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A NOTE ON SMOOTH TRANSCENDENTAL APPROXIMATION TO $|x|$

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Abstract. In this review paper, we present a pellucid proof of how $x\tanh(x/\mu)$ approximates $|x|$ and is better than $\sqrt{x^2+\mu}$ when we are concerned with accuracy.

1 Introduction

The following limits of hyperbolic tangent (see [2]):

$$\lim_{x\to-\infty}\tanh(x)=-1$$

and

$$\lim_{x\to\infty}\tanh(x)=1$$

are known. It is easy to see that for $\mu>0$,

$$\lim_{\mu\to0}\tanh\left(\frac{x}{\mu}\right)=-1\ ;\quad \text{when } x<0$$
$$\text{and } \lim_{\mu\to0}\tanh\left(\frac{x}{\mu}\right)=1;\quad \text{when } x>0.$$

As a consequence for $\mu\to0$ one can write

$$x\tanh\left(\frac{x}{\mu}\right)\approx|x|.$$

$x\tanh\left(\frac{x}{\mu}\right)$ being differentiable can be a good approximation for $|x|$. The following theorem [1] in this connection was recently proposed by first author.

Theorem 1.1. *([1, Theorem 1]) The approximation* $h(x)=x\tanh\left(\frac{x}{\mu}\right);\mu>0\in\mathbf{R}$ *to* $|x|$ *satisfies*

$$x\tanh\left(\frac{x}{\mu}\right)-\mu<|x|<x\tanh\left(\frac{x}{\mu}\right)+\mu. \tag{1.1}$$

The proof of Theorem 1.1 in [1] is somewhat cumbersome and doesn't sound much convincing. The initial goal of this paper is to provide new pellucid proof of Theorem 1.1 and then to show how $x\tanh(x/\mu)$ is better approximation of $|x|$ than $\sqrt{x^2+\mu^2}$ or $\sqrt{x^2+\mu}$ in terms of accuracy. The details about the approximations $\sqrt{x^2+\mu^2}$ and $\sqrt{x^2+\mu}$ can be found in [4] and [3] respectively.

2 Main Result

We need the following lemma for our promising proof.

Lemma 2.1. *For* $x\in\mathbf{R}$ *such that* $x\neq0$ *we have*

$$|\tanh(x)|+\frac{1}{|x|}>1. \tag{2.1}$$

Proof: We consider the following two cases:

Case(1): For $x > 0$, we introduce the function $f(x) = \tanh(x) + \frac{1}{x} - 1$ which on differentiation gives

$$f'(x) = \frac{1}{\cosh^2(x)} - \frac{1}{x^2}.$$

Therefore $f'(x) < 0$, since $\cosh(x) > x$. Hence $f(x)$ is decreasing on $(0, \infty)$ and we have that

$$f(x) > f(\infty^-) \text{ for any } x > 0.$$

So

$$\tanh(x) + \frac{1}{x} - 1 > 0.$$

Case(2): For $x < 0$ we introduce the function $g(x) = \tanh(x) + \frac{1}{x} + 1$. As in Case 1, $g'(x) < 0$ and is decreasing in $(-\infty, 0)$.

$$\text{Hence } g(x) < g(-\infty^+) \text{ for any } x < 0.$$

So we get

$$\tanh(x) + \frac{1}{x} + 1 < 0,$$

which proves our lemma.

Proof of Theorem 1: Clearly for $x = 0$ the theorem holds. For $x \neq 0$ we prove (1.1) by making use of Lemma 2.1 as follows:
consider

$$\begin{aligned} \left| |x| - x\tanh\left(\frac{x}{\mu}\right) \right| &= \left| |x| - \left| x\tanh\left(\frac{x}{\mu}\right) \right| \right| \\ &= |x| \left| 1 - \left| \tanh\left(\frac{x}{\mu}\right) \right| \right| \\ &< |x| \left| \frac{\mu}{x} \right| = \mu \end{aligned}$$

by Lemma 2.1. This completes the proof. □

In the same paper, it is claimed that the approximation $x\tanh(x/\mu)$ to $|x|$ is better than $\sqrt{x^2 + \mu^2}$. The claim is supported by graphs; but is not proved. We prove this claim as follows:

Comparison between two approximations: All three functions being positive and

$$x\tanh\left(\frac{x}{\mu}\right) < |x| < \sqrt{x^2 + \mu^2}$$

that is

$$x^2\tanh^2\left(\frac{x}{\mu}\right) < x^2 < x^2 + \mu^2.$$

It is enough to prove that

$$x^2 - x^2\tanh^2\left(\frac{x}{\mu}\right) < \mu^2$$

which is equivalent to

$$x^2\,\text{sech}^2\left(\frac{x}{\mu}\right) < \mu^2.$$

This follows immediately due to

$$\cosh\left(\frac{x}{\mu}\right) > \frac{x}{\mu}.$$

Now as $\mu \to 0$, $\sqrt{x^2+\mu^2} < \sqrt{x^2+\mu}$, proving that $x\tanh(x/\mu)$ is far better than $\sqrt{x^2+\mu}$ as far as accuracy is concerned.

This fact is illustrated in the following table by investigating global L_2 error which is given by

$$e(h) = \int_{-\infty}^{\infty} [|x| - h(x)]^2 \, dx$$

where $h(x)$ is approximation to $|x|$.

Table 1. Global L_2 errors $e(h)$ for the functions $h(x)$

	$\mu = 0.1$	
$h(x)$	$x\tanh(x/\mu)$	$\sqrt{x^2+\mu}$
$e(h)$	≈ 0.000158151	≈ 0.042164

	$\mu = 0.01$	
$h(x)$	$x\tanh(x/\mu)$	$\sqrt{x^2+\mu}$
$e(h)$	$\approx 1.58151 \times 10^{-7}$	≈ 0.00133333

Again it is easy to verify the following by above formula:
$e(x\tanh(x/\mu)) = 0.158151 \times \mu^3$, while $e(\sqrt{x^2+\mu^2}) = \frac{4}{3} \times \mu^3$ supporting the claim.

3 Conclusion

A new crystal clear proof of old theorem is presented in a simple way and two approximations are compared analytically as well as by numerical illustrations.

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