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MEASURE OF GROWTH OF ENTIRE FUNCTIONS FROM THE VIEW POINT OF THEIR RELATIVE L^*-TYPE AND RELATIVE L^*-WEAK TYPE

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Abstract In this paper we study some growth properties of entire functions on the basis of relative L^*-type and relative L^*-weak type of an entire function with respect to another entire function.

1 Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function corresponding to entire f is defined as $M_f(r) = \max\{|f(z)| : |z| = r\}$. To start our paper we just recall the following definitions:

Definition 1.1. The *order* ρ_f and *lower order* λ_f of an entire function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r},$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

Definition 1.2. The *type* σ_f and *lower type* $\overline{\sigma}_f$ of an entire function f are defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}},\ 0 < \rho_f < \infty\ .$$

Datta and Jha [4] introduced the definition of *weak type* of an entire function of finite positive *lower order* in the following way:

Definition 1.3. [4] The *weak type* τ_f and the growth indicator $\overline{\tau}_f$ of an entire function f of finite *positive lower order* λ_f are defined by

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}},\ 0 < \lambda_f < \infty\ .$$

Somasundaram and Thamizharasi [6] introduced the notions of L-order and L-type for entire function where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalised concept for L-order and L-type for entire functions are L^*-order and L^*-type respectively. Their definitions are as follows:

Definition 1.4. [6] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

Definition 1.5. [6] The L^*-type $\sigma_f^{L^*}$ and L^*-lower type $\overline{\sigma}_f^{L^*}$ of an entire function f are defined as

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \quad \text{and} \quad \overline{\sigma}_f^{L^*} = \liminf_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}},\ 0 < \rho_f^{L^*} < \infty\,.$$

In order to determine the relative growth of two entire functions of same non zero finite L^*-lower order one may define the L^*-weak type in the following way:

Definition 1.6. The L^*-weak type $\tau_f^{L^*}$ of an entire function f is defined as follows:

$$\tau_f^{L^*} = \liminf_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}}\ ,\ 0 < \lambda_f^{L^*} < \infty\,.$$

Likewise the growth indicator $\overline{\tau}_f^{L^*}$ of an entire function f can be defined in the following manner:

$$\overline{\tau}_f^{L^*} = \limsup_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}}\ ,\ 0 < \lambda_f^{L^*} < \infty\,.$$

If an entire function g is non-constant then $M_g(r)$ is strictly increasing and continuous and its inverse $M_g{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} M_g^{-1}(s) = \infty$. In the line of Somasundaram and Thamizharasi [6] and Bernal [1], one may define the relative L^*-order of an entire function in the following manner :

Definition 1.7. {[3], [5]} The relative L^*-order $\rho_g^{L^*}(f)$ and relative L^*-lower $\lambda_g^{L^*}(f)$ of an entire function f with respect to another entire function g are defined as

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]}\,.$$

In order to determine the relative growth of two entire functions having same non zero finite relative L^*-order with respect to another entire function, one may define the concept of the relative L^*-type and relative L^*-lower type in the following manner:

Definition 1.8. The relative L^*-type $\sigma_g^{L^*}(f)$ and relative L^*-lower type $\overline{\sigma}_g^{L^*}(f)$ of an entire function f with respect to g are defined as follows:

$$\sigma_g^{L^*}(f) = \limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}} \quad \text{and} \quad \overline{\sigma}_g^{L^*}(f) = \liminf_{r\to\infty} \frac{M_g^{-1} M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}},\ 0 < \rho_g^{L^*}(f) < \infty\,.$$

Analogusly, in order to determine the relative growth of two entire functions having same non zero finite relative L^*-lower order with respect to another entire function, one can define the relative L^*-weak type in the following way:

Definition 1.9. The relative L^*-weak type $\tau_g^{L^*}(f)$ of an entire function f with respect to g of finite positive relative L^*-lower order $\lambda_g^{L^*}(f)$ is defined as:

$$\tau_g^{L^*}(f) = \liminf_{r\to\infty} \frac{M_g^{-1} M_f(r)}{\left[re^{L(r)}\right]^{\lambda_g^{L^*}(f)}},\ 0 < \lambda_g^{L^*}(f) < \infty.$$

Similarly, the growth indicator $\overline{\tau}_g^{L^*}(f)$ of an entire function f with respect to another entire function g can be defined in the following manner:

$$\overline{\tau}_g^{L^*}(f) = \limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{\left[re^{L(r)}\right]^{\lambda_g^{L^*}(f)}},\ 0 < \lambda_g^{L^*}(f) < \infty.$$

In the paper we study some relative growth properties of entire functions with respect to another entire function on the basis of relative L^*-type and relative L^*-weak type. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [7].

2 Lemma

In this section we present a lemma due to Datta et al. [2]:

Lemma 2.1. *[2] Let f and g be any two entire functions such that $0 \leq \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and $0 \leq \lambda_g \leq \rho_g < \infty$. Then*

$$\frac{\lambda_f^{L^*}}{\rho_g} \leq \lambda_g^{L^*}(f) \leq \min\left\{\frac{\lambda_f^{L^*}}{\lambda_g}, \frac{\rho_f^{L^*}}{\rho_g}\right\} \leq \max\left\{\frac{\lambda_f^{L^*}}{\lambda_g}, \frac{\rho_f^{L^*}}{\rho_g}\right\} \leq \rho_g^{L^*}(f) \leq \frac{\rho_f^{L^*}}{\lambda_g} .$$

3 Theorems

In this section we state the main results of the paper.

Theorem 3.1. *Let f and g be any two entire functions such that $0 \leq \rho_f^{L^*} < \infty$ and $0 \leq \lambda_g \leq \rho_g < \infty$. Then*

$$\max\left\{\left[\frac{\overline{\sigma}_f^{L^*}}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\sigma_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\} \leq \sigma_g^{L^*}(f) \leq \left[\frac{\sigma_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}} .$$

Proof. From the definitions of $\sigma_f^{L^*}$ and $\overline{\sigma}_f^{L^*}$, we have for all sufficiently large values of r that

$$M_f(r) \leq \exp\left[\left(\sigma_f^{L^*} + \varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right], \tag{3.1}$$

$$M_f(r) \geq \exp\left[\left(\overline{\sigma}_f^{L^*} - \varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right] \tag{3.2}$$

and also for a sequence of values of r tending to infinity, we get that

$$M_f(r) \geq \exp\left[\left(\sigma_f^{L^*} - \varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right], \tag{3.3}$$

$$M_f(r) \leq \exp\left[\left(\overline{\sigma}_f^{L^*} + \varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right] . \tag{3.4}$$

Similarly from the definitions of σ_g and $\overline{\sigma}_g$, it follows for all sufficiently large values of r that

$$\begin{aligned} M_g(r) &\leq \exp\left[(\sigma_g + \varepsilon) \cdot r^{\rho_g}\right] \\ i.e.,\ r &\leq M_g^{-1}\left[\exp\left[(\sigma_g + \varepsilon) \cdot r^{\rho_g}\right]\right] \\ i.e.,\ M_g^{-1}(r) &\geq \left[\left(\frac{\log r}{(\sigma_g + \varepsilon)}\right)^{\frac{1}{\rho_g}}\right] \text{ and} \end{aligned} \tag{3.5}$$

$$M_g^{-1}(r) \leq \left[\left(\frac{\log r}{(\overline{\sigma}_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}\right] . \tag{3.6}$$

Also for a sequence of values of r tending to infinity, we obtain that

$$M_g^{-1}(r) \leq \left[\left(\frac{\log r}{(\sigma_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}\right] \text{ and} \tag{3.7}$$

$$M_g^{-1}(r) \geq \left[\left(\frac{\log r}{(\overline{\sigma}_g + \varepsilon)}\right)^{\frac{1}{\rho_g}}\right] . \tag{3.8}$$

From the definitions of $\overline{\tau}_f^{L^*}$ and $\tau_f^{L^*}$, we have for all sufficiently large values of r that

$$M_f(r) \leq \exp\left[\left(\overline{\tau}_f^{L^*}+\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_f^{L^*}}\right], \tag{3.9}$$

$$M_f(r) \geq \exp\left[\left(\tau_f^{L^*}-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_f^{L^*}}\right] \tag{3.10}$$

and also for a sequence of values of r tending to infinity, we get that

$$M_f(r) \geq \exp\left[\left(\overline{\tau}_f^{L^*}-\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_f^{L^*}}\right], \tag{3.11}$$

$$M_f(r) \leq \exp\left[\left(\tau_f^{L^*}+\varepsilon\right)\left[re^{L(r)}\right]^{\lambda_f^{L^*}}\right]. \tag{3.12}$$

Similarly from the definitions of $\overline{\tau}_g$ and τ_g, it follows for all sufficiently large values of r that

$$\begin{aligned} M_g(r) &\leq \exp\left[(\overline{\tau}_g+\varepsilon)\cdot r^{\lambda_g}\right] \\ i.e.,\ r &\leq M_g^{-1}\left[\exp\left[(\overline{\tau}_g+\varepsilon)\cdot r^{\lambda_g}\right]\right] \\ i.e.,\ M_g^{-1}(r) &\geq \left[\left(\frac{\log r}{(\overline{\tau}_g+\varepsilon)}\right)^{\frac{1}{\lambda_g}}\right] \text{ and} \end{aligned} \tag{3.13}$$

$$M_g^{-1}(r) \leq \left[\left(\frac{\log r}{(\tau_g-\varepsilon)}\right)^{\frac{1}{\lambda_g}}\right]. \tag{3.14}$$

Also for a sequence of values of r tending to infinity, we obtain that

$$M_g^{-1}(r) \leq \left[\left(\frac{\log r}{(\overline{\tau}_g-\varepsilon)}\right)^{\frac{1}{\lambda_g}}\right] \text{ and} \tag{3.15}$$

$$M_g^{-1}(r) \geq \left[\left(\frac{\log r}{(\tau_g+\varepsilon)}\right)^{\frac{1}{\lambda_g}}\right]. \tag{3.16}$$

Now from (3.3) and in view of (3.13), we get for a sequence of values of r tending to infinity that

$$M_g^{-1}M_f(r) \geq M_g^{-1}\left[\exp\left[\left(\sigma_f^{L^*}-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \geq \left[\left(\frac{\log\exp\left[\left(\sigma_f^{L^*}-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]}{(\overline{\tau}_g+\varepsilon)}\right)^{\frac{1}{\lambda_g}}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \geq \left[\frac{\left(\sigma_f^{L^*}-\varepsilon\right)}{(\overline{\tau}_g+\varepsilon)}\right]^{\frac{1}{\lambda_g}} \cdot \left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\lambda_g}}.$$

Since in view of Lemma 2.1, $\frac{\rho_f^{L^*}}{\lambda_g} \geq \rho_g^{L^*}(f)$ and as $\varepsilon\,(>0)$ is arbitrary, therefore it follows from above that

$$\limsup_{r\to\infty}\frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}} \geq \left[\frac{\sigma_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e.,\ \sigma_g^{L^*}(f) \geq \left[\frac{\sigma_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}. \tag{3.17}$$

Similarly from (3.2) and in view of (3.16), it follows for a sequence of values of r tending to infinity that

$$M_g^{-1}M_f(r) \geq M_g^{-1}\left[\exp\left[\left(\overline{\sigma}_f^{L^*} - \varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \geq \left[\left(\frac{\log\exp\left[\left(\overline{\sigma}_f^{L^*} - \varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]}{(\tau_g + \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \geq \left[\frac{\left(\overline{\sigma}_f^{L^*} - \varepsilon\right)}{(\tau_g + \varepsilon)}\right]^{\frac{1}{\lambda_g}} \cdot \left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\lambda_g}}.$$

Since in view of Lemma 2.1, it follows that $\frac{\rho_f^{L^*}}{\lambda_g} \geq \rho_g^{L^*}(f)$. Also $\varepsilon\,(>0)$ is arbitrary, so we get from above that

$$\limsup_{r\to\infty}\frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}} \geq \left[\frac{\overline{\sigma}_f^{L^*}}{\tau_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e.,\ \sigma_g^{L^*}(f) \geq \left[\frac{\overline{\sigma}_f^{L^*}}{\tau_g}\right]^{\frac{1}{\lambda_g}}. \tag{3.18}$$

Again in view of (3.6), we have from (3.1) for all sufficiently large values of r that

$$M_g^{-1}M_f(r) \leq M_g^{-1}\left[\exp\left[\left(\sigma_f^{L^*} + \varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]\right]$$

$$i.e., M_g^{-1}M_f(r) \leq \left[\left(\frac{\log\exp\left[\left(\sigma_f^{L^*} + \varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]}{(\overline{\sigma}_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \leq \left[\frac{\left(\sigma_f^{L^*} + \varepsilon\right)}{(\overline{\sigma}_g - \varepsilon)}\right]^{\frac{1}{\rho_g}} \cdot \left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}}. \tag{3.19}$$

As in view of Lemma 2.1, it follows that $\frac{\rho_f^{L^*}}{\rho_g} \leq \rho_g^{L^*}(f)$. Since $\varepsilon\,(>0)$ is arbitrary, we get from (3.19) that

$$\limsup_{r\to\infty}\frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}} \leq \left[\frac{\sigma_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e.,\ \sigma_g^{L^*}(f) \leq \left[\frac{\sigma_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}. \tag{3.20}$$

Thus the theorem follows from (3.17), (3.18) and (3.20). □

The conclusion of the following corollary can be carried out from (3.6) and (3.9); (3.9) and (3.14) respectively after applying the same technique of Theorem 3.1 and with the help of Lemma 2.1. Therefore its proof is omitted.

Corollary 3.2. *Let f and g be any two entire functions such that $0 \leq \lambda_f^{L^*} < \infty$ and $0 \leq \lambda_g \leq \rho_g < \infty$. Then*

$$\sigma_g^{L^*}(f) \leq \min\left\{ \left[\frac{\overline{\tau}_f^{L^*}}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}} \right\}.$$

Similarly in the line of Theorem 3.1 and with the help of Lemma 2.1, one may easily carried out the following theorem from pairwise inequalities numbers (3.10) and (3.13); (3.7) and (3.9); (3.6) and (3.12) respectively and therefore its proofs is omitted:

Theorem 3.3. *Let f and g be any two entire functions such that $0 \leq \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and $0 \leq \lambda_g \leq \rho_g < \infty$. Then*

$$\left[\frac{\tau_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} \leq \tau_g^{L^*}(f) \leq \min\left\{ \left[\frac{\tau_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\overline{\tau}_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}} \right\}.$$

Corollary 3.4. *Let f and g be any two entire functions such that $0 \leq \rho_f^{L^*} < \infty$ and $0 \leq \lambda_g \leq \rho_g < \infty$. Then*

$$\tau_g^{L^*}(f) \geq \max\left\{ \left[\frac{\overline{\sigma}_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} \right\}.$$

With the help of Lemma 2.1, the conclusion of the above corollary can be carry out from (3.2), (3.5) and (3.2), (3.13) respectively after applying the same technique of Theorem 3.1 and therefore its proof is omitted.

Theorem 3.5. *Let f and g be any two entire functions such that $0 \leq \rho_f^{L^*} < \infty$ and $0 \leq \lambda_g \leq \rho_g < \infty$. Then*

$$\left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} \leq \overline{\sigma}_g^{L^*}(f) \leq \min\left\{ \left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}} \right\}.$$

Proof. From (3.2) and in view of (3.13), we get for all sufficiently large values of r that

$$M_g^{-1}M_f(r) \geq M_g^{-1}\left[\exp\left[\left(\overline{\sigma}_f^{L^*} - \varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \geq \left[\left(\frac{\log\exp\left[\left(\overline{\sigma}_f^{L^*} - \varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]}{(\overline{\tau}_g + \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \geq \left[\frac{\left(\overline{\sigma}_f^{L^*} - \varepsilon\right)}{(\overline{\tau}_g + \varepsilon)}\right]^{\frac{1}{\lambda_g}} \cdot \left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\lambda_g}}.$$

Now in view of Lemma 2.1, it follows that $\frac{\rho_f^{L^*}}{\lambda_g} \geq \rho_g^{L^*}(f)$. Since $\varepsilon(>0)$ is arbitrary, we get from above that

$$\liminf_{r\to\infty}\frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}} \geq \left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e.,\ \overline{\sigma}_g^{L^*}(f) \geq \left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}. \tag{3.21}$$

Further in view of (3.7), we get from (3.1) for a sequence of values of r tending to infinity that

$$M_g^{-1}M_f(r) \leq M_g^{-1}\left[\exp\left[\left(\sigma_f^{L^*}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \leq \left[\left(\frac{\log\exp\left[\left(\sigma_f^{L^*}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]}{(\sigma_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \leq \left[\frac{\left(\sigma_f^{L^*}+\varepsilon\right)}{(\sigma_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}}. \tag{3.22}$$

Again as in view of Lemma 2.1, $\frac{\rho_f^{L^*}}{\rho_g} \leq \rho_g^{L^*}(f)$ and $\varepsilon\,(>0)$ is arbitrary, therefore we get from (3.22) that

$$\liminf_{r\to\infty}\frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}} \leq \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e.,\ \overline{\sigma}_g^{L^*}(f) \leq \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}. \tag{3.23}$$

Likewise from (3.4) and in view of (3.6), it follows for a sequence of values of r tending to infinity that

$$M_g^{-1}M_f(r) \leq M_g^{-1}\left[\exp\left[\left(\overline{\sigma}_f^{L^*}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \leq \left[\left(\frac{\log\exp\left[\left(\overline{\sigma}_f^{L^*}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]}{(\overline{\sigma}_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}\right]$$

$$i.e.,\ M_g^{-1}M_f(r) \leq \left[\frac{\left(\overline{\sigma}_f^{L^*}+\varepsilon\right)}{(\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}}. \tag{3.24}$$

Analogously, we get from (3.24) that

$$\liminf_{r\to\infty}\frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}} \leq \left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e.,\ \overline{\sigma}_g^{L^*}(f) \leq \left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \tag{3.25}$$

since in view of Lemma 2.1, $\frac{\rho_f^{L^*}}{\rho_g} \leq \rho_g^{L^*}(f)$ and $\varepsilon\,(>0)$ is arbitrary.

Thus the theorem follows from (3.21), (3.23) and (3.25). □

Corollary 3.6. *Let f and g be any two entire functions such that $0 \leq \lambda_f^{L^*} < \infty$ and $0 \leq \lambda_g \leq \rho_g < \infty$. Then*

$$\overline{\sigma}_g^{L^*}(f) \leq \min\left\{\left[\frac{\tau_f^{L^*}}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\tau_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}\right\}.$$

The conclusion of the above corollary can be carried out from pairwise inequalities no (3.6) and (3.12) ; (3.7) and (3.9) ; (3.12) and (3.14); (3.9) and (3.15) respectively after applying the same technique of Theorem 3.5 and with the help of Lemma 2.1. Therefore its proof is omitted.

Similarly in the line of Theorem 3.1 and with the help of Lemma 2.1, one may easily carried out the following theorem from pairwise inequalities no (3.11) and (3.13) ; (3.10) and (3.16); (3.6) and (3.9) respectively and therefore its proofs is omitted:

Theorem 3.7. *Let f and g be any two entire functions such that $0 \leq \lambda_f^{L^*} < \infty$ and $0 \leq \lambda_g \leq \rho_g < \infty$. Then*

$$\max\left\{\left[\frac{\overline{\tau}_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\tau_f^{L^*}}{\tau_g}\right]^{\frac{1}{\lambda_g}}\right\} \leq \overline{\tau}_g^{L^*}(f) \leq \left[\frac{\overline{\tau}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}.$$

Corollary 3.8. *Let f and g be any two entire functions such that $0 \leq \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and $0 \leq \lambda_g \leq \rho_g < \infty$. Then*

$$\overline{\tau}_g^{L^*}(f) \geq \max\left\{\left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\sigma}_f^{L^*}}{\tau_g}\right]^{\frac{1}{\lambda_g}}\right\}.$$

The conclusion of the above corollary can be carried out from pairwise inequalities no (3.3) and (3.5) ; (3.2) and (3.8) ; (3.3) and (3.13); (3.2) and (3.16) respectively after applying the same technique of Theorem 3.5 and with the help of Lemma 2.1. Therefore its proof is omitted.

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