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HERMITE-HADAMARD INEQUALITY FOR COORDINATED F_h-CONVEX FUNCTIONS ON TIME SCALES

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Abstract In this paper, partial diamond-F_h dynamic derivative and double diamond-F_h integral calculus for two-variable functions are introduced on time scales. Also, two-dimensional Hermite-Hadamard-type integral inequalities for the generalized class of co-ordinated F_h-convex functions on time scales are established. The applicability of our results ranges from Optimization problems to Calculus of Variations and to Economics.

1 Introduction

A set $K \subseteq \mathbb{R}$ is said to be convex if $\forall\, x, y \in K,\ \lambda \in [0,\ 1]$, we have

$$(1-\lambda)x + \lambda y \in K.$$

A function $f : K \to \mathbb{R}$ is said to be convex in the classical sense if $\forall\, x, y \in K,\ \lambda \in [0, 1]$, we have

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y).$$

The inequality

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R},\ a < b, \tag{1.1}$$

holds for any convex function f defined on $\mathbb{R}$. It was first suggested by Hermite in 1881. But this result was nowhere mentioned in literature and was not widely known as Hermite's result. A leading expert on the history and theory of convex functions, Beckenbach [3], wrote that the inequality (1.1) was proven by Hadamard in 1893. In general, (1.1) is now known as the Hermite-Hadamard inequality. It has several generalizations and extensions for single, double and multivariable-time scales and other related integral inequalities for convex functions and different classes of convex functions on classical intervals with recent extensions to time scales theory, see for example, [8], [11], [13], [14], [18].

The concept of the theory of time scales was initiated by Stefen Hilger [16] in order to unify and extend the theory of difference and differential calculus consistently. In this theory, the delta and nabla calculus for single and two-variable functions are introduced (see [5], [6], [14], [22]). A linear combination of these delta and nabla dynamics-the diamond-α calculus on time scales-was developed by Sheng et al. [23]. Since the advent of this notion, several authors have extended many classical mathematical inequalities to time scales via the diamond-α dynamic calculus for univariate, bivariate and multivariate functions. For more information, see [8], [17], [19], [21], [24].

The concepts of the delta and nabla calculus on time scales with applications to Economics, Optimization and the Calculus of Variations have been introduced and employed in different directions. For details, interested readers are referred to [1], [3], [4], [5], [7], [9], [10], [11], [14],

[15]. Among these applications, Bohner [4] presented the time scale version of the simplest variational problem of finding the function $y = u(t) \in C^1[a, b]$, a weak extremum minimized by the functional

$$J[u] = \int_a^b L(t, u^\sigma(t), u^\Delta(t))\Delta(t), \tag{1.2}$$

satisfying the Dirichlet boundary conditions $u(a) = A, u(b) = B$, provided the Lagrangian $L(t, u, u^\Delta)$ is a class C^2 function with respect to all its arguments t, u, u^Δ. An Economic application of the calculus of Variations, using (1.2) can be found in Guzowska et al. [15] while the nabla version of the variational problem (1.2) can also be found in the paper by Atici et al. [2].

It is worthy to note that a two-variable delta calculus of variations on time scales was initiated by Ahlbrandt and Morian [1], where an Euler Lagrange equation for double integral variational problems on time scales was obtained in case of rectangular regions of integration. Meanwhile, Bohner and Guseinov [7] reformulated Ahlbrandt and Morian's [1] variational problem, for the case of ω-type region of integration.

In 2008, Dinu [8] employed the diamond-α calculus of Sheng et al. [23] to establish a full variant of the classical Hermite-Hadamard inequality (1.1) involving single variable function for the class of convex functions on time scales.

Nwaeze [17], employed Theorem 3.9 of Dinu [8] for a univariate function on time scales to prove the following Hadamard's type result, via the combined diamond-α dynamics, extending (1.1), for functions defined on a rectangle, that are convex on the coordinates. The result of Nwaeze [17] reads as follows:

Theorem 1.1. *[17] Let $a, b, x \in \mathbb{T}_1, c, d, y \in \mathbb{T}_2$, with $a < b, c < d$ and $f : [a, b] \times [c, d] \to \mathbb{R}$ be such that the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) := f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) := f(x, v)$ defined for all $y \in [c, d]$ and $x \in [a, b]$, are continuous and convex. Then the following inequalities hold*

$$\begin{aligned} &\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x, s_\alpha) \diamond_\alpha x + \frac{1}{d-c}\int_c^d f(t_\alpha, y) \diamond_\alpha y\right] \\ \leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y) \diamond_\alpha x \diamond_\alpha y \\ \leq\ & \frac{1}{2(b-a)(d-c)}\int_a^b [(d - s_\alpha)f(x, c) + (s_\alpha - c)f(x, d)] \diamond_\alpha x \\ +\ & \frac{1}{2(b-a)(d-c)}\int_c^d [(b - t_\alpha)f(a, y) + (t_\alpha - a)f(b, y)] \diamond_\alpha y, \end{aligned} \tag{1.3}$$

where $t_\alpha = \frac{1}{b-a}\int_a^b t \diamond_\alpha t$, and $s_\alpha = \frac{1}{d-c}\int_c^d s \diamond_\alpha s$.

Recently, the authors [10] introduced a more generalized class of convex function on time scales and a more general, combined dynamic calculus, referred to as the diamond-F_h calculus, which includes the delta, nabla and diamond-α calculi of [2], [6] and [23].

Definition 1.2. [10]. Let $\mathbb{T}$ be a time scale and let $h : \mathbb{J}_\mathbb{T} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_\mathbb{T}$ is a F_h-convex subset of the real $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *diamond-F_h differentiable* on $\mathbb{T}_k^k$(derived set from $\mathbb{T}$) in the sense of Δ and ∇, if $f^{\diamond_{F_h}}(t)$ exists for all $t \in \mathbb{T}_k^k$, and the diamond-F_h derivative is given by

$$f^{\diamond_{F_h}}(t) = \left(\frac{\lambda}{h(\lambda)}\right)^s f^\Delta(t) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f^\nabla(t), \quad where\ \ s \in [0, 1], \quad and \quad 0 \leq \lambda \leq 1. \tag{1.4}$$

If f is defined in $t \in \mathbb{T}_k^k$ for any $\epsilon > 0$, there is a neighbourhood U of m and $n \in U$, with $\mu_{mn} = \sigma(m) - n$ and $\nu_{mn} = \rho(m) - n$ such that

$$\left|\left(\frac{\lambda}{h(\lambda)}\right)^s [f(\sigma(m)) - f(n)]\nu_{mn} + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s [f(\rho(m)) - f(n)]\mu_{mn} - f^{\diamond_{F_h}}(t)\mu_{mn}\nu_{mn}\right| < \epsilon|\mu_{mn}\nu_{mn}|,$$

where $s \in [0, 1]$ *and* $\lambda \in [0, 1]$.

Remark 1.3. (i) $f^{\diamond_{F_h}}(t)$ reduces to the diamond-α derivative of Sheng et al. [23] for $F_h = \alpha, s = 1$ and $h(\lambda) = 1$. Thus every diamond-α differentiable function on $\mathbb{T}$ is diamond-F_h differentiable but the converse is not true (see [13]).

(ii) If f is diamond-F_h differentiable for $0 \leq s \leq 1$, and $0 \leq \lambda \leq 1$, then f is both Δ and ∇ differentiable.

(iii) For $F_h = 1$, $s = 1$ and $h(\lambda) = 1$, the diamond-F_h derivative reduces to the standard Δ derivative or the standard ∇ derivative for $F_h = 0$, $s = 1$ and $h(\lambda) = 1$ while it representes a 'weighted dynamic derivative' for $F_h \in (0, 1)$, $s = 1$ and $h(\lambda) = 1$ (see [3], [5]).

(iv) The combined dynamic derivative (1.4) gives a centralized derivative formula on any uniformly discrete time scale $\mathbb{T}$ when $F_h = \frac{1}{2}$, $s = 1$ and $h(\lambda) = 1$. This feature is particularly useful in many computational applications (see [23]).

(v) When $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$ and $f^{\diamond_{F_h}}(t)$ becomes the total differential operator (Ordinary derivative)(see [20]).

Definition 1.4. [10] Let $\mathbb{T}$ be a time scale and let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. The *diamond-F_h integral* of a function $f : \mathbb{T} \to \mathbb{R}$ from a to b, where $a, b \in \mathbb{T}$ is given by

$$\int_a^b f(t) \diamond_{F_h} t = \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b f(t)\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b f(t)\nabla t, \qquad s \in [0, 1], \quad 0 \leq \lambda \leq 1, \tag{1.5}$$

provided that f has a Δ and ∇ integral on $[a, b]_{\mathbb{T}}$ or $I_{\mathbb{T}}$.

Remark 1.5. (i) The equality (1.5) reduces to the diamond-α integral of Sheng et al. [23], if $F_h = \alpha; h(\lambda) = 1$, $s = 1$ and $\lambda = 1$. Thus, every diamond-α integrable function on $\mathbb{T}$ is diamond-F_h integrable but the converse is not true, see [13].

(ii) If f is diamond-F_h integrable for $0 \leq s \leq 1$, and $0 \leq \lambda \leq 1$, then f is both Δ and ∇ integrable.

Recently, Fagbemigun et al. [13] employed the concept of Definitions 1.2 and 1.4 to prove the following Hadamard's type result, among others, for a univariate function involving the class of F_h-convex functions of the authors [10], to obtain several generalizations of the Hermite-Hadamard inequality (1.1) on time scales.

Theorem 1.6. *[13] Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$ and $f : I_{\mathbb{T}} \to \mathbb{R}$ be a continuous F_h-convex function, $a, b, t \in I_{\mathbb{T}}$, with $a < b$. Then*

$$\begin{aligned} 2^s \left(h(\tfrac{1}{2})\right)^s f\left(\tfrac{a+b}{2}\right) &\leq \tfrac{1}{b-a} \int_a^b f(x) \diamond_{F_h} x \\ &\leq f(a) \int_0^1 \left(\tfrac{\lambda}{h(\lambda)}\right)^s \Delta\lambda + f(b) \int_0^1 \left(\tfrac{1-\lambda}{h(1-\lambda)}\right)^s \nabla\lambda. \end{aligned} \tag{1.6}$$

Remark 1.7. (i) By choosing $x = \lambda a + (1-\lambda)b$, $s = 1, F_h = \alpha$ and $h(\cdot) = 1$ in (1.6), we recover the second inequality of Theorem 3.9 of Dinu [8].

(ii) When $\mathbb{T} = \mathbb{R}$, $h(\frac{1}{2}) = \frac{1}{2}$ and $s = 1$ in inequality (1.6), the first part of the Hermite-Hadamard inequality (1.1) on classical intervals is recovered.

(iii) The nabla integral version of the first part of Theorem 1.6 is obtained if we choose $F_h = 0$.

Interestingly, in a more recent paper of Fagbemigun et al. [11], these concepts of the generalized class of F_h-convex functions with a more general, combined diamond-F_h calculus have been extended to establish double integral inequalities of Hermite-Hadamard-type for functions defined on time-scaled linear spaces, while solutions of the problems of the calculus of variations and varying dynamic optimization problems in Economics were obtained with the aid of these concepts in more recent papers of Fagbemigun et al. [12], [13].

It is the purpose of this paper to establish two-dimensional Hermite-Hadamard-type integral inequalities for coordinated F_h-convex functions on time scales. An application of our results to Economic models is also discussed.

2 Preliminaries

In the sequel, we shall first discuss the following new concepts and definitions.

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales with $\mathbb{T}_1 \times \mathbb{T}_2 = \{(x,y) : x \in \mathbb{T}_1, y \in \mathbb{T}_2\}$ which is a complete metric space with the metric d defined by

$$d((x,y),(x^{'},y^{'})) = ((x-x^{'})^2 + (y-y^{'})^2)^{\frac{1}{2}}, \qquad \forall \quad (x,y),(x^{'},y^{'}) \in \mathbb{T}_1 \times \mathbb{T}_2.$$

Let σ_i, ρ_i, $(i = 1, 2)$ denote respectively the forward jump operator, backward jump operator, and the diamond-F_h dynamic differentiation operator on $\mathbb{T}_i$.

Definition 2.1. Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. f is said to have a partial $\diamond_{(F_h)_1}$ derivative $\frac{\partial f(t_1,t_2)}{\diamond_{(F_h)_1} t_1}$(wrt t_1), at $(t_1,t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$, if for each $\epsilon > 0$, there exists a neighbourhood Ut_1 of t_1 such that

$$\Bigg| \left(\frac{\lambda}{h(\lambda)}\right)_1^s [f(\sigma_1(t_1),t_2) - f(m,t_2,)]\mu t_1 m$$
$$+ \left(\frac{1-\lambda}{h(1-\lambda)}\right)_1^s [f(\rho_1(t_1),t_2) - f(m,t_2)]\nu t_1 m - f^{\diamond_{(F_h)_1}}(t_1,\, t_2)\mu t_1 m \nu t_1 m \Bigg| < \epsilon |\mu t_1 m \nu t_1 m| \tag{2.1}$$

for $s \in [0,1], 0 \leq \lambda \leq 1$ and for all $m \in Ut_1$, where $Ut_1 m = \sigma_1(t_1) - m$, $\nu t_1 m = \rho_1(t_1) - m$.

Definition 2.2. Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$ and $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ an increasing function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. f is said to have a "partial $\diamond_{(F_h)_2}$ derivative" $\frac{\partial f(t_1,t_2)}{\diamond_{(F_h)_2} t_2}$(wrt t_2), at $(t_1,t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$, if for each $\epsilon > 0$, there exists a neighbourhood Ut_2 of t_2 such that

$$\Bigg| \left(\frac{\lambda}{h(\lambda)}\right)_2^s [f(t_1,\, \sigma_2(t_2) - f(t_1,\, m)]\mu t_2 m$$
$$+ \left(\frac{1-\lambda}{h(1-\lambda)}\right)_2^s [f(t_1,\, \rho_2(t_2) - f(t_1,\, m)]\nu t_2 m - f^{\diamond_{(F_h)_2}}(t_1,\, t_2)\mu t_2 m \nu t_2 m \Bigg| < \epsilon |\mu t_2 m \nu t_2 m|, \tag{2.2}$$

for $s \in [0,1], 0 \leq \lambda \leq 1$ and for all $n \in Ut_2$, where $Ut_2 m = \sigma_2(t_2) - m$, $\nu t_2 m = \rho_2(t_2) - m$.

These derivatives can also be denoted by $f^{\diamond_{(F_h)_1}}(t_1,\, t_2)$ and $f^{\diamond_{(F_h)_2}}(t_1,\, t_2)$ respectively.

Before we define the double diamond-F_h dynamic integral, we shall employ the following remark of [6].

Remark 2.3. [6] Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$. If the Δ and ∇ integrals of f exist on $\mathbb{T}_1 \times \mathbb{T}_2$, then the following types of integrals can be defined:

(i) $\Delta\Delta$-integral over $R^0 = [a,b) \times [c,d)$, which is introduced by using partitions consisting of subrectangles of the form $[\alpha,\beta) \times [\gamma,\partial)$;

(ii) $\nabla\nabla$-integral over $R^1 = (a,b] \times (c,d]$, which is introduced by using partitions consisting of subrectangles of the form $(\alpha,\beta] \times (\gamma,\partial]$;

(iii) $\Delta\nabla$-integral over $R^2 = [a,b) \times (c,d]$, which is introduced by using partitions consisting of subrectangles of the form $[\alpha,\beta) \times (\gamma,\partial]$;

(iv) $\nabla\Delta$-integral over $R^3 = (a,b] \times [c,d)$, which is introduced by using partitions consisting of subrectangles of the form $(\alpha,\beta] \times [\gamma,\partial)$.

Now let $\bar{U}(f)$ and $\bar{L}(f)$ denote the upper and lower Darboux Δ-integral of f from a to b ; $\underline{\mathrm{U}}(f)$ and $\underline{\mathrm{L}}(f)$ denote the upper and lower Darboux ∇-integral of f from a to b respectively. Given the construction of $U(f)$ and $L(f)$, which follows from the properties of supremum and infimum, we give the following new definition.

Definition 2.4. Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. If f is Δ-integrable on $R^0 = [a, b) \times [c, d)$ and ∇-integrable on $R^1 = (a, b] \times (c, d]$, then it is $\diamond_{F_h}$-integrable on $R = [a, b] \times [c, d]$ and

$$\begin{aligned}\int_R f(t,\, k) \diamond_{(F_h)_1} t \diamond_{(F_h)_2} k \quad &= \left(\tfrac{\lambda}{h(\lambda)}\right)^s \int\int_{R^0} f(t,\, k)\Delta_1 t\Delta_2 k \\ &+ \left(\tfrac{1-\lambda}{h(1-\lambda)}\right)^s \int\int_{R^1} f(t,\, k)\nabla_1 t\nabla_2 k,\end{aligned} \tag{2.3}$$

for all $s \in [0, 1]$, $0 \leq \lambda \leq 1$ and $t, k \in J_{\mathbb{T}}$.

Since $\bar{U}(f) \geq \bar{L}(f)$ and $\underline{\mathrm{U}}(f) \geq \underline{\mathrm{L}}(f)$, we may state the following Theorem.

Theorem 2.5. *Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. If f be $\diamond_{F_h}$-integrable on $R = [a, b] \times [c, d]$, provided its Δ and ∇ integrals exist, then*

(i) If $F_h = 1$, f is $\Delta\Delta$-integrable on $R^0 = [a, b) \times [c, d)$;

(ii) If $F_h = 0$, f is $\nabla\nabla$-integrable on $R^1 = (a, b] \times (c, d]$;

(iii) If $F_h = \frac{1}{2}$, f is $\Delta\Delta$-integrable and $\nabla\nabla$-integrable on R^0 and R^1

(iv) If $F_h = \alpha$, f is double $\diamond_\alpha$-integrable on $R = [a, b] \times [c, d]$.

With the introduction of these new concepts, Fagbemigun et al.'s [13] single diamond-F_h integral variational calculus is now extended to double diamond-F_h integral variational calculus on time scales as follows.

Let $R = [a, b] \times [c, d]$ define a rectangle on $\mathbb{T}_1 \times \mathbb{T}_2$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. Consider the functional defined by

$$J_{(\diamond_{F_h})_1(\diamond_{F_h})_2}[u] = \left(\frac{\lambda}{h(\lambda)}\right)^s J_{\Delta_1\Delta_2}[u] + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s J_{\nabla_1\nabla_2}[u], \tag{2.4}$$

for all $s,\, \lambda \in [0, 1]$, where

$$J_{\Delta_1\Delta_2}[u] = \int_c^d \int_a^b L(x, y, u((\sigma_1(x), \sigma_2(y)), u^{\Delta_1}(x, \sigma_2(y)), u^{\Delta_2}(\sigma_1(x), y))\Delta_1 x, \Delta_2 y$$

and

$$J_{\nabla_1\nabla_2}[u] = \int_c^d \int_a^b L(x, y, u((\rho_1(x), \rho_2(y)), u^{\nabla_1}(x, \rho_2(y)), u^{\nabla_2}(\rho_1(x), y))\nabla_1 x, \nabla_2 y,$$

$L(x, y, u, p, q)$ is a continuous functional, together with its partial delta and nabla derivatives of the first and second order with respect to x, y and partial usual derivatives of the first and second order with respect to its arguments u, p, q in the domain $D(J)$ of variation of the independent variables.

3 Main Results

Consider the bi-dimensional time scale interval $I^2_{\mathbb{T}} : [a, b]_{I_{\mathbb{T}}} \times [c, d]_{I_{\mathbb{T}}}$ in $\mathbb{T}^2$ with $a < b$, $c < d$.

Definition 3.1. Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. A monotonically increasing function $f : I^2_{\mathbb{T}} \to \mathbb{R}$ on $I^2_{\mathbb{T}}$ is F_h-convex on time scale co-ordinates if the partial mappings

$f_y : [a, b]_{I_{\mathbb{T}}} \to \mathbb{R}, \quad f_y(u) := f(u, y), \qquad \forall\ y \in [c, d]_{I_{\mathbb{T}}}$

and

$f_x : [c, d]_{I_{\mathbb{T}}} \to \mathbb{R}, \quad f_x(v) := f(x, v), \qquad \forall\ x \in [a, b]_{I_{\mathbb{T}}}$

are continuous and F_h-convex.

Definition 3.2. Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. We say that a monotonically increasing function $f : I^2_{\mathbb{T}} \to \mathbb{R}$ is F_h-convex on time scale co-ordinates if for all $x, y \in I_{\mathbb{T}}$, we have

$$\begin{aligned} f(\lambda x + (1-\lambda)y,\, tu + (1-t)v) \;&\leq \left(\tfrac{t}{h(t)}\right)^s \left(\tfrac{\lambda}{h(\lambda)}\right)^s f(x,u) \\ &+ \left(\tfrac{\lambda}{h(\lambda)}\right)^s \left(\tfrac{1-t}{h(1-t)}\right)^s f(x,v) \\ &+ \left(\tfrac{1-\lambda}{h(1-\lambda)}\right)^s \left(\tfrac{t}{h(t)}\right)^s f(y,u) \\ &+ \left(\tfrac{1-t}{h(1-t)}\right)^s \left(\tfrac{1-\lambda}{h(1-\lambda)}\right)^s f(y,v), \end{aligned}$$

holds for $s \in [0,1]$, $0 \leq \lambda \leq 1$, $0 \leq t \leq 1$ and $(x,u), (x,v), (y,u), (y,v) \in I^2_{\mathbb{T}}$.

It is easy to see that the mapping $f : I^2_{\mathbb{T}} \to \mathbb{R}$ is F_h-convex in $I^2_{\mathbb{T}}$ satisfying the following inequality:

$$f(\lambda x + (1-\lambda)u,\, \lambda y + (1-\lambda)v) \leq \left(\frac{\lambda}{h(\lambda)}\right)^s f(x,\, y) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(u,\, v) \tag{3.1}$$

holds for all $(x,\, y)$, $(u,\, v) \in I^2_{\mathbb{T}}$, $s \in [0,1]$ and $0 \leq \lambda \leq 1$.

We state and prove the following Lemma.

Lemma 3.3. *Every F_h-convex mapping $f : I^2_{\mathbb{T}} \to \mathbb{R}$ on $I^2_{\mathbb{T}}$ is F_h-convex on the co-ordinates.*

Proof. Suppose that the mapping $f : I^2_{\mathbb{T}} \to \mathbb{R}$ is F_h-convex in $I^2_{\mathbb{T}}$ by (3.1).
Consider the partial mapping $f_x : [c,\, d]_{I_{\mathbb{T}}} \to \mathbb{R}, \quad f_x(v) := f(x,v)$.
Then for all $s \in [0,1]$, $0 \leq \lambda \leq 1$ and $f(u,v)$ monotonically increasing functions on $I_{\mathbb{T}}$, we have

$$\begin{aligned} f_x(\lambda u + (1-\lambda)v) \;&=\; f\left(x, \left(\tfrac{\lambda}{h(\lambda)}\right)^s u + \left(\tfrac{1-\lambda}{h(1-\lambda)}\right)^s v\right) \\ &=\; f\left(\left(\tfrac{\lambda}{h(\lambda)}\right)^s x + \left(\tfrac{1-\lambda}{h(1-\lambda)}\right)^s x, \left(\tfrac{\lambda}{h(\lambda)}\right)^s u + \left(\tfrac{1-\lambda}{h(1-\lambda)}\right)^s v\right) \\ &\leq\; \left(\tfrac{\lambda}{h(\lambda)}\right)^s f(x,u) + \left(\tfrac{1-\lambda}{h(1-\lambda)}\right)^s f(x,v) \\ &=\; \left(\tfrac{\lambda}{h(\lambda)}\right)^s f_x u + \left(\tfrac{1-\lambda}{h(1-\lambda)}\right)^s f_x v, \end{aligned}$$

which shows F_h-convexity of f_x.
By a similar argument, the partial mappings
$f_y : [a,b]_{I_{\mathbb{T}}} \to \mathbb{R}, \quad f_y(u) := f(u,y)$, is also F_h-convex for all $s \in [0,1]$, $0 \leq \lambda \leq 1$ and $f(v,r)$ monotonically increasing functions on $I_{\mathbb{T}}$ goes likewise and the proof is omitted. □

Note that in some special cases, some co-ordinated F_h-convex functions may not necessarily be F_h-convex on time scales.

Firstly, we discuss and establish a double integral inequality of Hermite-Hadamard type for a F_h-convex function on time scale co-ordinates.

Theorem 3.4. *Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. Let $f : I^2_{\mathbb{T}} \to \mathbb{R}$ be a continuous and an integrable F_h-convex function with respect to the function F_h on the co-ordinates on $I^2_{\mathbb{T}}$. Then for any $a, b, c, d \geq 0$, with $b > a, d > c$ and $s \in [0,1]$,*

$$\begin{aligned} f(M_{F_h}, N_{F_h}) \;&\leq\; \left(\tfrac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \tfrac{I_{\lambda,t}(a,b;c,d)}{(b-a)(d-c)} \\ &\leq\; \left(\tfrac{t}{h(t)}\right)^s \tfrac{I_{M,y}(a,b;c,d)}{(b-a)(d-c)} + \left(\tfrac{1-t}{h(1-t)}\right)^s \tfrac{I_{M,N}(a,b;c,d)}{(b-a)(d-c)} \\ &\leq\; \left(\tfrac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \tfrac{4I_{x,y}(a,b;c,d)}{(b-a)(d-c)}, \end{aligned} \tag{3.2}$$

where
$M_{F_h} = \int_a^b u \diamond_{F_h} u,\ N_{F_h} = \int_a^b v \diamond_{F_h} v,$
$I_{\lambda,t}(a,b;c,d) = \int_a^b \int_c^d f(\lambda x + (1-\lambda) M_{F_h}, ty + (1-t) N_{F_h}) \diamond_{F_h} y \diamond_{F_h} x,$
$I_{M,y}(a,b;c,d) = \int_a^b \int_c^d f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s x + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s M_{F_h}, y\right) \diamond_{F_h} x \diamond_{F_h} y,$
$I_{M,N}(a,b;c,d) = \int_a^b \int_c^d f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s x + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s M_{F_h}, N_{F_h})\right) \diamond_{F_h} x \diamond_{F_h} y,$
and
$I_{x,y}(a,b;c,d) = \int_a^b \int_c^d f(x,y) \diamond_{F_h} x \diamond_{F_h} y.$

Proof. (A) To show the first inequality in (3.2).
We have

$$\begin{aligned}
& f(M_{F_h}, N_{F_h}) \\
\leq\ & \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s f\left(\frac{1}{b-a}\int_a^b [\lambda x + (1-\lambda) M_{F_h}],\ N_{F_h}\right) \diamond_{F_h} x \\
\leq\ & \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{b-a}\int_a^b f[\lambda x + (1-\lambda) M_{F_h},\ N_{F_h}] \diamond_{F_h} x \\
\leq\ & \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{b-a}\int_a^b f\left(\lambda x + (1-\lambda) M_{F_h},\ \frac{1}{d-c}\int_c^d [ty + (1-t) N_{F_h}] \diamond_{F_h} y\right) \diamond_{F_h} x \\
\leq\ & \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{b-a}\int_a^b \left[\frac{1}{d-c}\int_c^d f\left(\lambda x + (1-\lambda) M_{F_h},\ ty + (1-t) N_{F_h}\right) \diamond_{F_h} y\right] \diamond_{F_h} x.
\end{aligned}$$

This proves the first inequality in (3.2).
Then by Definition 3.2, we have that

$$\begin{aligned}
& \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{b-a}\int_a^b \left[\frac{1}{d-c}\int_c^d f\left(\lambda x + (1-\lambda) M_{F_h},\ ty + (1-t) N_{F_h}\right) \diamond_{F_h} y\right] \diamond_{F_h} x \\
\leq\ & \left(\frac{t}{h(t)}\right)^s \frac{1}{b-a}\int_a^b \left(\frac{1}{d-c}\int_c^d f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s x + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s M_{F_h},\ y\right) \diamond_{F_h} y\right) \diamond_{F_h} x \\
+\ & \left(\frac{1-t}{h(1-t)}\right)^s \times \frac{1}{d-c}\int_c^d f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s x + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s M_{F_h},\ N_{F_h})\right) \diamond_{F_h} y \diamond_{F_h} x, \quad (*)
\end{aligned}$$

satisfying the second inequality in (3.2).

Thus from the right hand side of (*), we have

$$\begin{aligned}
& \left(\frac{t}{h(t)}\right)^s \frac{1}{b-a}\int_a^b \left(\frac{1}{d-c}\int_c^d f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s x + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s M_{F_h},\ y\right) \diamond_{F_h} y\right) \diamond_{F_h} x \\
+\ & \left(\frac{1-t}{h(1-t)}\right)^s \times \frac{1}{d-c}\int_c^d f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s x + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s M_{F_h},\ N_{F_h})\right) \diamond_{F_h} y \diamond_{F_h} x \\
\leq\ & \left(\frac{t}{h(t)}\right)^s \times \frac{1}{d-c}\int_c^d [\left(\frac{\lambda}{h(\lambda)}\right)^s \frac{1}{b-a}\int_a^b f(x,y) \diamond_{F_h} y \diamond_{F_h} x \\
+\ & \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \frac{1}{b-a}\int_a^b f\left(M_{F_h}, y\right) \diamond_{F_h} x] \diamond_{F_h} y \\
+\ & \left(\frac{1-t}{h(1-t)}\right)^s \times \frac{1}{d-c}\int_c^d [\left(\frac{\lambda}{h(\lambda)}\right)^s \cdot \frac{1}{b-a}\int_a^b f\left(x, N_{F_h}\right) \diamond_{F_h} x \\
+\ & \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f\left(M_{F_h}, N_{F_h}\right)] \diamond_{F_h} y \\
\leq\ & \left(\frac{t}{h(t)}\right)^s \left(\frac{\lambda}{h(\lambda)}\right)^s \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y) \diamond_{F_h} x \diamond_{F_h} y \\
+\ & \left(\frac{t}{h(t)}\right)^s \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \frac{1}{d-c}\int_c^d f(M_{F_h}, y) \diamond_{F_h} y \\
+\ & \left(\frac{1-t}{h(1-t)}\right)^s \left(\frac{\lambda}{h(\lambda)}\right)^s \frac{1}{b-a}\int_a^b f\left(x, N_{F_h}\right) \diamond_{F_h} x \\
+\ & \left(\frac{1-t}{h(1-t)}\right)^s \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f\left(M_{F_h}, N_{F_h}\right). \qquad (3.3)
\end{aligned}$$

Also, from the first inequality in Theorem 1.6, inequality (3.1) and Lemma 3.3, we have

$$f(M_{F_h}, y) \leq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{b-a}\int_a^b f(x,y) \diamond_{F_h} x \qquad (3.4)$$

and

$$f(x, N_{F_h})) \leq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{d-c}\int_c^d f(x,y) \diamond_{F_h} y. \qquad (3.5)$$

Integrating (3.4) over $\diamond_{F_h} y$ on $[c,d]_{I_{\mathbb{T}}}$ and (3.5) over $\diamond_{F_h} x$ on $[a,b]_{I_{\mathbb{T}}}$, we have

$$\frac{1}{d-c}\int_c^d f(M_{F_h},y)\diamond_{F_h} y \leq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\diamond_{F_h} x\diamond_{F_h} y \tag{3.6}$$

and

$$\frac{1}{b-a}\int_a^b f(x,N_{F_h})\diamond_{F_h} x \leq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\diamond_{F_h} y\diamond_{F_h} x. \tag{3.7}$$

Using (3.4), (3.5),(3.6) and (3.7), we deduce that (3.3) becomes

$$\begin{array}{ll} & \left(\frac{t}{h(t)}\right)^s\left(\frac{\lambda}{h(\lambda)}\right)^s \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\diamond_{F_h} x\diamond_{F_h} y \\ + & \left(\frac{t}{h(t)}\right)^s\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \frac{1}{d-c}\int_c^d f(M_{F_h},y)\diamond_{F_h} y \\ + & \left(\frac{1-t}{h(1-t)}\right)^s\left(\frac{\lambda}{h(\lambda)}\right)^s \frac{1}{b-a}\int_a^b f\left(x,N_{F_h}\right)\diamond_{F_h} x \\ + & \left(\frac{1-t}{h(1-t)}\right)^s\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f\left(M_{F_h},N_{F_h}\right) \\ \leq & \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s [\left(\frac{t}{h(t)}\right)^s\left(\frac{\lambda}{h(\lambda)}\right)^s + \left(\frac{t}{h(t)}\right)^s\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \\ + & \left(\frac{1-t}{h(1-t)}\right)^s\left(\frac{\lambda}{h(\lambda)}\right)^s + \left(\frac{1-t}{h(1-t)}\right)^s\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s] \\ \times & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\diamond_{F_h} x\diamond_{F_h} \\ \leq & \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{4}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\diamond_{F_h} x\diamond_{F_h} y. \end{array}$$

This proves the third inequality in (3.2). □

Theorem 3.5. *Let $h:\mathbb{J}_{\mathbb{T}}\subset\mathbb{T}\to\mathbb{R}$ be a non zero non negative function with the property that $h(t)>0$ for all $t\geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. Let $f:I^2_{\mathbb{T}}=[a,b]_{I_{\mathbb{T}}}\times[c,d]_{I_{\mathbb{T}}}\to\mathbb{R}$ be continuous, integrable and co-ordinated F_h-convex on $I^2_{\mathbb{T}}$. Then for any $a,b,c,d\geq 0$, with $b>a$, $d>c$, the following inequalities hold*

$$\begin{array}{ll} & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,\,M_{F_h})\diamond_{F_h} x + \frac{1}{d-c}\int_c^d f(N_{F_h},y)\diamond_{F_h} y\right] \\ \leq & \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{I_{x,y}(a,b;c,d)}{(b-a)(d-c)} \\ \leq & \frac{1}{2}\left[\left(\frac{\frac{d-M_{F_h}}{(b-a)(d-c)}}{h\left(\frac{d-M_{F_h}}{(b-a)(d-c)}\right)}\right)^s + \left(\frac{\frac{y-N_{F_h}}{(b-a)(d-c)}}{h\left(\frac{y-N_{F_h}}{(b-a)(d-c)}\right)}\right)^s\right] I_{x_\Delta,y_\Delta}(a,b;c,d) \\ + & \frac{1}{2}\left[\left(\frac{\frac{M_{F_h}-c}{(b-a)(d-c)}}{h\left(\frac{M_{F_h}-c}{(b-a)(d-c)}\right)}\right)^s + \left(\frac{\frac{N_{F_h}-x}{(b-a)(d-c)}}{h\left(\frac{N_{F_h}-x}{(b-a)(d-c)}\right)}\right)^s\right] I_{x_\nabla,y_\nabla}(a,b;c,d), \end{array} \tag{3.8}$$

where

$$\begin{array}{rl} M_{F_h} & = \int_a^b u\diamond_{F_h} u,\ N_{F_h}=\int_a^b v\diamond_{F_h} v, \\ I_{x,y}(a,b;c,d) & = \int_a^b\int_c^d f(x,y)\diamond_{F_h} x\diamond_{F_h} y, \\ I_{x_\Delta,y_\Delta}(a,b;c,d) & = \int_a^b\int_c^d f(x,y)\Delta x\Delta y \end{array}$$

and

$$I_{x_\nabla,y_\nabla}(a,b;c,d) = \int_a^b\int_c^d f(x,y)\nabla x\nabla y.$$

Proof. Since $x=\lambda a+(1-\lambda)b$ for $\lambda\in[0,1]$, then (1.6) can be written as

$$\begin{array}{rl} f(M_{F_h}) & \leq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{b-a}\int_a^b f(x)\diamond_{F_h} x \\ & \leq \left(\frac{\frac{b-M_{F_h}}{b-a}}{h\left(\frac{b-M_{F_h}}{b-a}\right)}\right)^s \int_a^b f(x)\Delta x + \left(\frac{\frac{M_{F_h}-a}{b-a}}{h\left(\frac{M_{F_h}-a}{b-a}\right)}\right)^s \int_a^b f(x)\nabla x, \end{array}$$

where M_{F_h} is as defined above.
By Definition 3.1, we have

$$\begin{aligned} f_x(M_{F_h}) &\leq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{d-c}\int_c^d f_x(y)\diamond_{F_h} y \\ &\leq \left(\frac{\frac{d-M_{F_h}}{d-c}}{h\left(\frac{d-M_{F_h}}{d-c}\right)}\right)^s \int_c^d f_x(y)\Delta y + \left(\frac{\frac{M_{F_h}-c}{d-c}}{h\left(\frac{M_{F_h}-c}{d-c}\right)}\right)^s \int_c^d f_x(y)\nabla y, \end{aligned}$$

That is,

$$\begin{aligned} f(x,M_{F_h}) &\leq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{d-c}\int_c^d f(x,y)\diamond_{F_h} y \\ &\leq \left(\frac{\frac{d-M_{F_h}}{d-c}}{h\left(\frac{d-M_{F_h}}{d-c}\right)}\right)^s \int_c^d f(x,y)\Delta y + \left(\frac{\frac{M_{F_h}-c}{d-c}}{h\left(\frac{M_{F_h}-c}{d-c}\right)}\right)^s \int_c^d f(x,y)\nabla y. \end{aligned} \tag{3.9}$$

Integrating both sides of (3.9) over $\diamond_{F_h}x,\ \Delta x$ and ∇x on $[a,b]_{I_{\mathbb{T}}}$, we obtain

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x,M_{F_h})\diamond_{F_h} x &\leq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\diamond_{F_h} x\diamond_{F_h} y \\ &\leq \left(\frac{\frac{d-M_{F_h}}{(b-a)(d-c)}}{h\left(\frac{d-M_{F_h}}{(b-a)(d-c)}\right)}\right)^s \int_a^b\int_c^d f(x,y)\Delta x\Delta y \\ &+ \left(\frac{\frac{M_{F_h}-c}{(b-a)(d-c)}}{h\left(\frac{M_{F_h}-c}{(b-a)(d-c)}\right)}\right)^s \int_a^b\int_c^d f(x,y)\nabla x\nabla y. \end{aligned} \tag{3.10}$$

By a similar argument, for the partial mapping $f_y:[a,b]\to\mathbb{R},\quad f_y(u):=f(u,y)$, we obtain

$$\begin{aligned} f(N_{F_h},y) &\leq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{b-a}\int_a^b f(x,y)\diamond_{F_h} x \\ &\leq \left(\frac{\frac{y-N_{F_h}}{b-a}}{h\left(\frac{y-N_{F_h}}{b-a}\right)}\right)^s \int_a^b f(x,y)\Delta x + \left(\frac{\frac{N_{F_h}-x}{b-a}}{h\left(\frac{N_{F_h}-x}{b-a}\right)}\right)^s \int_a^b f(x,y)\nabla x. \end{aligned} \tag{3.11}$$

Integrating both sides of (3.11) over $\diamond_{F_h}y,\ \Delta y$ and ∇y on $[c,d]_{I_{\mathbb{T}}}$, we obtain

$$\begin{aligned} \frac{1}{d-c}\int_c^d f(N_{F_h},y)\diamond_{F_h} x &\leq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\diamond_{F_h} x\diamond_{F_h} y \\ &\leq \left(\frac{\frac{y-N_{F_h}}{(b-a)(d-c)}}{h\left(\frac{y-N_{F_h}}{(b-a)(d-c)}\right)}\right)^s \int_a^b\int_c^d f(x,y)\Delta x\Delta y \\ &+ \left(\frac{\frac{N_{F_h}-x}{(b-a)(d-c)}}{h\left(\frac{N_{F_h}-x}{(b-a)(d-c)}\right)}\right)^s \int_a^b\int_c^d f(x,y)\nabla x\nabla y. \end{aligned} \tag{3.12}$$

Adding (3.10) and (3.12), we get the desired result (3.8). □

Remark 3.6. (i) If $F_h=\alpha,\ s=1, h(\frac{1}{2})=\frac{1}{2}$, then the first and second inequalities of Theorem 1.1. above are recovered (see Nwaeze [17]).

(ii) If we take $I_{\mathbb{T}_1}=I_{\mathbb{T}_2}=\mathbb{R}$ in Theorem 3.5, we get the second inequality of Theorem 1.1 of Dragomir [9].

4 Applications

Most dynamic optimization problems in Economics are set up in the following form: a representative consumer seeks to maximize his lifetime utility u subject to certain budget constraints A. There is the (constant) discount factor δ, which satisfies $0\leq\delta\leq 1$, C_s is consumption during period s, $u(C_s)$ is the utility the consumer derives from consuming C_s units of consumption in periods $s=0,1,2,...,T$. Utility is assumed to be concave: $u(C_s)$ has $u(C_s)'>0$

and $u(C_s)'' < 0$. The consumer receives some income Y in a time period s and decides how much to consume and save during that same period. If the consumer consumes more today, the utility or satisfaction he derives from consumption, is forgone tomorrow as the 'punishment'. The consumer would always like to consume more but each additional unit consumed during the same period generates less utility than the previous unit consumed within the same period. This property of utility function is called the law of diminishing marginal utility (LMDU). This means that the first unit of consumption of a good or service yields more utility than the second or subsequent units, with a continuing reduction for greater amounts.

The individual is constrained by the fact that the value function of his consumption, $u(C)$ must be equal to the value function of his income Y_s, plus the assets/debts, A_s that he might accommodate in a period s. A_{s+1} is the amount of assets held at the beginning of period $s+1$. A could be positive or negative; the consumer might save for the future or borrow against the future at interest rate r in any given period s but the value of A_T, which is the debt accrued with limit or the last period asset holding, has to be nonnegative(the optimal level is naturally zero, we want to spend all the money we have got and we do not care to leave money behind after death).

Thus, a simple utility maximization model of household consumption in Economics for a function of two variables can be set up and solved in time scales settings, using the same intuition as that of the dynamic optimization problem presented above, by employing our developed concepts in sections 2 and 3 as follows. The model assumes a perfect foresight.

Theorem 4.1. *Let* $R = [0, T_1] \times [0, T_2]$ *define a rectangle on* $\mathbb{T}_1 \times \mathbb{T}_2$ *and* $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ *be a nonzero non negative function with the property that* $h(t) > 0$ *for all* $t \geq 0$*, where* $\mathbb{J}_{\mathbb{T}}$ *is a* F_h*-convex subset of the real* $\mathbb{T}_1 \times \mathbb{T}_2$*. The value function of the lifetime utility* $U_{\diamond_{F_h}}$ *to be maximized subject to certain constraints is;*

$$Maximize\ U_{\diamond_{F_h}} = \int_R u(C(t_1, t_2))e_{-\delta}((t_1, 0), (t_2, 0)) \diamond_{F_h} t_1 \diamond_{F_h} t_2, \tag{4.1}$$

subject to the budget constraints

$$\begin{aligned} A^{\nabla\nabla}(t) &= [(rA + Y - C)(\rho(t))]^{\nabla}, \\ A^{\Delta\Delta}(t) &= \left[\frac{r}{1 + r\mu(t)} A^{\sigma}(t) + \frac{1}{1+r\mu(t)} y^{\sigma}(t) - \frac{1}{1+r\mu(t)} c^{\sigma}(t)\right]^{\Delta}, \\ a(0) &= a_0, \qquad a(T) = a_T, \end{aligned} \tag{4.2}$$

where u *is co-ordinated* F_h*-concave (*$u^{'}(C(t_1, t_2)) > 0$ *and* $u^{''}(C(t_1, t_2)) < 0$*),* $0 \leq \lambda \leq 1$*,* $s \in [0, 1]$*,* $A^{\Delta\Delta}$ *and* $A^{\nabla\nabla}$ *are the partial delta and nabla derivatives of the budget constraints,* e *is the exponential function,* $r, \delta, A,$ *and* Y *are as defined above.*

Proof. Let $f(t_1, t_2)$ be a function satisfied by the consumption function path that would maximize lifetime utility $u(C(t_1, t_2))e_{-\delta}((t_1, 0), (t_2, 0))$, then the condition for a functional (2.4) to have a local extremum for a function $u(t_1, t_2)$ and the sufficient condition for an absolute maximum(minimum) of (2.4) hold and hence satisfies the sufficient conditions for optimization, which in turn satisfies Theorem 3.5. □

Therefore, the model (4.1)-(4.2) can be analysed by writing (4.1) in terms of (2.3), stating the maximum principle and giving the Hamiltonian function for the model.

Remark 4.2. The new double diamond-F_h time scale model (4.1)-(4.2) unifies the convensional discrete and continuous models for $R = \mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ and $R = \mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ respectively. It equally unifies nabla, delta, double delta and single variable diamond-F_h models of [2], [7], [8] and [15] within a much more general framework.

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