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On the Fekete-Szegö coefficient functional for quasi-subordination class

S R Swamy and Y Sailaja

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Abstract In this work, considering a special subclass of the family of holomorphic functions in an open unit disk, defined by means of quasi-subordination, we determine sharp bounds for Fekete-Szegö functional $|d_3 - \mu d_2^2|$ of functions in this class. Several results for new classes and connections to known classes are mentioned.

1 Introduction, preliminaries and definitions

Let A be the family of normalized functions that have the form

$$s(z) = z + \sum_{k=2}^{\infty} d_k z^k, \tag{1.1}$$

which are holomorphic in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the collection of all members from A that are univalent in $\mathfrak{D}$. Let $\eta(z)$ be holomorphic function in $\mathfrak{D}$ with $|\eta(z)| \leq 1, z \in \mathfrak{D}$, such that

$$\eta(z) = R_0 + R_1 z + R_2 z^2 + ..., \tag{1.2}$$

where $R_0, R_1, R_2, ...$ are real. Let $\mathfrak{h}(z)$ be holomorphic function in $\mathfrak{D}$, with $\mathfrak{h}(0) = 1, \mathfrak{h}'(0) > 0$, having the positive real part, such that

$$\mathfrak{h}(z) = 1 + Q_1 z + Q_2 z^2 + ..., \tag{1.3}$$

where $Q_1, Q_2, Q_3, ...$ are real and $Q_1 > 0$. Through out this work we shall assume that the functions η and $\mathfrak{h}$ follow the above conditions unless otherwise mentioned.

It is known that for $s \in \mathcal{S}$ given by (1.1), there holds upper bounds for $|d_3 - \mu d_2^2|$ when μ is real, which are sharp (see[8]). Since then, the estimation of the sharp upper bounds for $|d_3 - \mu d_2^2|$ with μ being an arbitrary real or complex number for any compact collection $\mathfrak{F}$ of elements in $\mathcal{S}$ is well- known as the Fekete- Szegö problem for $\mathfrak{F}$. Several researchers including [3], [5], [6], [11], [12], [17] and [20] have estimated sharp Fekete-Szegö bounds for many subclasses of $\mathcal{S}$. Additional informations about Fekete-Szegö problem associated with q- derivative operator are available in the works of Alsoboh and Darus [2] and Elhaddad and Darus [7]. Very interesting resource about Fekete-Szegö inequality associated with the Haradam polynomials may be found in the investigation by Srivastava et. al [18].

We recall the principle of subordinatin and also the principle of majorization, between two holomorphic functions $s(z)$ and $\nu(z)$ in $\mathfrak{D}$. We say that $s(z)$ is subordinate to $\nu(z)$, written $s(z) \prec \nu(z)$, $z \in \mathfrak{D}$, if there is a holomorphic function $\omega(z)$ in $\mathfrak{D}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathfrak{D}$, such that $s(z) = \nu(\omega(z))$. Moreover $s(z) \prec \nu(z)$ is equivalent to $s(0) = \nu(0)$ and $s(\mathfrak{D}) \subset \nu(\mathfrak{D})$, if ν is univalent in $\mathfrak{D}$. We know that $s(z)$ is majorized by $\nu(z)$, written $s(z) \prec\prec \nu(z)$, $z \in \mathfrak{D}$, if there exists a holomorphic function $\eta(z)$, $z \in \mathfrak{D}$, with $|\eta(z)| \leq 1$, such that $s(z) = \eta(z)\nu(z)$, $z \in \mathfrak{D}$.

Robertson [16] introduced a new concept called quasi-subordination, which generalizes both subordination and majorization. For two holomorphic functions $s(z)$ and $\nu(z)$, $s(z)$ is quasi-subordinate to $\nu(z)$, written as $s(z) \prec_q \nu(z)$, $z \in \mathfrak{D}$, if there exists holomorphic functions η and ω with $|\eta(z)| \leq 1, \omega(0) = 0$ and $|\omega(z)| < 1$ such that $s(z) = \eta(z)\nu(\omega(z)), z \in \mathfrak{D}$. Observe that if $\eta(z) = 1$, then $s(z) = \nu(\omega(z)), z \in \mathfrak{D}$, so that $s(z) \prec \nu(z)$ in $\mathfrak{D}$. Also note that if $\omega(z) = z$, then $s(z) = \eta(z)\nu(z)$, $z \in \mathfrak{D}$ and hence $s(z) \prec\prec \nu(z)$ in $\mathfrak{D}$. There are more studies related to quasi-subordination such as [1], [4], [9], [14] and [15].

Let Υ be the family of holomorphic functions in $\mathfrak{D}$ of the form

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + ... \tag{1.4}$$

satisfying the condition $|\omega(z)| < 1$, $z \in \mathfrak{D}$. We require the following lemma [10] to prove our main result.

Lemma 1.1. *If $\omega \in \Upsilon$, then for any complex number μ, we have $|\omega_1| \leq 1$, $|\omega_2 - \mu\omega_1^2| \leq 1 + (|\mu| - 1)|\omega_1|^2 \leq max\{1, |\mu|\}$. $\omega(z) = z$ or $\omega(z) = z^2$ exhibit the sharpness of the result.*

Inspired by recent trends on quasi-subordination, we define the following new subclasses of the family A.

Definition 1.2. A function $s(z)$ in A is said to be in the family $\mathfrak{R}_q(\tau, \gamma, \mathfrak{h})$, $\tau \geq 0$, $\gamma \in \mathbb{C} - \{0\}$, if

$$\frac{1}{\gamma}(s'(z) + \tau z s''(z) - 1) \prec_q (\mathfrak{h}(z) - 1), z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

Definition 1.3. A function $s(z)$ in A is said to be in the family $\mathfrak{L}_q(\tau, \gamma, \mathfrak{h})$, $\tau \geq 1$, $\gamma \in \mathbb{C} - \{0\}$, if

$$1 + \frac{\tau}{\gamma}\left(\frac{zs''(z)}{s'(z)}\right) \prec_q \mathfrak{h}(z),\ z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

Motivated by the paper [19] and earlier works on quasi-subordination, we now define a new special class $M_q(\tau, \gamma, \mu, \mathfrak{h})$.

Definition 1.4. A function $s(z)$ in A is said to be in the family $M_q(\tau, \xi, \gamma, \mathfrak{h})$, $0 \leq \xi \leq 1$, $\tau \geq 0$, $\tau \geq \xi$, $\gamma \in \mathbb{C} - \{0\}$, if

$$\frac{1}{\gamma}\left(\frac{zs'(z) + \tau z^2 s''(z)}{(1-\xi)z + \xi z s'(z)} - 1\right) \prec_q (\mathfrak{h}(z) - 1), z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

Clearly a function s is in $M_q(\tau, \xi, \gamma, \mathfrak{h})$ if and only if there exits a holomorphic function $\eta(z)$ with $|\eta(z)| \leq 1$, $z \in \mathfrak{D}$ such that

$$\frac{\frac{1}{\gamma}\left(\frac{zs'(z)+\tau z^2 s''(z)}{(1-\xi)z+\xi z s'(z)} - 1\right)}{\eta(z)} \prec (\mathfrak{h}(z) - 1), z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

If we set $\eta(z) \equiv 1$, then $M_q(\tau, \xi, \gamma, \mathfrak{h})$ is denoted by $M(\tau, \xi, \gamma, \mathfrak{h})$ satisfying the condition

$$1 + \frac{1}{\gamma}\left(\frac{zs'(z) + \tau z^2 s''(z)}{(1-\xi)z + \xi z s'(z)} - 1\right) \prec \mathfrak{h}(z), z \in \mathfrak{D}.$$

The family $M_q(\tau, \xi, \gamma, \mathfrak{h})$ is of special interest . In view of this, we deem it worth while to note the relevance of $M_q(\tau, \xi, \gamma, \mathfrak{h})$ with classes defined above as well as some known ones. Indeed we have $i) M_q(\tau, 0, \gamma, \mathfrak{h}) = \mathfrak{R}_q(\tau, \gamma, \mathfrak{h})$, $ii) M_q(\tau, 1, \gamma, \mathfrak{h}) = \mathfrak{L}_q(\tau, \gamma, \mathfrak{h})$, and $iii) M_q(\xi, \xi, \gamma, \mathfrak{h})$, $0 \leq \xi \leq 1$ is investigated by Kant and Vyas [9].

In the second section, we find Fekete-Szegö functional $|d_3 - \mu d_2^2|$ for elements in $M_q(\tau, \xi, \gamma, \mathfrak{h})$. Many new consequences of this result are pointed out.

2 Main results

Theorem 2.1. *Let* $0 \leq \xi \leq 1, \tau \geq 0, \tau \geq \xi$ *and* $\gamma \in \mathbb{C} - \{0\}$. *If* $s(z) \in M_q(\tau, \xi, \gamma, \mathfrak{h})$, *then*

$$|d_2| \leq \frac{|\gamma| Q_1}{2(1-\xi+\tau)} \tag{2.1}$$

and for any complex number $\mu \in \mathbb{C}$,

$$|d_3 - \mu d_2^2| \leq \frac{|\gamma| Q_1}{3(1-\xi+2\tau)} max \left(1, \left| J Q_1 - \frac{Q_2}{Q_1} \right| \right), \tag{2.2}$$

where

$$J = \gamma \left(\frac{3\mu(1-\xi+2\tau)}{4(1-\xi+\tau)^2} - \frac{\xi}{1-\xi+\tau} \right). \tag{2.3}$$

The result is sharp.

Proof. Let $s \in M_q(\tau, \xi, \gamma, \mathfrak{h})$. Then there exists a schwarz function $\omega(z)$ and a holomorphic function $\eta(z)$ such that

$$\frac{1}{\gamma} \left(\frac{zs'(z) + \tau z^2 s''(z)}{(1-\xi)z + \xi z s'(z)} - 1 \right) = \eta(z)(\mathfrak{h}(\omega(z)) - 1),\ z \in \mathfrak{D}. \tag{2.4}$$

Series expansions of s and its successive derivatives from (1.1) gives

$$\begin{aligned} &\frac{1}{\gamma} \left(\frac{zs'(z) + \tau z^2 s''(z)}{(1-\xi)z + \xi z s'(z)} - 1 \right) = \\ &\quad \frac{1}{\gamma} \left[(1-\xi+\tau) 2 d_2 z + \left((1-\xi+2\tau) 3 d_3 - (1-\xi+\tau) 4 \xi d_2^2 \right) z^2 + ... \right]. \end{aligned} \tag{2.5}$$

Similarly from (1.2), (1.3) and (1.4), we obtain

$$\eta(z)(\mathfrak{h}(\omega(z)) - 1) = R_0 Q_1 \omega_1 z + \left[R_1 Q_1 \omega_1 + R_0 (Q_1 \omega_2 + Q_2 \omega_1^2) \right] z^2 + \dots . \tag{2.6}$$

Making use of (2.5) and (2.6) in (2.4), we get

$$d_2 = \frac{\gamma R_0 Q_1 \omega_1}{2(1-\xi+\tau)} \tag{2.7}$$

and

$$d_3 = \frac{\gamma Q_1}{3(1-\xi+2\tau)} \left[R_1 \omega_1 + R_0 \left\{ \omega_2 + \left(\frac{\xi \gamma R_0 Q_1}{1-\xi+\tau} + \frac{Q_2}{Q_1} \right) \omega_1^2 \right\} \right]. \tag{2.8}$$

Thus, for any $\mu \in \mathbb{C}$, we get

$$d_3 - \mu d_2^2 = \frac{\gamma Q_1}{3(1-\xi+2\tau)} \left[R_1 \omega_1 + \left(\omega_2 + \frac{Q_2}{Q_1} \omega_1^2 \right) R_0 - J R_1 Q_0^2 \omega_1^2 \right], \tag{2.9}$$

where J is as stated in (2.3).

Since $\eta(z)$ is holomorphic and bounded by one in $\mathfrak{D}$, we have (see [13],p.172)

$$|R_0| \leq 1 \quad \text{and} \quad R_1 = (1 - R_0^2) x \qquad x \leq 1. \tag{2.10}$$

The assertion (2.1) follows from (2.7) using (2.10) and Lemma 1.1. From (2.9) and (2.10), we obtain

$$d_3 - \mu d_2^2 = \frac{\gamma Q_1}{3(1-\xi+2\tau)} \left[x \omega_1 + \left(\omega_2 + \frac{Q_2}{Q_1} \omega_1^2 \right) R_0 - (J Q_1 \omega_1^2 + x \omega_1) R_0^2 \right]. \tag{2.11}$$

If $R_0 = 0$, then (2.11) yields

$$|d_3 - \mu d_2^2| \leq \frac{|\gamma| Q_1}{3(1-\xi+2\tau)}. \tag{2.12}$$

On the other side, if $R_0 \neq 0$, we define a function

$$L(R_0) = x\omega_1 + \left(\omega_2 + \frac{Q_2}{Q_1}\omega_1^2\right) R_0 - (JQ_1\omega_1^2 + x\omega_1)R_0^2. \tag{2.13}$$

The equation (2.13) is a quadratic in R_0 and hence holomorphic in $|R_0| \leq 1$. Clearly $|L(R_0)|$ attains its maximum value at $R_0 = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Thus

$$\begin{aligned} max|L(R_0)| &= \max_{0\leq\theta\leq 2\pi} |L(e^{i\theta})| = |L(1)| \\ &= |\omega_2 - \left(JQ_1 - \frac{Q_2}{Q_1}\right) \omega_1^2|. \end{aligned}$$

Therefore, it follows from (2.11) that

$$|d_3 - \mu d_2^2| \leq \frac{|\gamma|Q_1}{3(1-\xi+2\tau)} \left|\omega_2 - \left(JQ_1 - \frac{Q_2}{Q_1}\right)\omega_1^2\right|. \tag{2.14}$$

By virtue of Lemma 1.1, we obtain

$$|d_3 - \mu d_2^2| \leq \frac{|\gamma|Q_1}{3(1-\xi+2\tau)} max\left(1, \left|JQ_1 - \frac{Q_2}{Q_1}\right|\right). \tag{2.15}$$

The assertion (2.2) now follows from (2.12) and (2.15). We exhibit the sharpness by defining $s(z)$ as

$$\frac{1}{\gamma}\left(\frac{zs'(z) + \tau z^2 s''(z)}{(1-\xi)z + \xi z s'(z)} - 1\right) = \mathfrak{h}(z),$$

or

$$\frac{1}{\gamma}\left(\frac{zs'(z) + \tau z^2 s''(z)}{(1-\xi)z + \xi z s'(z)} - 1\right) = \mathfrak{h}(z^2),$$

or

$$\frac{1}{\gamma}\left(\frac{zs'(z) + \tau z^2 s''(z)}{(1-\xi)z + \xi z s'(z)} - 1\right) = z(\mathfrak{h}(z) - 1).$$

This ends the proof. □

Remark 2.2. We obtain Theorem 1 of [9] from Theorem 2.1, when $\tau = 1$.

We conclude the below sharp result for the class $\mathfrak{R}_q(\tau,\gamma,\mathfrak{h})$, by putting $\xi = 0$ in Theorem 2.1.

Corollary 2.3. *Let $\gamma \in \mathbb{C} - \{0\}$ and $\tau \geq 0$. If $s(z) \in \mathfrak{R}_q(\tau,\gamma,\mathfrak{h})$, then $|d_2| \leq \frac{|\gamma|Q_1}{2(1+\tau)}$ and for some $\mu \in \mathbb{C}$, $|d_3 - \mu d_2^2| \leq \frac{|\gamma|Q_1}{3(1+2\tau)} max\left(1, \left|\frac{3\mu|\gamma|(1+2\tau)}{4(1+\tau)^2}Q_1 - \frac{Q_2}{Q_1}\right|\right)$.*

Remark 2.4. For $\tau = 1$, Corollary 2.3 reduces to Corollary 1 of [9].

We conclude the following sharp result for the class $\mathfrak{L}_q(\tau,\gamma,\mathfrak{h})$, on putting $\xi = 1$ in Theorem 2.1.

Corollary 2.5. *Let $\gamma \in \mathbb{C} - \{0\}$ and $\tau \geq 1$. If $s(z) \in \mathfrak{L}_q(\tau,\gamma,\mathfrak{h})$, then $|d_2| \leq \frac{|\gamma|Q_1}{2\tau}$ and for some $\mu \in \mathbb{C}$, $|d_3 - \mu d_2^2| \leq \frac{|\gamma|Q_1}{6\tau} max\left(1, \left|\frac{|\gamma|\mu Q_1}{2\tau} - \frac{Q_2}{Q_1}\right|\right)$.*

Our next sharp result is based on majorization.

Theorem 2.6. *Let $\gamma \in \mathbb{C} - \{0\}, 0 \leq \xi \leq 1, \tau \geq 0$ and $\tau \geq \xi$. If $s(z) \in A$ satisfies*

$$\frac{1}{\gamma}\left(\frac{zs'(z) + \tau z^2 s''(z)}{(1-\xi)z + \xi z s'(z)} - 1\right) \prec\prec (\mathfrak{h}(z) - 1), z \in \mathfrak{D}, \tag{2.16}$$

then

$$|d_2| \leq \frac{|\gamma|Q_1}{2(1-\xi+\tau)} \tag{2.17}$$

and for any complex number μ,

$$|d_3-\mu d_2^2| \le \frac{|\gamma|Q_1}{3(1-\xi+2\tau)} max\left(1, \left|JQ_1-\frac{Q_2}{Q_1}\right|\right), \tag{2.18}$$

where J *is as stated by (2.3).*

Proof. Assume that (2.16) holds. There exists a holomorphic function $\eta(z)$, from the principle of majorization, such that

$$\frac{1}{\gamma}\left(\frac{zs'(z)+\tau z^2 s''(z)}{(1-\xi)z+\xi zs'(z)}-1\right)=\eta(z)(\mathfrak{h}(z)-1), z\in\mathfrak{D}. \tag{2.19}$$

Setting $\omega(z)\equiv z$ (so that $\omega_1=1, \omega_n=0, n\ge 2$), we obtain the desired results (2.17) and (2.18), following the proof of Theorem 2.1. We exhibit the sharpness by defining $s(z)$ as

$$1+\frac{1}{\gamma}\left(\frac{zs'(z)+\tau z^2 s''(z)}{(1-\xi)z+\xi zs'(z)}-1\right)=\mathfrak{h}(z),\ z\in\mathfrak{D},$$

which completes the proof. □

Our next sharp result is associated with $M(\tau,\xi,\gamma,\mathfrak{h})$

Theorem 2.7. *Let* $0\le\xi\le 1, \tau\ge 0, \tau\ge\xi$ *and* $\gamma\in\mathbb{C}-\{0\}$. *If* $s\in M(\tau,\xi,\gamma,\mathfrak{h})$, *then*

$$|d_2|\le\frac{|\gamma|Q_1}{2(1-\xi+\tau)}$$

and for any $\mu\in\mathbb{C}$,

$$|d_3-\mu d_2^2| \le \frac{|\gamma|Q_1}{3(1-\xi+2\tau)} max\left(1, \left|JQ_1-\frac{Q_2}{Q_1}\right|\right),$$

where J *is as stated in (2.3).*

Proof. Let $s\in M(\tau,\xi,\gamma,\mathfrak{h})$. Taking $\eta(z)\equiv 1,\ z\in\mathfrak{D}$, we get $R_0=1, R_n=0, n\in N$ and by following the proof of Theorem 2.1, we attain the desired results. We exhibit the sharpness by defining $s(z)$ as

$$\frac{1}{\gamma}\left(\frac{zs'(z)+\tau z^2 s''(z)}{(1-\xi)z+\xi zs'(z)}-1\right)=\mathfrak{h}(z),$$

or

$$\frac{1}{\gamma}\left(\frac{zs'(z)+\tau z^2 s''(z)}{(1-\xi)z+\xi zs'(z)}-1\right)=\mathfrak{h}(z^2),$$

which ends the proof. □

We now settle sharp bounds of $|d_3-\mu d_2^2|$ for real γ and μ,when $s\in M_q(\tau,\xi,\gamma,\mathfrak{h})$.

Theorem 2.8. *Let* $0\le\xi\le 1,\ \tau\ge 0,\ \tau\ge\xi$ *and* $\gamma\in\mathbb{C}-\{0\}$. *If* $s\in M_q(\tau,\xi,\gamma,\mathfrak{h})$, *then for real* γ *and* μ, *we have*

$$|d_3-\mu d_2^2|\le\begin{cases}\frac{|\gamma|Q_1}{3(1-\xi+2\tau)}\left[Q_1\gamma\left(\frac{\xi}{1-\xi+\tau}-\frac{3\mu(1-\xi+2\tau)}{4(1-\xi+\tau)^2}\right)+\frac{Q_2}{Q_1}\right] & (\mu\le\rho_1)\\ \frac{|\gamma|Q_1}{3(1-\xi+2\tau)} & (\rho_1\le\mu\le\rho_1+2\sigma)\\ -\frac{|\gamma|Q_1}{3(1-\xi+2\tau)}\left[Q_1\gamma\left(\frac{\xi}{1-\xi+\tau}-\frac{3\mu(1-\xi+2\tau)}{4(1-\xi+\tau)^2}\right)+\frac{Q_2}{Q_1}\right] & (\mu\ge\rho_1+2\sigma)\end{cases} \tag{2.20}$$

where

$$\rho_1=\frac{4\xi(1-\xi+\tau)}{3(1-\xi+2\tau)}-\frac{4(1-\xi+\tau)^2}{3\gamma(1-\xi+2\tau)}\left(\frac{1}{Q_1}-\frac{Q_2}{Q_1^2}\right) \tag{2.21}$$

and

$$\sigma=\frac{4(1-\xi+\tau)^2}{3\gamma(1-\xi+2\tau)Q_1}. \tag{2.22}$$

Proof. Let μ and γ be the real values. Then (2.20) can be obtained from (2.2) and (2.3), respectively, under the below cases:

$$JQ_1 - \frac{Q_2}{Q_1} \leq -1, -1 \leq JQ_1 - \frac{Q_2}{Q_1} \leq 1 \text{ and } JQ_1 - \frac{Q_2}{Q_1} \geq 1,$$

where J is as stated in (2.3). We also note the following:

(i) Equality holds for $\mu < \rho_1$ or $\mu > \rho_1 + 2\sigma$ if and only if $\eta(z) \equiv 1$ and $w(z) = z$ or one of its rotations.

(ii) Equality holds for $\rho_1 < \mu < \rho_1 + 2\sigma$ if and only if $\eta(z) \equiv 1$ and $w(z) = z^2$ or one of its rotations.

(iii) Equality holds for $\mu = \rho_1$ if and only if $\eta(z) \equiv 1$ and $w(z) = \frac{z(z+\theta)}{1+\theta z}, 0 \leq \theta \leq 1$, or one of its rotation, while for $\mu = \rho_1 + 2\sigma$, the equality holds if and only if $\eta(z) \equiv 1$ and $w(z) = -\frac{z(z+\theta)}{1+\theta z}, 0 \leq \theta \leq 1$, or one of its rotations.

□

The second part of assertion in (2.20) for real valus of μ and γ can be improved further as follows:

Theorem 2.9. *Let* $0 \leq \xi \leq 1, \tau \geq 0, \tau \geq \xi$ *and* $\mu \in \mathbb{C} - \{0\}$. *If* $s(z) \in M_q(\tau, \xi, \gamma, \mathfrak{h})$, *then for real* γ *and* μ, *we have*

$$|d_3 - \mu d_2^2| + (\mu - \rho_1)|d_2|^2 \leq \frac{|\gamma|Q_1}{3(1-\xi+2\tau)} \quad (\rho_1 \leq \mu \leq \rho_1 + \sigma) \tag{2.23}$$

and

$$|d_3 - \mu d_2^2| + (\rho_1 + 2\sigma - \mu)|d_2^2| \leq \frac{|\gamma|Q_1}{3(1-\xi+2\tau)} \quad (\rho_1 + \sigma \leq \mu \leq \rho_1 + 2\sigma), \tag{2.24}$$

where ρ_1 *and* σ *are given by (2.21) and (2.22), respectively.*

Proof. Let $s \in M_q(\tau, \xi, \gamma, \mathfrak{h})$. For real μ satisfying $\rho_1 \leq \mu \leq \rho_1 + \sigma$ and using (2.7) and (2.14), we get

$$\begin{aligned}|d_3 - \mu d_2^2| &+ (\mu - \rho_1)|d_2|^2 \\ &\leq \frac{|\gamma|Q_1}{3(1-\xi+2\tau)} \left[|w_2| - \frac{3|\gamma|Q_1(1-\xi+2\tau)}{4(1-\xi+\tau)^2}(\mu - \rho_1 - \sigma)|w_1|^2 \right. \\ &\left. + \frac{3|\gamma|Q_1(1-\xi+2\tau)}{4(1-\xi+\tau)^2}(\mu - \rho_1)|w_1|^2 \right].\end{aligned}$$

Therefore, by using Lemma 1.1, we obtain

$$|d_3 - \mu d_2^2| + (\mu - \rho_1)|d_2|^2 \leq \frac{|\gamma|Q_1}{3(1-\xi+2\tau)} \left[1 - |w_1|^2 + |w_1|^2\right],$$

which yields the assertion (2.23).

If $\rho_1 + \sigma \leq \mu \leq \rho_1 + 2\sigma$, then again from (2.7), (2.14) and Lemma 1.1, we have

$$\begin{aligned}|d_3 - \mu d_2^2| &+ (\rho_1 + 2\sigma - \mu)|d_2|^2 \\ &\leq \frac{|\gamma|Q_1}{3(1-\xi+2\tau)} \left[|w_2| + \frac{3|\gamma|Q_1(1-\xi+2\tau)}{4(1-\xi+\tau)^2}(\rho_1 + 2\sigma - \mu)|w_1|^2 \right. \\ &\left. + \frac{3|\gamma|Q_1(1-\xi+2\tau)}{4(1-\xi+\tau)^2}(\mu - \rho_1)|w_1|^2 \right] \\ &\leq \frac{|\gamma|Q_1}{3(1-\xi+2\tau)} \left[1 - |w_1|^2 + |w_1|^2\right],\end{aligned}$$

which estimates (2.24). □

Remark 2.10. Numerous consequences of Theorem 2.6 to Theorem 2.9 can be obtained for different choices of ξ and τ.

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Author information

S R Swamy and Y Sailaja,
Department of Computer Science and Engineering, RV College of Engineering, Bengaluru - 560 059, Karnataka, India.
Department of Mathematics, RV College of Engineering, Bengaluru - 560 059, Karnataka, India.
E-mail: swamysr@rvce.edu.in, sailajay@rvce.edu.in

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