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APPROXIMATION OF BOREL DERIVATIVES OF FUNCTIONS VIA NON-LINEAR SINGULAR INTEGRAL OPERATORS

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Abstract. In this paper, we consider convolution type nonlinear singular integral operators of the form

$$T_\lambda(f;x) = \int_a^b K_\lambda(t-x, f(t))\,dt\,,$$

where $\langle a,b\rangle$ is an arbitrary interval in $\mathbb{R}$, $\lambda \in \Lambda$, $f \in L_1\langle a,b\rangle$ and K_λ is a family of kernels satisfying suitable conditions. We give some approximation results with regard to the convergence of the operators T_λ to right, left, and symmetric Borel differentiable functions.

We note that our results extend some of the previous results obtained in [5] and [6] which they cope with the linear singular integral operators.

1 Introduction

The problem of approximation of r^{th} finite derivatives of functions belonging to $L_1(-\pi,\pi)$ by means of linear singular integral operators of convolution type considered in [15] as the following form

$$U(f;x,\lambda) = \int_{-\pi}^{\pi} f(t)\,K(t-x,\lambda)\,dt \quad , \quad x \in (-\pi,\pi)$$

that was investigated by Taberski, where the kernel $K(t,\lambda)$ is a singular function and satisfies suitable assumptions. In [11] and [12], Karsli and Ibikli extended Taberski's work to functions belonging to $L_1(a,b)$, under some weaker suitable assumptions. We also refer readers to another paper on this subject written by Gadjiev [4] as well as a monograph [3].

In paper [13], Musielak extended the concept of singularity over the case of nonlinear integral operators using the assumption of a Lipschitz condition for $K_\lambda(t, f(t))$ with respect to the second variable. In virtue of this study, we can use the classical way for linear integral operators[3] to obtain some convergence results for nonlinear integral operators.

Recently, the first author studied both pointwise convergence and rate of pointwise convergence of the nonlinear singular integral operators defined by

$$T_\lambda(f;x) = \int_a^b K_\lambda(t-x, f(t))\,dt, \qquad x \in \langle a,b\rangle \tag{1.1}$$

where $\langle a,b\rangle$ is an arbitrary interval in $\mathbb{R}$, $\lambda \in \Lambda$, $f \in L_1\langle a,b\rangle$ and K_λ is a family of kernels satisfying suitable properties (see [7]- [10] and [14]).

The goal of this paper is to obtain some convergence results for the derivatives of the operators (1.1). To the best of our knowledge, we can say that this study is the first paper on this subject by means of nonlinear singular integral operators.

2 Preliminaries

Let Λ be a nonempty set of indices with a suitable topology and λ_0 be an accumulation point of Λ in this topology. We denote the family of all neighborhoods of the neutral element θ of $\mathbb{R}$ by $\mathcal{U}(\theta)$ and x_0 is a fixed accumulation point of $\mathbb{R}$. We take a family $\mathcal{K}$ of functions $K_\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $K_\lambda(t, 0) = 0$ for all $t \in \mathbb{R}$ and $\lambda \in \Lambda$ such that $K_\lambda(t, u)$ integrable over $\mathbb{R}$ with respect to t, in the sense of Lebesgue measure, for all values of the index λ and second variable u. The family $\mathcal{K}$ will be called a kernel. In addition, if the kernel function $K_\lambda(t, u)$ is continuous in $\mathbb{R}$ for every $t \in \mathbb{R}$, then the kernel function is called Carathéodory kernel function.

We assume that $K_\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a kernel satisfying the following conditions: (‡)

a) Let $L_\lambda(\bullet)$ be any differentiable and integrable function such that

$$\left[\frac{\partial}{\partial x} K_\lambda(t-x, u) - \frac{\partial}{\partial x} K_\lambda(t-x, v)\right] = \frac{\partial}{\partial x} L_\lambda(t-x)\,[u-v]\,,$$

holds for every t and any $\lambda \in \Lambda$ where $u, v \in \mathbb{R}$.

b) $\lim\limits_{\lambda\to\lambda_0} \int\limits_{\mathbb{R}\setminus U} L_\lambda(z)dz = 0$, for every $U \in \mathcal{U}(0)$.

c) $\lim\limits_{\lambda\to\lambda_0} \left[\sup\limits_{|z|\geq\delta} L_\lambda(z)\right] = 0$, for every $\delta > 0$.

d) $\lim\limits_{\lambda\to\lambda_0} \int\limits_{\mathbb{R}} L_\lambda(z)\,dz = 1$.

e) There exists a $\delta_0 > 0$ such that $L_\lambda(z)$ is non-increasing on $[0, \delta_0)$ and non-decreasing on $(-\delta_0, 0]$ for any $\lambda \in \Lambda$.

Theorem 2.1. *[1] Let $1 \leq p < \infty$ and assume that a function $K_\lambda(t, u)$ is a kernel. If $f \in L_p\langle a, b\rangle$, then $T_\lambda(f) \in L_p\langle a, b\rangle$, for every $\lambda \in \Lambda$.*

Readers can find the following well-known definition in [6].

Definition 2.2. A function $f(t)$ has a right Borel derivative α ($\neq \infty$) at the point $x_0 \in \mathbb{R}$ if

$$\lim_{h\to 0^+} \frac{1}{h} \int_0^h \frac{f(x_0+t) - f(x_0)}{t} dt = \alpha$$

and we denote it $BD^+ f(x_0)$.

Similarly, a function $f(t)$ has a left Borel derivative β ($\neq \infty$) at the point $x_0 \in \mathbb{R}$ if

$$\lim_{h\to 0^+} \frac{1}{h} \int_0^h \frac{f(x_0) - f(x_0-t)}{t} dt = \beta$$

and we denote it $BD^- f(x_0)$.

A function $f(t)$ has a symmetric Borel derivative γ ($\neq \infty$) at the point $x_0 \in \mathbb{R}$ if

$$\lim_{h\to 0^+} \frac{1}{h} \int_0^h \frac{f(x_0+t) - f(x_0-t)}{2t} dt = \gamma$$

and we write it $BD_s f(x_0)$, where the integrals are taken in the sense of $\lim_{\varepsilon\to 0^+} \int\limits_{\varepsilon}^{h}$.

Note that, if $BD^+ f(x_0)$ and $BD^- f(x_0)$ exist, then clearly $BD_s f(x_0)$ exists and

$$BD_s f(x_0) = \frac{BD^+ f(x_0) + BD^- f(x_0)}{2}$$

holds true.

Obviously, if the ordinary derivative of $f(x)$ at x_0 exists, then so does Borel derivatives and the following equalities

$$BD^+ f(x_0) = BD^- f(x_0) = BD_s f(x_0) = f'(x_0)$$

hold.

3 Convergence of the derivatives

Now, we will investigate the approximation properties of finite first derivative of the operator T_λ in $L_1\langle a, b\rangle$.

Theorem 3.1. *Let assume that all the conditions of (‡) are satisfied by the kernel function $K_\lambda(t, u)$ and also $\frac{\partial}{\partial t} L_\lambda(t)$ are piecewise continuous function with respect to t on $(-\infty, \infty)$ having the following equality*

$$\lim_{\lambda\to\lambda_0} \sup_{\delta\le|t|} \left| \frac{\partial}{\partial t} L_\lambda(t) \right| = 0. \tag{3.1}$$

If the function $f \in L_1(a, b)$ has a finite derivative $f'(x)$ at x_0, then we obtain

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \frac{\partial}{\partial x} T_\lambda(f; x) = BD_s f(x_0).$$

Proof. Suppose that

$$x_0 + \delta < b, \;\; x_0 - \delta > a \;\text{ and }\; 0 \le x_0 - x < \frac{\delta}{2}, \tag{3.2}$$

for any $0 < \delta < \delta_0$. Define

$$g(t) := f(x_0) + (t - x_0) BD_s f(x_0). \tag{3.3}$$

Clearly, $g(t)$ is a linear polynomial.

At first, we start proving the theorem for $g(t)$ by introducing a new function $\widetilde{g} \in L_1(\mathbb{R})$ given by

$$\widetilde{g}(t) := \begin{cases} g(t) & , \quad t \in \langle a, b\rangle \\ 0 & , \quad t \notin \langle a, b\rangle \end{cases}. \tag{3.4}$$

If we apply the operator T_λ to the function $g(t)$, then we have

$$T_\lambda(g; x) = \int\limits_a^b K_\lambda(t - x, g(t))dt$$

and using (3.4) one can rewrite the last equality as follows:

$$T_\lambda(\widetilde{g}; x) = \int\limits_{\mathbb{R}} K_\lambda(t - x, \widetilde{g}(t))dt. \tag{3.5}$$

Differentiating the above equality with respect to x and using $(\ddagger\ a)$ yield

$$\frac{\partial}{\partial x}T_\lambda(\widetilde{g};x)=\int_{\mathbb{R}}\frac{\partial}{\partial x}K_\lambda(t-x,\widetilde{g}(t))dt=\int_{\mathbb{R}}\frac{\partial L_\lambda(t-x)}{\partial x}\widetilde{g}(t)dt=-\int_{\mathbb{R}}\widetilde{g}(t)\frac{\partial L_\lambda(t-x)}{\partial t}dt.$$

Using the integration by parts, we obtain

$$\frac{\partial}{\partial x}T_\lambda(\widetilde{g};x)=\int_{\mathbb{R}}\widetilde{g}'(t)L_\lambda(t-x)dt.$$

Using (3.3), (3.4) and $(\ddagger\ d)$ we have

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)}\frac{\partial}{\partial x}T_\lambda(\widetilde{g};x)=BD_sf(x_0).$$

Here, we set

$$I(x,\lambda):=\frac{\partial}{\partial x}T_\lambda(\widetilde{g};x)-\frac{\partial}{\partial x}T_\lambda(f;x). \tag{3.6}$$

it is enough to show the following equality for the proof of the theorem

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)}|I(x,\lambda)|=0.$$

For this purpose, we take absolute value of (3.6), then according to $(\ddagger\ a)$, (3.19) and (3.4), it is easy to see that

$$\begin{aligned}|I(x,\lambda)| &= \left|\int_a^b\frac{\partial}{\partial x}K_\lambda(t-x,f(t))dt-\int_a^b\frac{\partial}{\partial x}K_\lambda(t-x,g(t))dt\right|\\ &\leq \int_a^b|f(t)-g(t)|\left|\frac{\partial}{\partial x}L_\lambda(t-x)\right|dt.\end{aligned}$$

Since the function $f\in L_1(a,b)$ possess a finite derivative $f'(x)$ at x_0, we can divide the last integral into three terms as follows:

$$\begin{aligned}|I(x,\lambda)| &\leq \left\{\int_a^{x_0-\delta}+\int_{x_0-\delta}^{x_0+\delta}+\int_{x_0+\delta}^{b}\right\}|f(t)-g(t)|\left|\frac{\partial}{\partial t}L_\lambda(t-x)\right|dt\\ &=: I_1(x,\lambda)+I_2(x,\lambda)+I_3(x,\lambda).\end{aligned} \tag{3.7}$$

We now fix this δ and estimate $I_1(x,\lambda),I_2(x,\lambda)$ and $I_3(x,\lambda)$ as follows:

$$I_1(x,\lambda)=\int_a^{x_0-\delta}|f(t)-g(t)|\left|\frac{\partial}{\partial t}L_\lambda(t-x)\right|dt.$$

According to (3.2), we have $t-x<x_0-x-\delta<-\frac{\delta}{2}<0$. Thus, we get

$$I_1(x,\lambda)\leq\sup_{\frac{\delta}{2}<|u|}\left|\frac{\partial}{\partial u}L_\lambda(u)\right|\int_a^b|f(t)-g(t)|\,dt.$$

Since $f,\ g\in L_1(a,b)$, then $f-g\in L_1(a,b)$. So, there exists a positive constant M such that $\|f-g\|_{L_1(a,b)}\leq M$. Consequently, we have

$$I_1(x,\lambda)\leq M\sup_{\frac{\delta}{2}<|u|}\left|\frac{\partial}{\partial u}L_\lambda(u)\right|. \tag{3.8}$$

In the same way, we obtain

$$I_3(x,\lambda) \leq M \sup_{\frac{\delta}{2}<|u|} \left|\frac{\partial}{\partial u} L_\lambda(u)\right|. \tag{3.9}$$

Finally, we can tackle the integral $I_2(x,\lambda)$. For each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$I_2(x,\lambda) = \int_{x_0-\delta}^{x_0+\delta} |f(t)-g(t)| \left|\frac{\partial}{\partial t} L_\lambda(t-x)\right| dt \leq \varepsilon \int_{x_0-\delta}^{x_0+\delta} |t-x_0| \left|\frac{\partial}{\partial t} L_\lambda(t-x)\right| dt.$$

Setting

$$I_{2,1}(x,\lambda) := \int_{x_0-x-\delta}^{x_0-x+\delta} |x_0-x-t| \left|\frac{\partial}{\partial t} L_\lambda(t)\right| dt,$$

and using $(\ddagger\, e)$ together with (3.2), we can now obtain the following estimate:

$$\begin{aligned} I_{2,1}(x,\lambda) &= \int_{x_0-x-\delta}^{x_0-x} |x_0-x-t| \left|\frac{\partial}{\partial t} L_\lambda(t)\right| dt + \int_{x_0-x}^{x_0-x+\delta} |x_0-x-t| \left|\frac{\partial}{\partial t} L_\lambda(t)\right| dt \\ &= \int_{x_0-x-\delta}^{0} |x_0-x-t| \left|\frac{\partial}{\partial t} L_\lambda(t)\right| dt + \int_{0}^{x_0-x} |x_0-x-t| \left|\frac{\partial}{\partial t} L_\lambda(t)\right| dt \\ &\quad + \int_{x_0-x}^{x_0-x+\delta} (x_0-x-t) \frac{\partial}{\partial t} L_\lambda(t) dt \\ &= \int_{x_0-\delta}^{x_0} (t-x_0) \frac{\partial}{\partial t} L_\lambda(t-x) dt + \int_{x_0}^{x_0+\delta} (x_0-t) \frac{\partial}{\partial t} L_\lambda(t-x) dt. \end{aligned}$$

Using integration by parts and $(\ddagger\, e)$, we obtain:

$$\begin{aligned} \int_{x_0-\delta}^{x_0} (t-x_0) \frac{\partial}{\partial t} L_\lambda(t-x) dt &= (t-x_0) L_\lambda(t-x) \Big|_{x_0-\delta}^{x_0} + \int_{x_0-\delta}^{x_0} L_\lambda(t-x) dt \\ &= \delta L_\lambda(x_0-\delta-x) + \int_{x_0-\delta}^{x_0} L_\lambda(t-x) dt \end{aligned}$$

and

$$\begin{aligned} \int_{x_0}^{x_0+\delta} (x_0-t) \frac{\partial}{\partial t} L_\lambda(t-x) dt &= (x_0-t) L_\lambda(t-x) \Big|_{x_0}^{x_0+\delta} + \int_{x_0}^{x_0+\delta} L_\lambda(t-x) dt \\ &= \delta L_\lambda(x_0+\delta-x) + \int_{x_0}^{x_0+\delta} L_\lambda(t-x) dt. \end{aligned}$$

Thus, we get

$$I_2(x,\lambda) \leq \varepsilon\delta \left\{ L_\lambda(x_0-\delta-x) + L_\lambda(x_0+\delta-x) \right\} + \varepsilon \int_{\mathbb{R}} L_\lambda(t) dt. \tag{3.10}$$

Substituting (3.8)-(3.10) into (3.7), we obtain

$$\begin{aligned} |I(x,\lambda)| &\leq 2M \sup_{\frac{\delta}{2}<|u|} \left|\frac{\partial}{\partial u}L_\lambda(u)\right| \\ &+ \varepsilon\left[\delta\left\{L_\lambda(x_0-\delta-x)+L_\lambda(x_0+\delta-x)\right\}+\int_{\mathbb{R}} L_\lambda(t)dt\right] \end{aligned}$$

which in view of ($\ddagger$ $c-d$) and (3.1) approaches to zero as $(x,\lambda)\to(x_0,\lambda_0)$. The estimate $-\frac{\delta}{2}<x_0-x\leq 0$ is obtained in similar way. So this completes the proof. □

Theorem 3.2. *Suppose that the hypothesis of Theorem 3.1 are satisfied. Let $f\in L_1(a,b)$ be such that finite $f'_+(x)$ and $f'_-(x)$ derivatives exist at x_0, then*

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \frac{\partial}{\partial x}T_\lambda(f;x) = N\,BDf_+(x_0)+(1-N)BDf_-(x_0)$$

where

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \int_{x_0}^{\infty} L_\lambda(t-x)dt = N, \qquad 0\leq N\leq 1. \tag{3.11}$$

Proof. Suppose that (3.2) is satisfied for any $0<\delta<\delta_0$. Setting

$$g(t)=\begin{cases} f(x_0)+(t-x_0)BDf_-(x_0) & , \quad a\leq t<x_0 \\ f(x_0)+(t-x_0)BDf_+(x_0) & , \quad x_0\leq t\leq b \end{cases} \tag{3.12}$$

we also define a function $\widetilde{g}(t)$ as in (3.4). According to the proof of the Theorem 3.1, one can write

$$\frac{\partial}{\partial x}T_\lambda(g;x)=\int_{\mathbb{R}} \widetilde{g}'(t)L_\lambda(t-x)dt. \tag{3.13}$$

Substituting (3.12) into (3.4) and using it in (3.13), we have

$$\frac{\partial}{\partial x}T_\lambda(g;x)=BDf_-(x_0)\int_{-\infty}^{x_0} L_\lambda(t-x)dt+BDf_+(x_0)\int_{x_0}^{\infty} L_\lambda(t-x)dt$$

and by (3.11), we get

$$\frac{\partial}{\partial x}T_\lambda(g;x)=N\,BDf_+(x_0)+(1-N)BDf_-(x_0)$$

as $(x,\lambda)\to(x_0,\lambda_0)$. Setting

$$I(x,\lambda):=\frac{\partial}{\partial x}T_\lambda(g;x)-\frac{\partial}{\partial x}T_\lambda(f;x).$$

Since the function $f\in L_1(a,b)$ possess a finite $f'_+(x)$ and $f'_-(x)$ derivatives at x_0, we can divide the last integral into four terms as follows:

$$\begin{aligned} |I(x,\lambda)| &\leq \left\{\int_a^{x_0-\delta}+\int_{x_0-\delta}^{x_0}+\int_{x_0+\delta}^{b}+\int_{x_0}^{x_0+\delta}\right\}|f(t)-g(t)|\left|\frac{\partial}{\partial t}L_\lambda(t-x)\right|dt \\ &=: I_1(x,\lambda)+I_2(x,\lambda)+I_3(x,\lambda)+I_4(x,\lambda). \end{aligned}$$

Fix this δ and consider the integrals $I_2(x,\lambda)$ and $I_4(x,\lambda)$. For each $\varepsilon>0$ there exists a $\delta>0$ such that

$$\begin{aligned} I_2(x,\lambda) &= \int_{x_0-\delta}^{x_0}\left|\frac{f(t)-f(x_0)}{t-x_0}-BDf_-(x_0)\right||t-x_0|\left|\frac{\partial}{\partial t}L_\lambda(t-x)\right|dt \\ &\leq \varepsilon\int_{x_0-\delta}^{x_0}(x_0-t)\frac{\partial}{\partial t}L_\lambda(t-x)dt \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} I_4(x,\lambda) &= \int_{x_0}^{x_0+\delta} \left| \frac{f(t)-f(x_0)}{t-x_0} - BDf_+(x_0) \right| |t-x_0| \left| \frac{\partial}{\partial t} L_\lambda(t-x) \right| dt \\ &\leq \varepsilon \int_{x_0}^{x_0+\delta} (x_0-t) \frac{\partial}{\partial t} L_\lambda(t-x) dt \end{aligned} \tag{3.15}$$

Using (3.14) and (3.15) together, we have

$$I_2(x,\lambda) + I_4(x,\lambda) \leq \varepsilon \int_{x_0-\delta}^{x_0+\delta} (x_0-t) \frac{\partial}{\partial t} L_\lambda(t-x) dt.$$

Again using integration by parts, we obtain

$$I_2(x,\lambda) + I_4(x,\lambda) \leq \varepsilon \left[-\delta \left\{ L_\lambda(x_0+\delta-x) + L_\lambda(x_0-\delta-x) \right\} + \int_{\mathbb{R}} L_\lambda(t-x) dt \right]. \tag{3.16}$$

In view of (3.8) and (3.9), one has

$$I_1(x,\lambda) + I_3(x,\lambda) \leq 2M \sup_{\frac{\delta}{2}<|u|} \left| \frac{\partial}{\partial u} L_\lambda(u) \right|. \tag{3.17}$$

Hence, we get from (3.16) and (3.17)

$$\begin{aligned} |I(x,\lambda)| &\leq \varepsilon \left[-\delta \left\{ L_\lambda(x_0+\delta-x) + L_\lambda(x_0-\delta-x) \right\} + \int_{\mathbb{R}} L_\lambda(t-x) dt \right] \\ &\quad + 2M \sup_{\frac{\delta}{2}<|u|} \left| \frac{\partial}{\partial u} L_\lambda(u) \right| \end{aligned}$$

that, in the light of conditions ($\ddagger\ a-d$), approaches to zero as $(x,\lambda) \to (x_0,\lambda_0)$ and this completes the proof. □

Theorem 3.3. *Suppose that the hypothesis of Theorem 3.1 are satisfied. Let $f \in L_1(a,b)$ be such that $BDf_+(x)$ and $BDf_-(x)$ (right and left Borel derivatives) exist at x_0, then*

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \frac{\partial}{\partial x} T_\lambda(f;x) = N\, BDf_+(x_0) + (1-N) BDf_-(x_0)$$

where

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \int_{x_0}^{\infty} L_\lambda(t-x) dt = N, \qquad 0 \leq N \leq 1. \tag{3.18}$$

Proof. Suppose that

$$x_0+\delta < b, \ \ x_0-\delta > a \ \text{ and } \ 0 \leq x_0 - x < \frac{\delta}{2},$$

for any $0 < \delta < \delta_0$. Clearly, we have

$$T_\lambda(f;x) = \int_a^b K_\lambda(t-x, f(t)) dt.$$

Defining a new function $\widetilde{f} \in L_1(\mathbb{R})$ by

$$\widetilde{f}(t) := \begin{cases} f(t) & , \quad t \in (a,b) \\ 0 & , \quad t \notin (a,b) \end{cases} , \tag{3.19}$$

and using (3.19), we can rewrite the last equality as follows:

$$T_\lambda(f;x) = T_\lambda(\widetilde{f};x) = \int_{\mathbb{R}} K_\lambda(t-x, \widetilde{f}(t))dt.$$

Differentiating the inequality (3.5) with respect to x and using $(\ddagger\ a)$ yield

$$\frac{\partial}{\partial x} T_\lambda(f;x) = -\int_{\mathbb{R}} \widetilde{f}(t) \frac{\partial}{\partial t} L_\lambda(t-x)dt.$$

In view of the definitions of right and left Borel derivatives, we divide the last integral as follows:

$$\begin{aligned} I(x,\lambda) &= -\left\{ \int_{t \notin \langle a,b \rangle} + \int_{|t-x_0|>\delta,\, t\in\langle a,b\rangle} + \int_{x_0-\delta}^{x_0} + \int_{x_0}^{x_0+\delta} \right\} \widetilde{f}(t) \frac{\partial}{\partial t} L_\lambda(t-x)dt \\ &=: I_0(x,\lambda) + I_1(x,\lambda) + I_1'(x,\lambda) + I_2(x,\lambda) + I_3(x,\lambda). \end{aligned}$$

We now fix this δ and estimate $I_1(x,\lambda), I_1'(x,\lambda), I_2(x,\lambda)$ and $I_3(x,\lambda)$ as follows:

$$I_1(x,\lambda) \leq \int_a^{x_0-\delta} |f(t)| \left| \frac{\partial}{\partial t} L_\lambda(t-x) \right| dt.$$

According to (3.2), we have $t - x < x_0 - x - \delta < -\frac{\delta}{2} < 0$. Thus, we get

$$I_1(x,\lambda) \leq \sup_{\frac{\delta}{2}<|u|} \left| \frac{\partial}{\partial u} L_\lambda(u) \right| \int_a^b |f(t)|\, dt.$$

Since $f \in L_1(a,b)$, then there exists a positive constant M with $\|f\|_{L_1(a,b)} \leq M$. Consequently, we have

$$I_1(x,\lambda) \leq M \sup_{\frac{\delta}{2}<|u|} \left| \frac{\partial}{\partial u} L_\lambda(u) \right|.$$

In the same way, we obtain

$$I_1'(x,\lambda) = \int_{x_0+\delta}^{b} |f(t)| \left| \frac{\partial}{\partial t} L_\lambda(t-x) \right| dt \leq M \sup_{\frac{\delta}{2}<|u|} \left| \frac{\partial}{\partial u} L_\lambda(u) \right|.$$

Finally, let us consider the integrals $I_2(x,\lambda)$ and $I_3(x,\lambda)$. For each $\varepsilon > 0$ there exists a $\delta > 0$

such that

$$\begin{aligned} I_2(x,\lambda) &= \int_{x_0-\delta}^{x_0} -f(t)\frac{\partial}{\partial t}L_\lambda(t-x)dt = \int_{-\delta}^{0} -f(x_0+t)\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt \\ &= \int_{-\delta}^{0} [f(x_0)-f(x_0+t)]\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt - f(x_0)\int_{-\delta}^{0}\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt \\ &= \int_{-\delta}^{0}\left[\frac{f(x_0)-f(x_0+t)}{t} - BDf_-(x_0)\right] t\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt \\ &\quad + BDf_-(x_0)\int_{-\delta}^{0} t\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt - f(x_0)\int_{-\delta}^{0}\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt \\ &=: I_{2,1}(x,\lambda)+I_{2,2}(x,\lambda)+I_{2,3}(x,\lambda), \end{aligned}$$

and

$$\begin{aligned} I_3(x,\lambda) &= \int_{x_0}^{x_0+\delta} f(t)\frac{\partial}{\partial t}L_\lambda(t-x)dt = \int_{0}^{\delta} f(x_0+t)\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt \\ &= \int_{0}^{\delta} [f(x_0+t)-f(x_0)]\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt + f(x_0)\int_{0}^{\delta}\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt. \\ &= \int_{0}^{\delta}\left[\frac{f(x_0+t)-f(x_0)}{t} - BDf_+(x_0)\right]\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt \\ &\quad + BDf_+(x_0)\int_{0}^{\delta} t\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt + f(x_0)\int_{0}^{\delta}\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt \\ &=: I_{3,1}(x,\lambda)+I_{3,2}(x,\lambda)+I_{3,3}(x,\lambda), \end{aligned}$$

Setting

$$F(t) := \int_{0}^{t}\left[\frac{f(x_0+y)-f(x_0)}{y} - BDf_+(x_0)\right]dy,$$

then according to (3.1), for each $\varepsilon>0$ there exists a $\delta>0$ such that

$$|F(t)| \le \varepsilon\ t.$$

for all $0<t\le\delta$ and

$$G(t) := \int_{t}^{0}\left[\frac{f(x_0)-f(x_0-y)}{y} - BDf_-(x_0)\right]dy,$$

then according to (3.1), for each $\varepsilon>0$ there exists a $\delta>0$ such that

$$|G(t)| \le \varepsilon\ \ |t|$$

for all $-\delta\le t<0$.

We now fix this δ and estimate $I_2(x,\lambda)$ and $I_3(x,\lambda)$, respectively.

$$\begin{aligned} I_{3,1}(x,\lambda) &= \int\limits_0^\delta \left[t\frac{\partial}{\partial t}L_\lambda(x_0+t-x)\right]dF(t) \\ &= \left[t\frac{\partial}{\partial t}L_\lambda(x_0+t-x)\right]F(t)\,|_0^\delta - \int\limits_0^\delta F(t)\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt \\ &\quad -\int\limits_0^\delta F(t)t\frac{\partial^2}{\partial t^2}L_\lambda(x_0+t-x)dt \\ &\leq \varepsilon\delta^2\left[\frac{\partial}{\partial t}L_\lambda(x_0+t-x)\right]_\delta + \varepsilon\int\limits_0^\delta t\left|\frac{\partial}{\partial t}L_\lambda(x_0+t-x)\right|dt \\ &\quad +\varepsilon\int\limits_0^\delta t^2\left|\frac{\partial}{\partial t}L_\lambda(x_0+t-x)\right|dt. \end{aligned}$$

Integration by parts and using (‡ e), it is easy to see that the right hand side of the last inequality approaches to zero as $(x,\lambda)\to(x_0,\lambda_0)$.

Using the similar method,

$$|I_{2,1}(x,\lambda)| = \left|\int\limits_{-\delta}^0\left[\frac{f(x_0)-f(x_0+t)}{t} - BDf_-(x_0)\right]t\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt\right|$$

goes to zero as $(x,\lambda)\to(x_0,\lambda_0)$. Again integration by parts and using (3.1), one can easily shown that

$$I_{2,3}(x,\lambda)+I_{3,3}(x,\lambda)\to 0$$

as $(x,\lambda)\to(x_0,\lambda_0)$. For the terms $I_{2,2}(x,\lambda)$ and $I_{3,2}(x,\lambda)$, we can find the following equality.

$$\begin{aligned} I_{2,2}(x,\lambda)+I_{3,2}(x,\lambda) &= BDf_-(x_0)\int\limits_{-\delta}^0 t\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt \\ &\quad +BDf_+(x_0)\int\limits_0^\delta t\frac{\partial}{\partial t}L_\lambda(x_0+t-x)dt \\ &= BDf_-(x_0)\int\limits_{x_0-\delta}^{x_0} t\frac{\partial}{\partial t}L_\lambda(t-x)dt \\ &\quad +BDf_+(x_0)\int\limits_{x_0}^{x_0+\delta} t\frac{\partial}{\partial t}L_\lambda(t-x)dt. \end{aligned}$$

Integration by parts and using (3.18) yield

$$I_{2,2}(x,\lambda)+I_{3,2}(x,\lambda)\to N\,BDf_+(x_0)+(1-N)BDf_-(x_0).$$

The estimate $-\frac{\delta}{2}<x_0-x\leq 0$ is obtained in similar way. So this completes the proof. □

Corollary 3.4. *If we choose $N=\frac{1}{2}$ in Theorem 3.3, then we obtain*

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)}\frac{\partial}{\partial x}T_\lambda(f;x) = \frac{BDf_+(x_0)+BDf_-(x_0)}{2}.$$

Corollary 3.5. *Let $f \in L_1(a,b)$ has a symmetric Borel derivative $(BDf_s(x))$ at x_0. If the hypothesis of Theorem 3.3 are satisfied, then*

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \frac{\partial}{\partial x} T_\lambda(f;x) = BD_s f(x_0).$$

Example 3.6. As a particular case, let the function $K_\lambda(t,u)$ be linear with respect to the second variable, i.e.,

$$K_\lambda(t,u) = D_\lambda(t)\, u$$

where $D_\lambda(t)$ satisfies all the conditions of $(\ddagger)$. This form has been using in Approximation Theory since its beginning ([3]).

Example 3.7. Let consider the following function

$$K_\lambda(t,u) = \left(2\lambda^2 t\, u + H(u)\right) \chi_{[0,\frac{1}{\lambda}]}(t),$$

where $H(u)$ is a function independent of t, $\Lambda = [1,\infty)$ is a set of indices equipped with natural topology and let an accumulation point λ_0 of Λ be at infinity with regard to this topology. First of all, note that $K_\lambda(t,u)$ is a kernel, i.e., $K_\lambda(t,0) = 0$.
It is seen that for every $u \in \mathbb{R}$,

$$\frac{\partial}{\partial x} K_\lambda(t-x,u) = \begin{cases} -2\lambda^2 u & , \quad t-x \in [0,\frac{1}{\lambda}] \\ 0 & , \quad t-x \notin [0,\frac{1}{\lambda}] \end{cases}. \tag{3.20}$$

According to (3.20), we obtain $\frac{\partial}{\partial x} L_\lambda(t-x)$ as

$$\frac{\partial}{\partial x} L_\lambda(t-x) = \begin{cases} -2\lambda^2 & , \quad t-x \in [0,\frac{1}{\lambda}] \\ 0 & , \quad t-x \notin [0,\frac{1}{\lambda}] \end{cases} \tag{3.21}$$

which implies

$$L_\lambda(z) = \begin{cases} 2\lambda^2 z & , \quad z \in [0,\frac{1}{\lambda}] \\ 0 & , \quad z \notin [0,\frac{1}{\lambda}] \end{cases}.$$

In this case, one can easily seen that

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}} L_\lambda(z)dz = 1 < \infty, \ \lim_{\lambda\to\infty} \int_{\mathbb{R}\setminus U} L_\lambda(z)dz = 0$$

for every $U \in \mathcal{U}(0)$ and

$$\lim_{\lambda\to\infty} \left[\sup_{|z|\geq\delta} L_\lambda(z) \right] = 0$$

for every $\delta > 0$. In addition, by (3.21), we deduce that $L_\lambda(z)$, as a function of z, is non-increasing on $[0,\infty)$ and non-decreasing on $(-\infty,0]$ for each $\lambda \in \Lambda$.

Hence, all the conditions of $(\ddagger)$ are satisfied by the kernel function $K_\lambda(t,u)$.

Example 3.8. In a similar way, we introduce some kind of nonlinear moment kernels on some suitable sets, by setting

$$K_\lambda(t,u) = L_\lambda(t)u + G_\lambda(u),$$

where (G_λ) is family of functions independent of t. Such kind of kernels and their operator forms in the theory of nonlinear integral operators were deeply investigated (see [1], [2]).

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