Palestine Journal of Mathematics

Vol. 6(Special Issue: II) (2017) , 165–173 

# SOME TRANSFORMATIONS AND GENERATING RELATIONS OF MULTIVARIABLES HYPERGEOMETRIC FUNCTIONS

M. Ghayasuddin and N.U. Khan

Communicated by Jose Luis Lopez-Bonilla

MSC 2010 Classifications: 33C15, 33C65 and 33C70.

Keywords and phrases: Whittaker function, Generalized Whittaker function, Lauricella's function, Saran function, Appell's functions, Horn functions, Kampé de Fériet function, generalized hypergeometric functions and Laplace transform.

**Abstract**. This paper deals with an integral transformation associated with Whittaker functions into Lauricella hypergeometric function of $n$ variables. Its special cases yield various new transformations involving Saran's function, Appell's functions, Horn's functions, Kampé de Fériet function and generalized hypergeometric function. We also derive some (potentially) useful generating relations for the hypergeometric functions of several variables.

## 1 Introduction

Numerous integral transforms (for example, Laplace, Fourier, Mellin, Hankel etc.) involving a variety of special functions have been established by many researchers (for example, [1], [2], [3], [11], [13], [14], [15], [16] etc.). Such transforms play an important role in many diverse field of physics and engineering. In a sequel of such type of works, in this paper we present (presumably) a new integral transform involving the product of Whittaker function $M_{k,\mu}(x)$ and generalized Whittaker function $M_{k,\mu_1,\mu_2,\cdots,\mu_n}(x_1,x_2,\cdots,x_n)$, which is expressed in terms of Lauricella's hypergeometric function of $n$ variables $F_C^{(n)}$. Various new transformations (involving Saran's function, Appell's functions, Horn's functions, Kampé de Fériet function and generalized hypergeometric function) are also obtained as special cases of our main result. Further, by using the present transformations, we derive some (potentially) useful generating relations.

For the purposes of our present study, we begin by recalling here the definitions of some known functions.

The Whittaker function $M_{k,\mu}$ was introduced by Whittaker [4] (see also Whittaker and Watson [5]) in terms of confluent hypergeometric function ${}_1F_1$ (or Kummer's functions) as follows:

$$M_{k,\mu}(x)= x^{\mu+\frac{1}{2}}\, e^{-x/2}\, {}_1F_1\left(\frac{1}{2}+\mu-k\,;2\mu+1\,;x\right). \tag{1.1}$$

Further generalization of Whittaker function $M_{k,\mu}$ was introduced by Humbert [8, p.63, Eq.(15)] in the following form:

$$\begin{aligned}M_{k,\mu_1,\mu_2,\cdots,\mu_n}(x_1,x_2,\cdots,x_n) &= {x_1}^{\mu_1+\frac{1}{2}}{x_2}^{\mu_2+\frac{1}{2}}\cdots {x_n}^{\mu_n+\frac{1}{2}}\exp\left[-\frac{1}{2}\left(x_1+x_2+\cdots+x_n\right)\right]\\ &\times{\Psi_2}^{(n)}\left[\mu_1+\mu_2+\cdots+\mu_n-k+\frac{n}{2};\ 2\mu_1+1,\ 2\mu_2+1,\cdots,2\mu_n+1;\ x_1,\ x_2,\cdots,x_n\right],\end{aligned} \tag{1.2}$$

where ${\Psi_2}^{(n)}$ is the Humbert's confluent hypergeometric function of $n$ variables defined as follows (see [8, p.62, Eq.(11)]):

$${\Psi_2}^{(n)}[a;\ c_1,\cdots,c_n;\ x_1,\cdots,x_n] = \sum_{m_1,m_2,\cdots,m_n=0}^{\infty}\frac{(a)_{m_1+m_2+\cdots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}}\frac{{x_1}^{m_1}}{m_1!}\cdots\frac{{x_n}^{m_n}}{m_n!}$$

$$(\max\{|x_1|, |x_2|, \cdots |x_n|\} < \infty).$$

## 2 Main result

The following integral transform involving the product of Whittaker function $M_{k,\mu}(x)$ and generalized Whittaker function $M_{k,\mu_1,\mu_2,\cdots,\mu_n}(x_1, x_2, \cdots, x_n)$ will be evaluated in this section:

$$\int_0^\infty u^{\nu-1} e^{-pu} M_{\rho,\sigma}(xu) M_{k,\mu_1,\mu_2,\cdots,\mu_n}(y_1 u, y_2 u, \cdots, y_n u) du$$

$$= \frac{x^{\sigma+\frac{1}{2}} {y_1}^{\mu_1+\frac{1}{2}} \cdots {y_n}^{\mu_n+\frac{1}{2}} \Gamma(\sigma+b+\frac{1}{2})}{(p+Y+\frac{x}{2})^{\sigma+b+\frac{1}{2}}} \sum_{r=0}^{\infty} \frac{(\sigma+b+\frac{1}{2})_r (\sigma-\rho+\frac{1}{2})_r}{(2\sigma+1)_r} \frac{\{\frac{2x}{2(p+Y)+x}\}^r}{r!} \tag{2.1}$$

$$\times F_C^{(n)}\left[\sigma+b+\frac{1}{2}+r,\ a;\ 2\mu_1+1, \cdots, 2\mu_n+1;\ \frac{2y_1}{2(p+Y)+x}, \cdots, \frac{2y_n}{2(p+Y)+x}\right],$$

where $a = \mu_1 + \mu_2 + \cdots + \mu_n - k + \frac{n}{2}$, $b = \mu_1 + \mu_2 + \cdots + \mu_n + \nu + \frac{n}{2}$, $Y = \frac{y_1+y_2+\cdots+y_n}{2}$, $\Re(\sigma+b) > -\frac{1}{2}$, $\Re(p+Y) > \frac{1}{2}|\Re(x)|$ and $F_C^{(n)}$ is the Lauricella's hypergeometric function of $n$ variables (see [8, p.60]).

**Proof.** In order to derive (2.1), we denote the left-hand side of (2.1) by I, expressing $M_{k,\mu_1,\mu_2,\cdots,\mu_n}$ as a series with the help of (1.2) and (1.3) and then interchanging the order of integral sign and summation, which is guaranteed under the conditions, we get

$$\mathrm{I} = {y_1}^{\mu_1+\frac{1}{2}} {y_2}^{\mu_2+\frac{1}{2}} \cdots {y_n}^{\mu_n+\frac{1}{2}} \sum_{m_1,m_2,\cdots,m_n=0}^{\infty} \frac{(a)_{m_1+m_2+\cdots+m_n}}{(2\mu_1+1)_{m_1}(2\mu_2+1)_{m_2}\cdots(2\mu_n+1)_{m_n}}$$

$$\times \frac{{y_1}^{m_1}}{m_1!} \frac{{y_2}^{m_2}}{m_2!} \cdots \frac{{y_n}^{m_n}}{m_n!} \int_0^\infty u^{b+m_1+m_2+m_n-1} \exp[-(p+Y)u] M_{\rho,\sigma}(xu) du, \tag{2.2}$$

where $a = \mu_1 + \mu_2 + \cdots + \mu_n - k + \frac{n}{2}$, $b = \mu_1 + \mu_2 + \cdots + \mu_n + \nu + \frac{n}{2}$ and $Y = \frac{y_1+y_2+\cdots+y_n}{2}$.

Using the following known integral formula (see [2, p.215, Eq.(11)]):

$$\int_0^\infty u^{\nu-1} e^{-pu} M_{\rho,\sigma}(xu) du = \frac{x^{\sigma+\frac{1}{2}} \Gamma(\sigma+\nu+\frac{1}{2})}{(p+\frac{x}{2})^{\sigma+\nu+\frac{1}{2}}} {}_2F_1\left(\sigma+\nu+\frac{1}{2},\ \sigma-\rho+\frac{1}{2};\ 2\sigma+1;\ \frac{2x}{2p+x}\right)$$

$$\left(\Re(\sigma+\nu) > -\frac{1}{2}\right),$$

in the above equation, we obtain

$$\mathrm{I} = {y_1}^{\mu_1+\frac{1}{2}} {y_2}^{\mu_2+\frac{1}{2}} \cdots {y_n}^{\mu_n+\frac{1}{2}} \sum_{m_1,m_2,\cdots,m_n=0}^{\infty} \frac{(a)_{m_1+m_2+\cdots+m_n}}{(2\mu_1+1)_{m_1}(2\mu_2+1)_{m_2}\cdots(2\mu_n+1)_{m_n}}$$

$$\times \frac{{y_1}^{m_1}}{m_1!} \frac{{y_2}^{m_2}}{m_2!} \cdots \frac{{y_n}^{m_n}}{m_n!} \frac{x^{\sigma+\frac{1}{2}} \Gamma(\sigma+b+m_1+\cdots+m_n+\frac{1}{2})}{(p+Y+\frac{x}{2})^{\sigma+b+m_1+\cdots+m_n+\frac{1}{2}}}$$

$$\times {}_2F_1\left(\sigma+b+m_1+\cdots+m_n+\frac{1}{2},\ \sigma-\rho+\frac{1}{2};\ 2\sigma+1;\ \frac{2x}{2(p+Y)+x}\right).$$

Now expanding the hypergeometric function ${}_2F_1$ in its defining series, and arranging the resulting multiple series into the Lauricella hypergeometric function of $n$ variables $F_C^{(n)}$, we arrive at the right-hand side of (2.1). This completes the proof. □

**Remark 2.1.** On replacing $p$ by $p-\frac{1}{2}$ and $\nu$ by $\nu-\rho$, and setting $\sigma=\rho-\frac{1}{2}$ and $x=1$ in equation (2.1) and then using the relation

$$M_{\rho,\rho-\frac{1}{2}}(z)=\exp\left(-\frac{z}{2}\right)z^{\rho},$$

we get the known result of Kamarujjama and Khan [11, p.68, Eq.(2.1)].

## 3 Special Cases

This section deals with certain new transformations involving Saran's function, Appell's functions, Horn's functions, Kampé de Fériet function and generalized hypergeometric functions ${}_pF_q$.

**(1).** On setting $n=2$ in (2.1), we get the following transformation:

$$\int_0^\infty u^{\nu-1}e^{-pu}M_{\rho,\sigma}(xu)M_{k,\mu_1,\mu_2}(y_1u,\ y_2u)du=\frac{x^{\sigma+\frac{1}{2}}y_1{}^{\mu_1+\frac{1}{2}}y_2{}^{\mu_2+\frac{1}{2}}\Gamma(\lambda)}{(p+Y_1+\frac{x}{2})^{\lambda}}$$
$$\times F_E\Bigg(\lambda,\ \lambda,\ \lambda,\ \sigma-\rho+\frac{1}{2},\ \delta,\ \delta;\ 2\sigma+1,\ 2\mu_1+1,\ 2\mu_2+1;$$
$$\frac{2x}{2(p+Y_1)+x},\ \frac{2y_1}{2(p+Y_1)+x},\ \frac{2y_2}{2(p+Y_1)+x}\Bigg), \tag{3.1}$$

where $Y_1=\frac{y_1+y_2}{2}$, $\lambda=\sigma+\nu+\mu_1+\mu_2+\frac{3}{2}$, $\delta=\mu_1+\mu_2-k+1$, $\Re(\lambda)>0$, $\Re(p+Y_1)>\frac{1}{2}|\Re(x)|$ and $F_E$ is the Saran function defined as follows (see [8, p.66, Eq.(26)]):

$$F_E(\alpha_1,\alpha_1,\alpha_1,\ \beta_1,\beta_2,\beta_2;\ \gamma_1,\gamma_2,\gamma_3;\ x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha_1)_{m+n+p}(\beta_1)_m(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$|x|<r,\ |y|<s,\ |z|<t,\ r+(\sqrt{s}+\sqrt{t})^2=1.$$

**(2).** On setting $k=0$ and $\mu_2=\mu_1$ in (3.1), and then using the following known transformation (see [7, p.116, Eq.(4.1.16)]):

$$F_E(a,a,a,b_1,b_2,b_2;\ c_1,b_2,b_2;\ x,y,z)$$
$$=(1-y-z)^{-a}H_4\left[a,b_1;\ b_2,c_1;\ \frac{yz}{(1-y-z)^2},\frac{x}{(1-y-z)}\right], \tag{3.2}$$

we get a new transformation

$$\int_0^\infty u^{\nu-1}e^{-pu}M_{\rho,\sigma}(xu)M_{0,\mu_1,\mu_1}(y_1u,\ y_2u)du=\frac{x^{\sigma+\frac{1}{2}}(y_1y_2)^{\mu_1+\frac{1}{2}}\Gamma(\lambda_1)}{(p-Y_1+\frac{x}{2})^{\lambda_1}}$$
$$\times H_4\left[\lambda_1,\ \sigma-\rho+\frac{1}{2};\ 2\mu_1+1,\ 2\sigma+1;\ \frac{y_1y_2}{(p-Y_1+\frac{x}{2})^2},\ \frac{x}{(p-Y_1+\frac{x}{2})}\right], \tag{3.3}$$

where $\lambda_1=\sigma+\nu+2\mu_1+\frac{3}{2}$, $\Re(\lambda_1)>0$ and $H_4$ is the Horn's function (see [8, p.57]).

**(3).** On replacing $x$ by $4x$ and setting $y_1=y_2=x$, $\rho=k=0$ and $\mu_2=\mu_1$ in (3.1) and then using the following known transformation (see [12, p.130, Eq.(2.3)]):

$$F_E(a,a,a,b,c,c;\ 2b,c,c;\ 4y,y,y)$$
$$=(1-4y)^{-a}{}_4F_3\left[\begin{matrix}\frac{a}{2}, & \frac{a+1}{2}, & \frac{2b+2c-1}{4}, & \frac{2b+2c+1}{4}; \\ c, & \frac{2b+1}{2}, & \frac{2b+2c-1}{2}; & \end{matrix}\ \left(\frac{4y}{1-4y}\right)^2\right], \tag{3.4}$$

we get a new integral transform

$$\int_0^\infty u^{\nu-1}e^{-pu}M_{0,\sigma}(4xu)M_{0,\mu_1,\mu_1}(xu,\ xu)du = \frac{2^{2\sigma+1}x^{\sigma+2\mu_1+\frac{3}{2}}\Gamma(\lambda_1)}{(p+3x)^{\lambda_1}}\left(1-\frac{8x}{2p+3x}\right)^{-\lambda_1}$$

$$\times{}_4F_3\left[\begin{array}{cccc} \frac{\lambda_1}{2}, & \frac{\lambda_1+1}{2}, & \frac{\sigma+2\mu_1+1}{2}, & \frac{\sigma+2\mu_1+2}{2}; \\ 2\mu_1+1, & \sigma+1, & \sigma+2\mu_1+1; & \end{array} \left(\frac{8x}{2p-5x}\right)^2\right], \tag{3.5}$$

where $\lambda_1 = \sigma+\nu+2\mu_1+\frac{3}{2}$, $\Re(\lambda_1) > 0$ and ${}_pF_q$ is the generalized hypergeometric function (see [6, p.73]).

**(4).** On setting $n = 1$ in (2.1), we get the following integral transformation:

$$\int_0^\infty u^{\nu-1}e^{-pu}M_{\rho,\sigma}(xu)M_{k,\mu_1}(y_1u)du = \frac{x^{\sigma+\frac{1}{2}}{y_1}^{\mu_1+\frac{1}{2}}\Gamma(A)}{(p+\frac{y_1+x}{2})^A}$$

$$\times F_2\left(A,\ \sigma-\rho+\frac{1}{2},\ \mu_1-k+\frac{1}{2};\ 2\sigma+1,\ 2\mu_1+1;\ \frac{2x}{2p+y_1+x},\ \frac{2y_1}{2p+y_1+x}\right), \tag{3.6}$$

where $A = \sigma+\nu+\mu_1+1$, $\Re(A) > 0$, $\Re(p+\frac{y_1}{2}) > \frac{1}{2}|\Re(x)|$ and $F_2$ is the Appell's function defined as follows (see [8, p.53, Eq.(5)]):

$$F_2(a,b,b';\ c,c';\ x,y) = \sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$|x|+|y|<1.$$

**(5).** By using the following known transformation (see [9, p.270, Eq.(2)]):

$$F_2[a,b,c;\ d,e;\ x,y] = F^{1:1;1}_{0:1;1}\left[\begin{array}{ccc} a: & b; & c; \\ -: & d; & e; \end{array} x,y\right], \tag{3.7}$$

equation (3.6) reduces to

$$\int_0^\infty u^{\nu-1}e^{-pu}M_{\rho,\sigma}(xu)M_{k,\mu_1}(y_1u)du = \frac{x^{\sigma+\frac{1}{2}}{y_1}^{\mu_1+\frac{1}{2}}\Gamma(A)}{(p+\frac{y_1+x}{2})^A}$$

$$\times F^{1:1;1}_{0:1;1}\left[\begin{array}{ccc} A: & \sigma-\rho+\frac{1}{2}; & \mu_1-k+\frac{1}{2}; \\ -: & 2\sigma+1; & 2\mu_1+1; \end{array} \frac{2x}{2p+y_1+x},\ \frac{2y_1}{2p+y_1+x}\right], \tag{3.8}$$

where $A = \sigma+\nu+\mu_1+1$, $\Re(A) > 0$ and $F^{l:m;n}_{p:q;s}$ is the well known Kampé de Fériet function (see [8, p.63]).

**(6).** On setting $\rho = k = 0$ in (3.6) and then using the following transformation (see [1, p.381]):

$$F_2[\alpha,\beta,\beta';\ 2\beta,2\beta';\ 2x,y] = (1-x)^{-\alpha}H_4\left[\alpha,\beta';\ \beta+\frac{1}{2},2\beta';\ \frac{x^2}{4(1-x)^2},\frac{y}{(1-x)}\right], \tag{3.9}$$

we get

$$\int_0^\infty u^{\nu-1}e^{-pu}M_{0,\sigma}(xu)M_{0,\mu_1}(y_1u)du = \frac{x^{\sigma+\frac{1}{2}}{y_1}^{\mu_1+\frac{1}{2}}\Gamma(A)}{(p+\frac{y_1}{2})^A}$$

$$\times H_4\left[A,\ \mu_1+\frac{1}{2};\ \sigma+1,\ 2\mu_1+1;\ \frac{x^2}{4(2p+y_1)^2},\ \frac{2y_1}{(2p+y_1)}\right], \tag{3.10}$$

where $A = \sigma+\nu+\mu_1+1$, $\Re(A) > 0$ and $H_4$ is the Horn's function.

**(7).** Further, on setting $\rho = k = 0$ in (3.6) and then using the following known transformation (see [17, p.11]):

$$F_2\left[\alpha, \beta - \frac{1}{2}, \beta' - \frac{1}{2};\ 2\beta - 1, 2\beta' - 1;\ 2x, 2y\right] = (1 - x - y)^{-\alpha}$$

$$\times F_4\left[\frac{\alpha}{2}, \frac{\alpha + 1}{2};\ \beta, \beta';\ \frac{x^2}{(1 - x - y)^2}, \frac{y^2}{(1 - x - y)^2}\right], \tag{3.11}$$

we get

$$\int_0^\infty u^{\nu-1} e^{-pu} M_{0,\sigma}(xu) M_{0,\mu_1}(y_1 u) du = \frac{x^{\sigma+\frac{1}{2}} {y_1}^{\mu_1+\frac{1}{2}} \Gamma(A)}{p^A}$$

$$\times F_4\left[\frac{A}{2}, \frac{A + 1}{2};\ \sigma + 1, \mu_1 + 1;\ \frac{x^2}{4p^2}, \frac{{y_1}^2}{4p^2}\right], \tag{3.12}$$

where $A = \sigma + \nu + \mu_1 + 1$, $\Re(A) > 0$ and $F_4$ is the Appell's function (see [8, p.53]).

**(8).** If we consider $y_1 = x$ in (3.12) and then using the following transformation (see [10, p.101]):

$$F_4(a, b;\ c, d;\ x, x) = {}_4F_3\left[\begin{matrix} a, & b, & \frac{c+d-1}{2}, & \frac{c+d}{2}; \\ & & & & 4x \\ c, & d, & c + d - 1; & \end{matrix}\right], \tag{3.13}$$

equation (3.12) reduces to

$$\int_0^\infty u^{\nu-1} e^{-pu} M_{0,\sigma}(xu) M_{0,\mu_1}(xu) du = \frac{x^{\sigma+\mu_1+1} \Gamma(A)}{p^A}$$

$$\times {}_4F_3\left[\begin{matrix} \frac{A}{2}, & \frac{A+1}{2}, & \frac{\sigma+\mu_1+1}{2}, & \frac{\sigma+\mu_1+2}{2}; \\ & & & & \frac{x^2}{p^2} \\ \sigma + 1, & \mu_1 + 1, & \sigma + \mu_1 + 1; & \end{matrix}\right], \tag{3.14}$$

where $\Re(A) > 0$ and ${}_pF_q$ is the generalized hypergeometric function.

## 4 Generating Relations

This section deals with some generating relations of hypergeometric functions of several variables. We begin by recalling here the following known Kummer's first formula [6, p.125, Eq.(1)]:

$$e^{-w} {}_1F_1(\alpha; \beta; w) = \sum_{l=0}^{\infty} \frac{(\beta - \alpha)_l}{(\beta)_l} \frac{(-w)^l}{l!}. \tag{4.1}$$

On replacing $w$ by $wu$, multiplying both sides by $u^{\nu-1} e^{-pu} M_{\rho,\sigma}(xu) M_{k,\mu_1,\cdots,\mu_n}(y_1 u, \cdots, y_n u)$ in (4.1), and then integrating with respect to $u$ from 0 to $\infty$, we get

$$\int_0^\infty u^{\nu-1} e^{-(p+w)u} M_{\rho,\sigma}(xu) M_{k,\mu_1,\cdots,\mu_n}(y_1 u, \cdots, y_n u) {}_1F_1(\alpha; \beta; wu) du$$

$$= \sum_{l=0}^{\infty} \frac{(\beta - \alpha)_l}{(\beta)_l} \frac{(-w)^l}{l!} \int_0^\infty u^{\nu+l-1} e^{-pu} M_{\rho,\sigma}(xu) M_{k,\mu_1,\cdots,\mu_n}(y_1 u, \cdots, y_n u) du.$$

On expanding ${}_1F_1$ into series form, we arrive at

$$\sum_{i=0}^{\infty} \frac{(\alpha)_i}{(\beta)_i} \frac{w^i}{i!} \int_0^\infty u^{\nu+i-1} e^{-(p+w)u} M_{\rho,\sigma}(xu) M_{k,\mu_1,\cdots,\mu_n}(y_1 u, \cdots, y_n u) du$$

$$= \sum_{l=0}^{\infty} \frac{(\beta-\alpha)_l}{(\beta)_l} \frac{(-w)^l}{l!} \int_0^{\infty} u^{\nu+l-1} e^{-pu} M_{\rho,\sigma}(xu) M_{k,\mu_1,\cdots,\mu_n}(y_1 u, \cdots, y_n u) du. \tag{4.2}$$

By using the above equation (4.2), we derive various new generating relations for the hypergeometric functions of several variables as follows:

**(1).** Now using the result (2.1) on both sides of (4.2) and after a little simplification, we obtain the following generating relation for the Lauricella's hypergeometric function of $n$ variables $F_C^{(n)}$:

$$\left(\frac{P}{Q}\right)^L \sum_{i,r=0}^{\infty} \frac{(L)_{i+r}\,(\alpha)_i(\sigma-\rho+\frac{1}{2})_r}{(\beta)_i(2\sigma+1)_r} \frac{\left(\frac{w}{Q}\right)^i}{i!} \frac{\left(\frac{x}{Q}\right)^r}{r!}$$

$$\times F_C^{(n)}\left[L+i+r,\ a;\ 2\mu_1+1, 2\mu_2+1, \cdots, 2\mu_n+1;\ \frac{y_1}{Q}, \frac{y_2}{Q}, \cdots, \frac{y_n}{Q}\right]$$

$$= \sum_{l,r=0}^{\infty} \frac{(L)_{l+r}\,(\beta-\alpha)_l(\sigma-\rho+\frac{1}{2})_r}{(\beta)_l(2\sigma+1)_r} \frac{\left(-\frac{w}{P}\right)^l}{l!} \frac{\left(\frac{x}{P}\right)^r}{r!}$$

$$\times F_C^{(n)}\left[L+l+r,\ a;\ 2\mu_1+1, 2\mu_2+1, \cdots, 2\mu_n+1;\ \frac{y_1}{P}, \frac{y_2}{P}, \cdots, \frac{y_n}{P}\right], \tag{4.3}$$

where $P = p + Y + \frac{x}{2}$, $Q = p + w + Y + \frac{x}{2}$, $Y = \frac{y_1+y_2+\cdots+y_n}{2}$, $a = \mu_1 + \mu_2 + \cdots + \mu_n - k + \frac{n}{2}$, $b = \mu_1 + \mu_2 + \cdots + \mu_n + \nu + \frac{n}{2}$ and $L = \sigma + b + \frac{1}{2}$.

In (4.3), on expanding $F_C^{(n)}$ in its defining series and then taking $n = 2$, and after a little simplification, we get the following relation of Srivastava and Daoust function:

$$\left(\frac{P_1}{Q_1}\right)^{L_1} F^{2:1;1;0;0}_{0:1;1;1;1}\left[\begin{array}{l} (L_1:1,1,1,1),\ (a_1:0,0,1,1):\ (\alpha,1);\ (\sigma-\rho+\frac{1}{2},1);\ \_\_;\ \_\_; \\ \_\_\_\_\_\_ :\ (\beta,1);\ (2\sigma+1,1);\ (2\mu_1+1,1);\ (2\mu_2+1,1); \end{array} \frac{w}{Q_1}, \frac{x}{Q_1}, \frac{y_1}{Q_1}, \frac{y_2}{Q_1}\right]$$

$$= F^{2:1;1;0;0}_{0:1;1;1;1}\left[\begin{array}{l} (L_1:1,1,1,1),\ (a_1:0,0,1,1):\ (\beta-\alpha,1);\ (\sigma-\rho+\frac{1}{2},1);\ \_\_;\ \_\_; \\ \_\_\_\_\_\_ :\ (\beta,1);\ (2\sigma+1,1);\ (2\mu_1+1,1);\ (2\mu_2+1,1); \end{array} -\frac{w}{P_1}, \frac{x}{P_1}, \frac{y_1}{P_1}, \frac{y_2}{P_1}\right], \tag{4.4}$$

where $P_1 = p + Y_1 + \frac{x}{2}$, $Q_1 = p + w + Y_1 + \frac{x}{2}$, $Y_1 = \frac{y_1+y_2}{2}$, $L_1 = \sigma + \mu_1 + \mu_2 + \nu + \frac{3}{2}$, $a_1 = \mu_1 + \mu_2 - k + 1$ and $F^{2:1;1;0;0}_{0:1;1;1;1}$ is the Srivastava and Daoust function (see [8, p.65]).

Further, on expanding ${F_C}^{(n)}$ in its defining series and then taking $n = 1$ in (4.3), and after a little simplification, we get the following relation for Srivastava triple hypergeometric series:

$$\left(\frac{P_2}{Q_2}\right)^{L_2} F^{(3)}\left[\begin{array}{l} L_2 ::\ \_;\ \_;\ \_;\ \alpha;\ \sigma-\rho+\frac{1}{2};\ \mu_1-k+\frac{1}{2}; \\ \_ ::\ \_;\ \_;\ \_;\ \beta;\ 2\sigma+1;\ 2\mu_1+1; \end{array} \frac{w}{Q_2}, \frac{x}{Q_2}, \frac{y_1}{Q_2}\right]$$

$$= F^{(3)}\left[\begin{array}{l} L_2 ::\ \_;\ \_;\ \_;\ \beta-\alpha;\ \sigma-\rho+\frac{1}{2};\ \mu_1-k+\frac{1}{2}; \\ \_ ::\ \_;\ \_;\ \_;\ \beta;\ 2\sigma+1;\ 2\mu_1+1; \end{array} -\frac{w}{P_2}, \frac{x}{P_2}, \frac{y_1}{P_2}\right], \tag{4.5}$$

where $P_2 = p+\frac{y_1+x}{2}$, $Q_2 = p+w+\frac{y_1+x}{2}$, $L_2 = \sigma+\mu_1+\nu+1$ and $F^{(3)}[x,y,z]$ is the Srivastava triple hypergeometric series (see [8, p.69]).

**(2).** If we consider $n = 2$ in (4.2) and then using the result (3.1), and after a little simplification, we get the following generating relation for Saran's function:

$$\left(\frac{R}{S}\right)^{\lambda}\sum_{i=0}^{\infty}\frac{(\alpha)_i(\lambda)_i}{(\beta)_i}\frac{\left(\frac{w}{S}\right)^i}{i!}F_E\left[\lambda+i,\lambda+i,\lambda+i,\sigma-\rho+\frac{1}{2},\delta,\delta;\ 2\sigma+1,\right.$$
$$\left.2\mu_1+1,2\mu_2+1;\ \frac{x}{S},\frac{y_1}{S},\frac{y_2}{S}\right]$$
$$=\sum_{l=0}^{\infty}\frac{(\beta-\alpha)_l(\lambda)_l}{(\beta)_l}\frac{\left(-\frac{w}{R}\right)^l}{l!}F_E\left[\lambda+l,\lambda+l,\lambda+l,\sigma-\rho+\frac{1}{2},\delta,\delta;\ 2\sigma+1,\right.$$
$$\left.2\mu_1+1,2\mu_2+1;\ \frac{x}{R},\frac{y_1}{R},\frac{y_2}{R}\right], \tag{4.6}$$

where $R = p+Y_1+\frac{x}{2}$, $S = p+w+Y_1+\frac{x}{2}$, $Y_1 = \frac{y_1+y_2}{2}$, $\lambda = \sigma+\nu+\mu_1+\mu_2+\frac{3}{2}$ and $\delta = \mu_1+\mu_2-k+1$.

Next, if we consider $k = 0$ and $\mu_2 = \mu_1$ in (4.6) and then using the transformation (3.2), after some simplification, we obtain the following generating relation for Horn's function:

$$\left(\frac{R-y_1-y_2}{S-y_1-y_2}\right)^{\lambda_1}\sum_{i=0}^{\infty}\frac{(\alpha)_i(\lambda_1)_i}{(\beta)_i}\frac{\left(\frac{w}{S-y_1-y_2}\right)^i}{i!}$$
$$\times H_4\left[\lambda_1+i,\sigma-\rho+\frac{1}{2};\ 2\mu_1+1,2\sigma+1;\ \frac{y_1y_2}{(S-y_1-y_2)^2},\frac{x}{(S-y_1-y_2)}\right]$$
$$=\sum_{l=0}^{\infty}\frac{(\beta-\alpha)_l(\lambda_1)_l}{(\beta)_l}\frac{\left\{-\frac{w}{(R-y_1-y_2)}\right\}^l}{l!}$$
$$\times H_4\left[\lambda_1+l,\sigma-\rho+\frac{1}{2};\ 2\mu_1+1,2\sigma+1;\ \frac{y_1y_2}{(R-y_1-y_2)^2},\frac{x}{(R-y_1-y_2)}\right], \tag{4.7}$$

where $\lambda_1 = \sigma+\nu+2\mu_1+\frac{3}{2}$.

Further, on replacing $x$ by $4x$, setting $y_1 = y_2 = x$, $\rho = k = 0$ and $\mu_2 = \mu_1$ in (4.6), and then using the transformation (3.4), we get the following generating relation for the generalized hypergeometric function ${}_pF_q$:

$$\left(\frac{R_1-4x}{S_1-4x}\right)^{\lambda_1}\sum_{i=0}^{\infty}\frac{(\alpha)_i(\lambda_1)_i}{(\beta)_i}\frac{\left(\frac{w}{S_1-4x}\right)^i}{i!}$$
$$\times{}_4F_3\left[\begin{matrix}\frac{\lambda_1+i}{2}, & \frac{\lambda_1+i+1}{2}, & \frac{\sigma+2\mu_1+1}{2}, & \frac{\sigma+2\mu_1+2}{2}; \\ 2\mu_1+1, & \sigma+1 & \sigma+2\mu_1+1; & \end{matrix}\ \left(\frac{4x}{S_1-4x}\right)^2\right]$$
$$=\sum_{l=0}^{\infty}\frac{(\beta-\alpha)_l(\lambda_1)_l}{(\beta)_l}\frac{\left\{\frac{-w}{(R_1-4x)}\right\}^l}{l!}$$
$$\times{}_4F_3\left[\begin{matrix}\frac{\lambda_1+l}{2}, & \frac{\lambda_1+l+1}{2}, & \frac{\sigma+2\mu_1+1}{2}, & \frac{\sigma+2\mu_1+2}{2}; \\ 2\mu_1+1, & \sigma+1 & \sigma+2\mu_1+1; & \end{matrix}\ \left(\frac{4x}{R_1-4x}\right)^2\right], \tag{4.8}$$

where $\lambda_1 = \sigma + \nu + 2\mu_1 + \frac{3}{2}$, $R_1 = p + 3x$ and $S_1 = p + w + 3x$.

**(3).** If we consider $n = 1$ in (4.2) and then using the result (3.6), after some simplification, we get the following generating relation for Appell's function $F_2$:

$$\left(\frac{U}{V}\right)^A \sum_{i=0}^{\infty} \frac{(\alpha)_i (A)_i}{(\beta)_i} \frac{\left(\frac{w}{V}\right)^i}{i!} F_2\left(A+i,\, \sigma-\rho+\frac{1}{2},\, \mu_1-k+\frac{1}{2};\, 2\sigma+1,\, 2\mu_1+1;\, \frac{x}{V},\, \frac{y_1}{V}\right)$$

$$= \sum_{l=0}^{\infty} \frac{(\beta-\alpha)_l (A)_l}{(\beta)_l} \frac{\left(-\frac{w}{U}\right)^l}{l!} F_2\left(A+l,\, \sigma-\rho+\frac{1}{2},\, \mu_1-k+\frac{1}{2};\, 2\sigma+1,\, 2\mu_1+1;\, \frac{x}{U},\, \frac{y_1}{U}\right), \tag{4.9}$$

where $A = \sigma + \nu + \mu_1 + 1$, $U = p + \frac{y_1+x}{2}$ and $V = p + w + \frac{y_1+x}{2}$.

Further, by using the result (3.7) in (4.9), we obtain the following generating relation for Kampé de Fériet function:

$$\left(\frac{U}{V}\right)^A \sum_{i=0}^{\infty} \frac{(\alpha)_i (A)_i}{(\beta)_i} \frac{\left(\frac{w}{V}\right)^i}{i!} F^{1:1;1}_{0:1;1}\left[\begin{array}{lll} A+i: & \sigma-\rho+\frac{1}{2}; & \mu_1-k+\frac{1}{2}; \\ \_\_\_: & 2\sigma+1; & 2\mu_1+1; \end{array} \frac{x}{V},\, \frac{y_1}{V}\right]$$

$$= \sum_{l=0}^{\infty} \frac{(\beta-\alpha)_l (A)_l}{(\beta)_l} \frac{\left(-\frac{w}{U}\right)^l}{l!} F^{1:1;1}_{0:1;1}\left[\begin{array}{lll} A+l: & \sigma-\rho+\frac{1}{2}; & \mu_1-k+\frac{1}{2}; \\ \_\_\_: & 2\sigma+1; & 2\mu_1+1; \end{array} \frac{x}{U},\, \frac{y_1}{U}\right]. \tag{4.10}$$

Setting $\rho = k = 0$ in (4.9) and then using the transformation (3.9), after some simplification, we get the following generating relation for Horn's function $H_4$:

$$\left(\frac{2U-x}{2V-x}\right)^A \sum_{i=0}^{\infty} \frac{(\alpha)_i (A)_i}{(\beta)_i} \frac{\left(\frac{2w}{2V-x}\right)^i}{i!} H_4\left[A+i,\, \mu_1+\frac{1}{2};\, \sigma+1,\, 2\mu_1+1;\, \frac{x^2}{4(2V-x)^2},\, \frac{2y_1 V}{(2V-x)}\right]$$

$$= \sum_{l=0}^{\infty} \frac{(\beta-\alpha)_l (A)_l}{(\beta)_l} \frac{\left\{\frac{-2w}{(2U-x)}\right\}^l}{l!} H_4\left[A+l,\, \mu_1+\frac{1}{2};\, \sigma+1,\, 2\mu_1+1;\, \frac{x^2}{4(2U-x)^2},\, \frac{2y_1 U}{(2U-x)}\right]. \tag{4.11}$$

Further, if we consider $\rho = k = 0$ in (4.9) and then using the transformation (3.11), after some simplification, we obtain the following generating relation for Appell's function $F_4$:

$$\left(\frac{p}{p+w}\right)^A \sum_{i=0}^{\infty} \frac{(\alpha)_i (A)_i}{(\beta)_i} \frac{\left(\frac{w}{p+w}\right)^i}{i!} F_4\left[\frac{A+i}{2},\, \frac{A+i+1}{2};\, \sigma+1,\, \mu_1+1;\, \frac{x^2}{4(p+w)^2},\, \frac{{y_1}^2}{4(p+w)^2}\right]$$

$$= \sum_{l=0}^{\infty} \frac{(\beta-\alpha)_l (A)_l}{(\beta)_l} \frac{\left(\frac{-w}{p}\right)^l}{l!} F_4\left[\frac{A+l}{2},\, \frac{A+l+1}{2};\, \sigma+1,\, \mu_1+1;\, \frac{x^2}{4p^2},\, \frac{{y_1}^2}{4p^2}\right], \tag{4.12}$$

where $A = \sigma + \nu + \mu_1 + 1$.

In (4.12), on taking $y_1 = x$ and then by using the transformation (3.13), we get the following generating relation for generalized hypergeometric function ${}_pF_q$:

$$\left(\frac{p}{p+w}\right)^A \sum_{i=0}^{\infty} \frac{(\alpha)_i (A)_i}{(\beta)_i} \frac{\left(\frac{w}{p+w}\right)^i}{i!} {}_4F_3\left[\begin{array}{l} \frac{A+i}{2},\ \frac{A+i+1}{2},\ \frac{\sigma+\mu_1+1}{2},\ \frac{\sigma+\mu_1+2}{2}; \\ \sigma+1,\ \mu_1+1,\ \sigma+\mu_1+1; \end{array} \frac{x^2}{(p+w)^2}\right]$$

$$= \sum_{l=0}^{\infty} \frac{(\beta-\alpha)_l (A)_l}{(\beta)_l} \frac{\left(\frac{-w}{p}\right)^l}{l!} \; {}_4F_3 \left[ \begin{matrix} \frac{A+l}{2}, & \frac{A+l+1}{2}, & \frac{\sigma+\mu_1+1}{2}, & \frac{\sigma+\mu_1+2}{2}; & \frac{x^2}{p^2} \\ \sigma+1, & \mu_1+1, & \sigma+\mu_1+1; & & \end{matrix} \right]. \quad (4.13)$$

## References


[1] A. Erdélyi, *Transformation of hypergeometric functions of two variables,* Proc. Roy . Soc. Edinburg Sect. A, **62** (1948), 378-385.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Tables of integral transforms,* Vol. **I**, McGraw-Hill, New York, 1954.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Tables of integral transforms,* Vol. **II**, McGraw-Hill, New York, 1954.

[4] E.T. Whittaker, *An expression of certain known function as generalized hypergeometric functions,* Bull. Amer. Math. Soc, **10** (1903), 125-134.

[5] E.T. Whittaker and G.N. Watson, *A course of modern analysis*, 4th ed. Cambridge, England. Cambridge University, 1990.

[6] E.D. Rainville, *Special functions*, The Macmillan Company, New York, 1960.

[7] H. Exton, *Multiple hypergeometric functions and applications*, Ellis Horwood Ltd., Chichester, New York, 1976.

[8] H.M. Srivastava and H.L. Manocha, *A Treatise on generating functions,* Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1984.

[9] H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1985.

[10] J.L. Burchnall, *Differential equations associated with hypergeometric functions*, Quart, J. Math. Oxford, Sr., **13** (1942), 90-106.

[11] M. Kamarujjama and W.A. Khan, *On laplace transforms of generalized Whittaker function of multi-variables $M_{\lambda,\mu_1,\cdots,\mu_k}(x_1,\cdots,x_k)$*, Italian J. of Pure and Appl. Math., **32** (2014), 67-72.

[12] M.A. Pathan and M.I. Qureshi, *A note on the reducibility of the triple hypergeometric function $F_E$*, Math. Chronicle, **12** (1983), 129-133.

[13] N.U. Khan and T. Kashmin, *On infinite series of three variables involving Whittaker and Bessel functions*, Palestine J. Math., **5**(1) (2016), 185-190.

[14] N.U. Khan and M. Ghayasuddin, *Study of the unified double integral associated with generalized Bessel-Maitland function,* Pure and Appl. Math. Letters, Vol. **2016** (2016), 15-19.

[15] N.U. Khan and M. Ghayasuddin, *A new class of integral formulas associated with generalized Bessel functions,* Sohag J. Math., Vol. **3**, No. 2 (2016), 1-4.

[16] N.U. Khan, T. Usman and M. Ghayasuddin, *A new class of unified integral formulas associated with Whittaker functions,* New Trends in Math. Sci., Vol. **4**, No. 1 (2016), 160-167.

[17] W.N. Bailey, *The generating function of Jacobi polynomials*, J. London Math. Soc., **13** (1938), 8-12.


## Author information


M. Ghayasuddin and N.U. Khan, Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh-202002, India.
E-mail: ghayas.maths@gmail.com



Received: May 12, 2016.

Accepted: June 27, 2016.