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A STUDY ON GROWTH OF ITERATED ENTIRE FUNCTIONS

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Abstract. In this paper we consider iteration of three entire functions and study some growth properties.

1 Introduction

Let $f(z)$ and $g(z)$ be two entire functions. Lahiri and Banerjee in [7] form the iterations of $f(z)$ with respect to $g(z)$ as follows:

$$\begin{aligned} f_1(z) &= f(z) \\ f_2(z) &= f(g(z)) = f(g_1(z)) \\ f_3(z) &= f(g(f(z))) = f(g(f_1(z))) \\ &\vdots \\ f_n(z) &= f(g(f(g\ldots(f(z) \text{ or } g(z) \text{ according as } n \text{ is odd or even})\ldots))) \\ &= f(g_{n-1}(z)) = f(g(f_{n-2}(z))) \end{aligned}$$

and so

$$\begin{aligned} g_1(z) &= g(z) \\ g_2(z) &= g(f(z)) = g(f_1(z)) \\ g_3(z) &= g(f_2(z)) = g(f(g_1(z))) \\ &\vdots \\ g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))). \end{aligned}$$

Then all $f_n(z)$ and $g_n(z)$ are entire functions.

Let $f(z)$ and $g(z)$ be entire functions. Banerjee and Dutta [1] used the notations $M_{f_1}(r)$, $M_{f_2}(r)$, $M_{f_3}(r)$, etc to mean $M(r,f)$, $M(M(r,f),g)$, $M(M(M(r,f),g),f)$, respectively and $F(r) = O^*(G(r))$ to mean that there exist two positive costants K_1 and K_2 such that $K_1 \leq \frac{F(r)}{G(r)} \leq K_2$ for large r.

In 2003 Sun [9] proved the following theorem.

Theorem 1.1. *Let $f_1,\ f_2$ and $g_1,\ g_2$ be four transcendental entire functions with $T(r,f_1) = O^*\left((\log r)^{\nu} e^{(\log r)^{\alpha}}\right)$ and $T(r,g_1) = O^*\left((\log r)^{\beta}\right)$.*

If $T(r,f_1) \sim T(r,f_2)$ and $T(r,g_1) \sim T(r,g_2)$ $(r \to \infty)$, then

$$T(r, f_1(g_1)) \sim T(r, f_2(g_2)) \quad (r \to \infty,\ r \notin E),$$

where $\nu > 0$, $0 < \alpha < 1$, $\beta > 1$ and $\alpha\beta < 1$ and E is a set of finite logarithmic measure.

After this in 2011, Banerjee and Dutta [1] extend Theorem 1.1 for iterated entire functions in the following manner.

Theorem 1.2. *Let f, g, u, v be four transcendental entire functions with* $T(r,f) \sim T(r,u)$, $T(r,g) \sim T(r,v)$, $T(r,f) = O^*\left((\log r)^{\nu} e^{(\log r)^{\alpha}}\right)$ $(0<\alpha<1, \nu>0)$ *and* $T(r,g) = O^*\left((\log r)^{\beta}\right)$ *where* $\beta > 1$ *is a constant, then* $T(r,f_n) \sim T(r,u_n)$ *for* $n \geq 2$,
where $u_n(z) = u(v(u(v...(u(z)$ *or* $v(z))...)))$ *according as* n *is odd or even.*

In [8], Niino and Suita proved the following theorem.

Theorem 1.3. *Let* $f(z)$ *and* $g(z)$ *be entire functions. If* $M(r,g) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$ *for any* $\varepsilon > 0$, *then we have*

$$T(r, f(g)) \leq (1+\varepsilon) T(M(r,g), f).$$

In particular, if $g(0) = 0$, *then*

$$T(r, f(g)) \leq T(M(r,g), f) \text{ for all } r > 0.$$

As a generalisation of Theorem 1.3, Banerjee and Dutta [1] proved the following theorem.

Theorem 1.4. *Let* $f(z), g(z)$ *be two entire functions.Then*

$$T(R_2, f) \leq T(r, f_n) \leq T(R_3, f)$$

where $|f(z)| > R_1 > \frac{2+\varepsilon}{\varepsilon}|f(0)|$ *and* $|g(z)| > R_2 > \frac{2+\varepsilon}{\varepsilon}|g(0)|$, $R_3 = \max\{M_{f_{n-1}}(r), M_{g_{n-1}}(r)\}$ *for sufficiently large values of* r *and any integer* $n \geq 2$.

Further they showed Theorem 1.2 also true for less conditions and proved the following theorem.

Theorem 1.5. *Let f, g, u, v be four transcendental entire functions with* $T(r,f) \sim T(r,u)$, $T(r,g) \sim T(r,v)$, $T(r,f) = O^*\left((\log r)^{\beta}\right)$ *and* $T(r,g) = O^*\left((\log r)^{\beta}\right)$ *where* $\beta > 1$ *is a constant, then* $T(r,f_n) \sim T(r,u_n)$.

In this paper we consider three entire functions $f(z), g(z)$ and $h(z)$ and following Banerjee and Mandal [2] form the iterations of $f(z)$ with respect to $g(z)$ and $h(z)$ [defined below] and generalise the results of Banerjee and Dutta [1] in this direction.

$$\begin{aligned}
f_1(z) &= f(z) \\
f_2(z) &= f(g(z)) = f(g_1(z)) \\
f_3(z) &= f(g(h(z))) = f(g(h_1(z))) = f(g_2(z)) \\
f_4(z) &= f(g(h(f(z)))) = f(g(h_2(z))) = f(g_3(z)) \\
&\vdots \\
f_n(z) &= f(g(h(f..(f(z) \text{ or } g(z) \text{ or } h(z) \text{ according as } n = 3m-2 \text{ or } 3m-1 \\
&\quad \text{or } 3m)...))) \\
&= f(g_{n-1}(z)) = f(g(h_{n-2}(z))).
\end{aligned}$$

Similarly,

$$\begin{aligned} g_1(z) &= g(z) \\ g_2(z) &= g(h(z)) = g(h_1(z)) \\ g_3(z) &= g(h(f(z))) = g(h(f_1(z))) = g(h_2(z)) \\ g_4(z) &= g(h(f(g(z)))) = g(h(f_2(z))) = g(h_3(z)) \\ &\vdots \\ g_n(z) &= g(h(f(g...(g(z) \text{ or } h(z) \text{ or } f(z) \text{ according as } n = 3m-2 \text{ or } 3m-1 \\ &\quad \text{or } 3m)...))) \\ &= g(h_{n-1}(z)) = g(h(f_{n-2}(z))) \end{aligned}$$

and

$$\begin{aligned} h_1(z) &= h(z) \\ h_2(z) &= h(f(z)) = h(f_1(z)) \\ h_3(z) &= h(f(g(z))) = h(f(g_1(z))) = h(f_2(z)) \\ h_4(z) &= h(f(g(h(z)))) = h(f(g_2(z))) = h(f_3(z)) \\ &\vdots \\ h_n(z) &= h(f(g(h...(h(z) \text{ or } f(z) \text{ or } g(z) \text{ according as } n = 3m-2 \text{ or } 3m-1 \\ &\quad \text{or } 3m)...))) \\ &= h(f_{n-1}(z)) = h(f(g_{n-2}(z))). \end{aligned}$$

Clearly all f_n, g_n and h_n are entire functions.

We now use the following notation throughout the paper.

Let $f(z)$, $g(z)$ and $h(z)$ be entire functions. we use the notations $M_{f_1}(r)$, $M_{f_2}(r)$, $M_{f_3}(r)$, $M_{f_4}(r)$ etc to mean $M(r,f)$, $M(M(r,f),h)$, $M(M(M(r,f),h),g)$, $M(M(M(M(r,f),h),g),f)$ respectively. Similarly we use the notations $M_{g_1}(r)$, $M_{g_2}(r)$, $M_{g_3}(r)$, $M_{g_4}(r)$ etc to mean $M(r,g)$, $M(M(r,g),f)$, $M(M(M(r,g),f),h)$, $M(M(M(M(r,g),f),h),g)$ respectively and $M_{h_1}(r)$, $M_{h_2}(r)$, $M_{h_3}(r)$, $M_{h_4}(r)$ etc to mean $M(r,h)$, $M(M(r,h),g)$, $M(M(M(r,h),g),f)$, $M(M(M(M(r,h),g),f),h)$ respectively and $F(r) = O^*(G(r))$ to mean that there exist two positive costants K_1 and K_2 such that $K_1 \le \frac{F(r)}{G(r)} \le K_2$ for any large r.

2 Lemmas

Lemma 2.1. *[6] Let $f(z)$ be an entire function. For $0 \le r < R < \infty$, we have*

$$T(r,f) \le \log^+ M(r,f) \le \frac{R+r}{R-r} T(R,f).$$

Lemma 2.2. *[5] Let $f(z)$ be an entire function of order ρ $(\rho < \infty)$. If $k > \rho - 1$, then*

$$\log M(r,f) \sim \log M\left(r - r^{-k}, f\right) \quad (r \to \infty).$$

Lemma 2.3. *[8] Let $g(z)$ and $f(z)$ be two entire functions. Suppose that $|g(z)| > R > |g(0)|$ on the circumference $\{|z| = r\}$ for some $r > 0$. Then we have*

$$T(r, f(g)) \ge \frac{R - |g(0)|}{R + |g(0)|} T(R,f).$$

Lemma 2.4. *[9] Let f be a transcendental entire function with*

$$T(r,f)=O^*\left((\log r)^{\beta}e^{(\log r)^{\alpha}}\right)\qquad(0<\alpha<1,\beta>0).$$

Then

$$\begin{aligned}T(r,f) &\sim \log M(r,f) \qquad (r\to\infty, r\notin E)\ \textit{and}\\ T(\sigma r,f) &\sim T(r,f) \qquad (r\to\infty,\sigma\geq 2, r\notin E),\end{aligned}$$

where E is a set of finite logarithmic measure.

Lemma 2.5. *[1] Let f be a transcendental entire function with $T(r,f)=O^*\left((\log r)^{\beta}\right)$ where $\beta>1$. Then*

$$\begin{aligned}T(r,f) &\sim \log M(r,f) \qquad (r\to\infty, r\notin E)\ \textit{and}\\ T(\sigma r,f) &\sim T(r,f) \qquad (r\to\infty,\sigma\geq 2, r\notin E),\end{aligned}$$

where E is a set of finite logarithmic measure.

Lemma 2.6. *[1] Let f_1 and f_2 be two entire functions with $T(r,f_1)=O^*\left((\log r)^{\beta}\right)$ where $\beta>1$ and $T(r,f_1)\sim T(r,f_2)$ then $M(r,f_1)\sim M(r,f_2)$.*

Lemma 2.7. *[1] Let f_1 and f_2 be two entire functions with $T(r,f_1)=O^*\left((\log r)^{\nu}e^{(\log r)^{\alpha}}\right)$ where $\nu>1$, $0<\alpha<1$ and $T(r,f_1)\sim T(r,f_2)$ then $M(r,f_1)\sim M(r,f_2)$.*

3 Main Results

Theorem 3.1. *Let $f(z), g(z)$ and $h(z)$ be three entire functions.Then we have*

$$T(R_2,f)\leq T(r,f_n)\leq T(R_4,f)\tag{3.1}$$

where $|f(z)|>R_1>\frac{2+\varepsilon}{\varepsilon}|f(0)|$, $|g(z)|>R_2>\frac{2+\varepsilon}{\varepsilon}|g(0)|$, $|h(z)|>R_3>\frac{2+\varepsilon}{\varepsilon}|h(0)|$ and $R_4=\max\{M_{f_{n-1}}(r), M_{g_{n-1}}(r), M_{h_{n-1}}(r)\}$ for sufficiently large values of r and any integer $n\geq 3$.

Proof. **CASE-I :** When $n=3m, m\in\mathbb{N}$ and $\varepsilon>0$ arbitrary small, then by Theorem 1.3, we have

$$\begin{aligned}T(r,f_n) &= T(r,f_{n-1}(h))\\ &\leq (1+\varepsilon)T(M(r,h),f_{n-1})\\ &= (1+\varepsilon)T(M_{h_1}(r),f_{n-2}(g))\\ &\leq (1+\varepsilon)^2T(M(M_{h_1}(r),g),f_{n-2})\\ &= (1+\varepsilon)^2T(M_{h_2}(r),f_{n-2})\\ &= (1+\varepsilon)^2T(M_{h_2}(r),f_{n-3}(f))\\ &\leq (1+\varepsilon)^3T(M_{h_3}(r),f_{n-3})\\ &= (1+\varepsilon)^3T(M_{h_3}(r),f_{n-4}(h))\\ &\leq (1+\varepsilon)^4T(M_{h_4}(r),f_{n-4})\\ &\ \ \vdots\\ &\leq (1+\varepsilon)^{n-1}T\left(M_{h_{n-1}}(r),f\right)\\ &\leq (1+\varepsilon)^{n-1}T(R_4,f).\end{aligned}$$

CASE-II : When $n = 3m - 1, m \in \mathbb{N}$ we have

$$\begin{aligned}
T(r, f_n) &= T(r, f_{n-1}(g)) \\
&\leq (1+\varepsilon) T(M(r,g), f_{n-1}) \\
&= (1+\varepsilon) T(M_{g_1}(r), f_{n-2}(f)) \\
&\leq (1+\varepsilon)^2 T(M(M_{g_1}(r), f), f_{n-2}) \\
&= (1+\varepsilon)^2 T(M_{g_2}(r), f_{n-2}) \\
&= (1+\varepsilon)^2 T(M_{g_2}(r), f_{n-3}(h)) \\
&\leq (1+\varepsilon)^3 T(M_{g_3}(r), f_{n-3}) \\
&= (1+\varepsilon)^3 T(M_{g_3}(r), f_{n-4}(g)) \\
&\leq (1+\varepsilon)^4 T(M_{g_4}(r), f_{n-4}) \\
&\quad \vdots \\
&\leq (1+\varepsilon)^{n-1} T\left(M_{g_{n-1}}(r), f\right) \\
&\leq (1+\varepsilon)^{n-1} T(R_4, f).
\end{aligned}$$

CASE-III : When $n = 3m - 2, m \in \mathbb{N}$ we have

$$\begin{aligned}
T(r, f_n) &= T(r, f_{n-1}(f)) \\
&\leq (1+\varepsilon) T(M(r,f), f_{n-1}) \\
&= (1+\varepsilon) T(M_{f_1}(r), f_{n-2}(h)) \\
&\leq (1+\varepsilon)^2 T(M(M_{f_1}(r), h), f_{n-2}) \\
&= (1+\varepsilon)^2 T(M_{f_2}(r), f_{n-2}) \\
&= (1+\varepsilon)^2 T(M_{f_2}(r), f_{n-3}(g)) \\
&\leq (1+\varepsilon)^3 T(M_{f_3}(r), f_{n-3}) \\
&= (1+\varepsilon)^3 T(M_{f_3}(r), f_{n-4}(f)) \\
&\leq (1+\varepsilon)^4 T(M_{f_4}(r), f_{n-4}) \\
&\quad \vdots \\
&\leq (1+\varepsilon)^{n-1} T\left(M_{f_{n-1}}(r), f\right) \\
&\leq (1+\varepsilon)^{n-1} T(R_4, f).
\end{aligned}$$

Therefore $T(r, f_n) \leq (1+\varepsilon)^{n-1} T(R_4, f)$ for any integer $n \geq 3$.
Since $\varepsilon > 0$ was arbitrary, for sufficiently large values of r we have

$$T(r, f_n) \leq T(R_4, f). \tag{3.2}$$

Also using Lemma 2.3, we have
When $n = 3m, m \in \mathbb{N}$

$$\begin{aligned}
T(r, f_n) &= T(r, f_{n-1}(h)) \\
&\geq \left(\frac{R_3 - |h(0)|}{R_3 + |h(0)|}\right) T(R_3, f_{n-1}) \\
&> (1-\varepsilon) T(R_3, f_{n-2}(g)) \\
&\geq (1-\varepsilon)\left(\frac{R_2 - |g(0)|}{R_2 + |g(0)|}\right) T(R_2, f_{n-2}) \\
&> (1-\varepsilon)^2 T(R_2, f_{n-2}) \\
&\geq (1-\varepsilon)^3 T(R_1, f_{n-3}) \\
&\vdots \\
&\geq (1-\varepsilon)^{n-2} T(R_3, f(g)) \\
&\geq (1-\varepsilon)^{n-1} T(R_2, f).
\end{aligned}$$

When $n = 3m - 1$, $m \in \mathbb{N}$ we have

$$\begin{aligned}
T(r, f_n) &= T(r, f_{n-1}(g)). \\
&\geq \left(\frac{R_2 - |g(0)|}{R_2 + |g(0)|}\right) T(R_2, f_{n-1}) \\
&> (1-\varepsilon) T(R_2, f_{n-2}(f)) \\
&\geq (1-\varepsilon)\left(\frac{R_1 - |f(0)|}{R_1 + |f(0)|}\right) T(R_1, f_{n-2}) \\
&> (1-\varepsilon)^2 T(R_1, f_{n-3}(h)) \\
&\geq (1-\varepsilon)^3 T(R_3, f_{n-3}) \\
&\vdots \\
&\geq (1-\varepsilon)^{n-2} T(R_3, f(g)) \\
&\geq (1-\varepsilon)^{n-1} T(R_2, f).
\end{aligned}$$

When $n = 3m - 2$, $m \in \mathbb{N}$ we have

$$\begin{aligned}
T(r, f_n) &= T(r, f_{n-1}(f)) \\
&\geq \left(\frac{R_1 - |f(0)|}{R_1 + |f(0)|}\right) T(R_1, f_{n-1}) \\
&> (1-\varepsilon) T(R_1, f_{n-2}(h)) \\
&\geq (1-\varepsilon)\left(\frac{R_3 - |h(0)|}{R_3 + |h(0)|}\right) T(R_3, f_{n-2}) \\
&> (1-\varepsilon)^2 T(R_3, f_{n-3}(g)) \\
&\geq (1-\varepsilon)^3 T(R_2, f_{n-3}) \\
&\vdots \\
&\geq (1-\varepsilon)^{n-2} T(R_3, f(g)) \\
&\geq (1-\varepsilon)^{n-1} T(R_2, f).
\end{aligned}$$

So

$$T(r, f_n) \geq (1-\varepsilon)^{n-1} T(R_2, f).$$

Therefore for sufficiently large values of r and for $\varepsilon > 0$ arbitrarily, small we have

$$T(r, f_n) \geq T(R_2, f) \tag{3.3}$$

Hence from (3.2) and (3.3) we obtain (3.1).
This completes the proof. □

Theorem 3.2. *Let f, g, h ; u, v, w be six transcendental entire functions with $T(r, f) \sim T(r, u)$, $T(r, g) \sim T(r, v)$, $T(r, h) \sim T(r, w)$, $T(r, f) = O^*\left((\log r)^{\nu} e^{(\log r)^{\alpha}}\right)$ $(0 < \alpha < 1, \nu > 0)$, $T(r, g) = O^*\left((\log r)^{\beta}\right)$ and $T(r, h) = O^*\left((\log r)^{\lambda}\right)$ where $\beta, \lambda > 1$ are constants. Then $T(r, f_n) \sim T(r, u_n)$ for $n \geq 3$, where $u_n(z) = u(v(w(u...(u(z)$ or $v(z)$ or $w(z)...))))$ according as $n = 3m - 2$ or $3m - 1$ or $3m$, $m \in \mathbb{N}$.*

Proof. We have from Theorem 3.1,

$$\begin{aligned} T(R_1, f) &\leq T(r, f_n) \leq T(R_2, f) && (3.4) \\ T\left(R_1', u\right) &\leq T(r, u_n) \leq T\left(R_2', u\right) && (3.5) \end{aligned}$$

where R_1 and R_1' are such that $|g(z)| > R_1 > \frac{2+\varepsilon}{\varepsilon}|g(0)|$, $|v(z)| > R_1' > \frac{2+\varepsilon}{\varepsilon}|v(0)|$ and $T(R_1, f) \sim T\left(R_1', f\right)$, $R_2 = \max\left\{M_{f_{n-1}}(r), M_{g_{n-1}}(r), M_{h_{n-1}}(r)\right\}$ and $R_2' = \max\left\{M_{u_{n-1}}(r), M_{v_{n-1}}(r), M_{w_{n-1}}(r)\right\}$ for sufficiently large values of r and arbitrary small $\varepsilon > 0$.

Since

$$T(r, f) \sim T(r, u)$$

so

$$T(R_1, f) \sim T\left(R_1', f\right) \sim T\left(R_1', u\right)$$

$$\text{i.e.} \quad T(R_1, f) \sim T\left(R_1', u\right) \quad (r \to \infty,\ r \notin E). \tag{3.6}$$

From Lemma 2.7, we have

$$M(r, f) \sim M(r, u).$$

Also we have

$$\begin{aligned} M(M(r, f), h) &\sim M(M(r, u), w) \quad (r \to \infty) \\ \text{i.e } M(M(M(r, f), h), g) &\sim M(M(M(r, u), w), v) \quad (r \to \infty). \end{aligned}$$

So, for $n = 3m - 2$, $m \in \mathbb{N}$ we have

$$M_{f_{n-1}}(r) \sim M_{u_{n-1}}(r) \quad (r \to \infty). \tag{3.7}$$

Similarly for $n = 3m - 1$, $m \in \mathbb{N}$ we have

$$M_{g_{n-1}}(r) \sim M_{v_{n-1}}(r) \quad (r \to \infty) \tag{3.8}$$

and for $n = 3m$, $m \in \mathbb{N}$ we have

$$M_{h_{n-1}}(r) \sim M_{w_{n-1}}(r) \quad (r \to \infty). \tag{3.9}$$

Combining (3.7), (3.8) and (3.9) and for $n \geq 3$, $n \in \mathbb{N}$ we obtain $R_2 \sim R_2'$ for large r.
So combining $T(r, f) \sim T(r, u)$ and $R_2 \sim R_2'$ we have

$$T(R_2, f) \sim T\left(R_2', u\right) \quad (r \to \infty). \tag{3.10}$$

So from (3.4), (3.5), (3.6) and (3.10) we get

$$T(r, f_n) \sim T(r, u_n).$$

This completes the proof. □

Theorem 3.3. *Let* f, g, h ; u, v, w *be six transcendental entire functions with* $T(r, f) \sim T(r, u)$, $T(r, g) \sim T(r, v)$, $T(r, h) \sim T(r, w)$, $T(r, f) = O^*\left((\log r)^\beta\right)$, $T(r, g) = O^*\left((\log r)^\beta\right)$ *and* $T(r, h) = O^*\left((\log r)^\beta\right)$ *where* $\beta > 1$ *is a constant. Then* $T(r, f_n) \sim T(r, u_n)$ *for* $n \geq 3$, *where* $u_n(z) = u(v(w(u...(u(z) \text{ or } v(z) \text{ or } w(z)...))))$ *according as* $n = 3m-2$ *or* $3m-1$ *or* $3m$, $m \in \mathbb{N}$.

The conditions of Theorem 3.2 and Theorem 3.3 are not strictly sharp, which are illustrated by the following examples.

Example 3.4. Let $f(z) = e^{2z}$, $g(z) = 2z$, $h(z) = 3z$ and $u(z) = 2e^{2z}$, $v(z) = 4z$, $w(z) = 6z$.
Then we have $f_3(z) = f(g(h(z))) = f(g(3z)) = f(6z) = e^{12z}$ and $u_3(z) = u(v(w(z))) = u(v(6z)) = u(24z) = 2e^{48z}$.
Now

$$\begin{aligned} T(r, f) &= \frac{2r}{\pi},\ T(r, u) = \frac{2r}{\pi} + \log 2 \\ T(r, g) &= \log r + \log 2,\ T(r, v) = \log r + \log 4 \\ T(r, h) &= \log r + \log 3,\ T(r, w) = \log r + \log 6 \end{aligned}$$

Thus $T(r, f) \sim T(r, u)$, $T(r, g) \sim T(r, v)$ and $T(r, h) \sim T(r, w)$.
Again

$$T(r, f_3) = \frac{12r}{\pi}, T(r, u_3) = \frac{48r}{\pi} + \log 2.$$

Therefore

$$\frac{T(r, f_3)}{T(r, u_3)} = \frac{\frac{12r}{\pi}}{\frac{48r}{\pi} + \log 2} = \frac{1}{4} \quad \text{as } r \to \infty.$$

So

$$T(r, f_3) \nsim T(r, u_{3.}) \text{ as } r \to \infty.$$

Example 3.5. Let $f(z) = 2e^z$, $g(z) = e^z$, $h(z) = z$ and $u(z) = 2z$, $v(z) = e^{\frac{z}{2}}$, $w(z) = 2e^z$.
Then we have $f_3(z) = f(g(h(z))) = f(g(z)) = f(e^z) = 2e^{e^z}$ and $u_3(z) = u(v(w(z))) = u(v(2e^z)) = u\left(e^{e^z}\right) = 2e^{e^z}$.
Then we have

$$\begin{aligned} T(r, f) &= \frac{r}{\pi} + \log 2,\ T(r, u) = \log 2 + \log r \\ T(r, g) &= \frac{r}{\pi},\ T(r, v) = \frac{r}{2\pi} \\ T(r, h) &= \log r,\ T(r, w) = \log 2 + \frac{r}{\pi}.. \end{aligned}$$

Therefore we have $T(r, f) \nsim T(r, u)$, $T(r, g) \nsim T(r, v)$, $T(r, h) \nsim T(r, w)$ as $r \to \infty$.
But still

$$T(r, f_3) \sim T(r, u_3) \text{ as } r \to \infty.$$

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