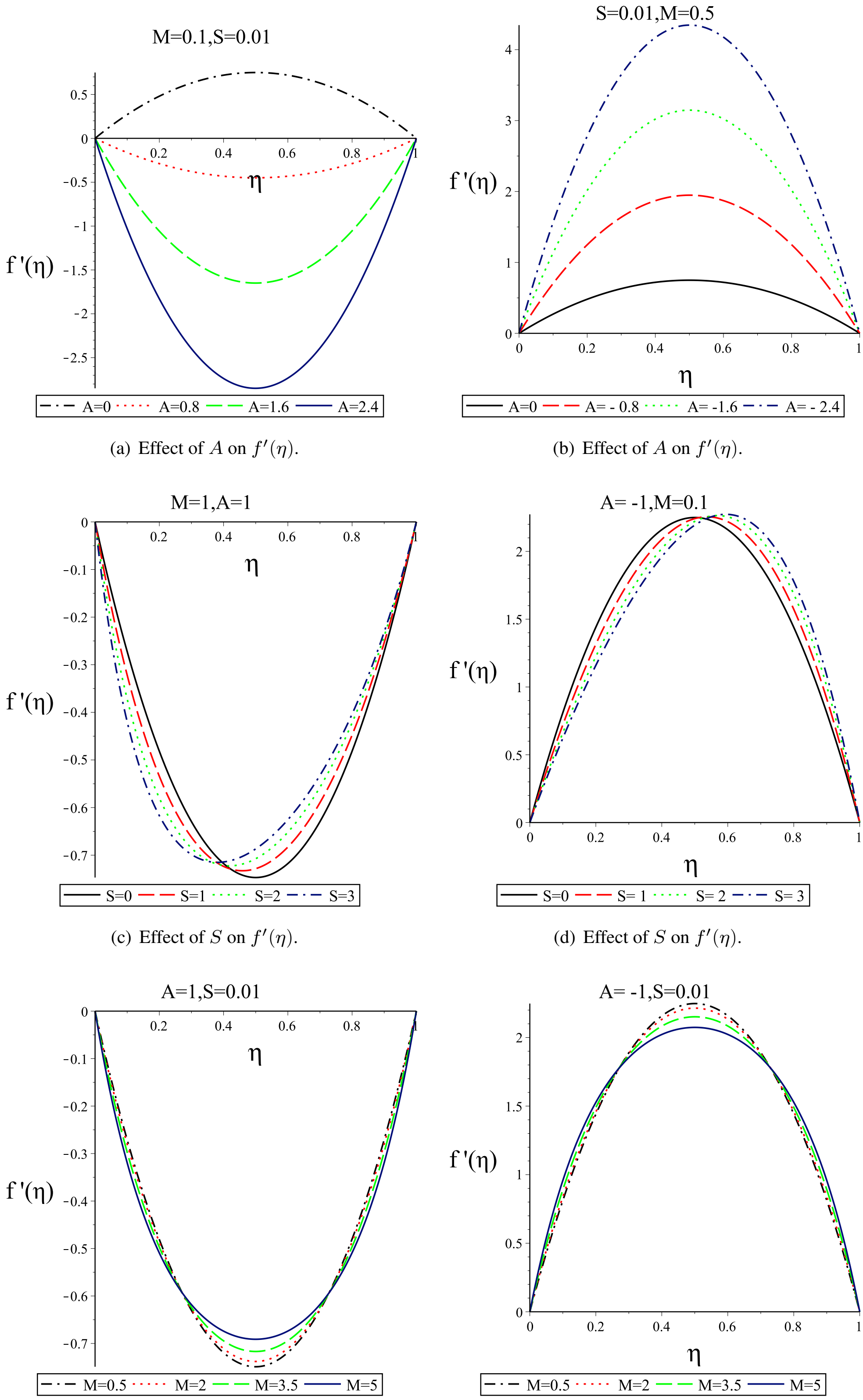


(a) Effect of A on $f'(\eta)$.

(b) Effect of A on $f'(\eta)$.

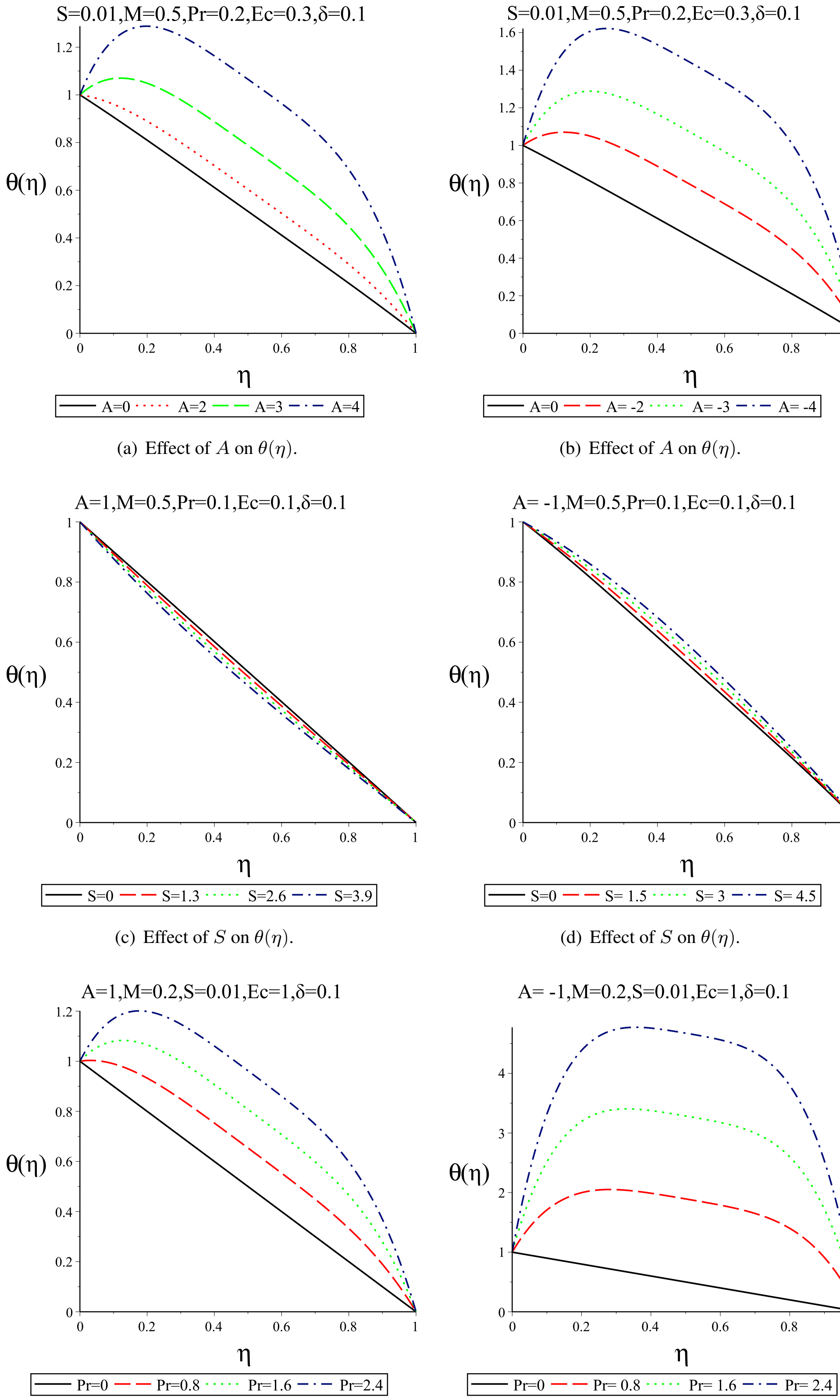
(c) Effect of S on $f'(\eta)$.

(d) Effect of S on $f'(\eta)$.

(e) Effect of M on $f'(\eta)$.

(f) Effect of M on $f'(\eta)$.

Figure 4. Effect of A, S, and M on $f'(\eta)$ for both the suction and injection cases



(a) Effect of A on $\theta(\eta)$.

(b) Effect of A on $\theta(\eta)$.

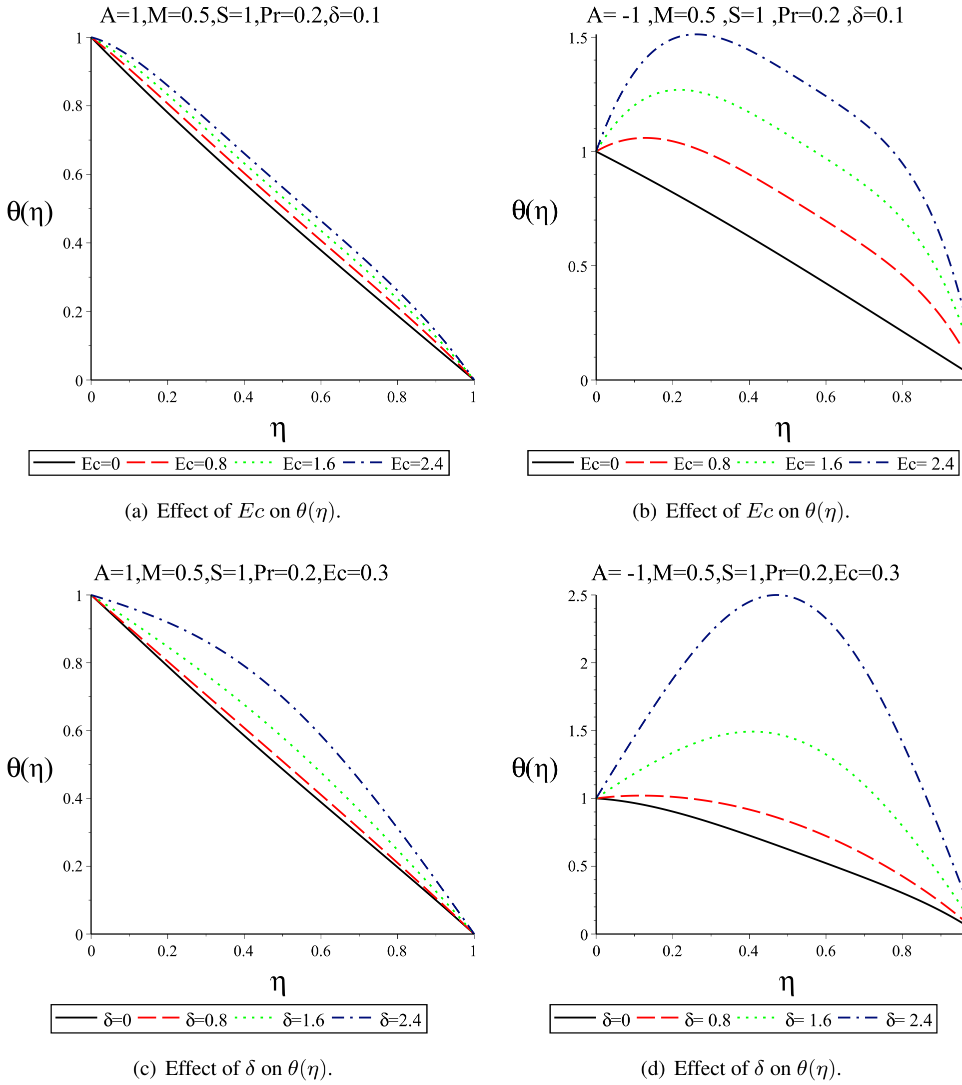
(c) Effect of S on $\theta(\eta)$.

(d) Effect of S on $\theta(\eta)$.

(e) Effect of Pr on $\theta(\eta)$.

(f) Effect of Pr on $\theta(\eta)$.

Figure 5. Effect of A, S, and Pr on $\theta(\eta)$ for both the suction and injection cases



(a) Effect of Ec on $\theta(\eta)$.

(b) Effect of Ec on $\theta(\eta)$.

(c) Effect of δ on $\theta(\eta)$.

(d) Effect of δ on $\theta(\eta)$.

Figure 6. Effect of Ec and δ on $\theta(\eta)$ for both the suction and injection cases

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