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On Certain Subclass Of Meromorphic Functions With Positive Coefficients

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Abstract In this paper we introduce and study a new subclass of meromorphically uniformly convex functions with positive coefficients defined by a differential operator and obtain coefficient estimates, growth and distortion theorem, radius of convexity,integral transforms, convex linear combinations, convolution properties and $\delta-$neighborhoods for the class $\sigma_p(\varrho, \upsilon, \varsigma)$.

1 Introduction

Let Σ denote the class of the functions of the form

$$\vartheta(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \tag{1.1}$$

which are regular in domain $E = \{z \in \mathbb{C} : 0 < |z| < 1\}$ with a simple pole at the origin with residue 1 there. Let Σ_s and $\Sigma^*(\varrho)$ and $\Sigma_k(\varrho), 0 \leq \varrho < 1$, denote the subclasses of Σ that are univalent, meromorphically starlike of order ϱ and meromorphically convex of order ϱ respectively. Analytically $\vartheta(z)$ of the form (1.1) is in $\Sigma^*(\varrho)$ if and only if

$$Re\left\{-\frac{z\vartheta'(z)}{\vartheta(z)}\right\} > \varrho, \quad z \in E. \tag{1.2}$$

Similarly, $\vartheta \in \Sigma_k(\varrho)$ if and only if $\vartheta(z)$ is of the form (1.1) and satisfies

$$Re\left\{-\left(1 + \frac{z\vartheta''(z)}{\vartheta'(z)}\right)\right\} > \varrho, \quad z \in E. \tag{1.3}$$

It being understood that if $\varrho = 1$ then $\vartheta(z) = \frac{1}{z}$ is the only function which is $\Sigma^*(1)$ and $\Sigma_k(1)$. The classes $\Sigma^*(\varrho)$ and $\Sigma_k(\varrho)$ have been extensively studied by Pommerenke [9], Clunie [1], Royster [12] and others.

Since, to a certain extent the work in the meromorphic univalent case has paralleled that of regular univalent case, it is natural to search for a subclass of Σ_s that has properties analogous to those of $T^*(\varrho)$. Juneja et al. [5] introduced the class Σ_p of functions of the form

$$\vartheta(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m, a_m \geq 0, \tag{1.4}$$

$$\Sigma_p^*(\varrho) = \Sigma_p \cap \Sigma^*(\varrho).$$

For functions $\vartheta(z)$ in the class Σ_p, we define a linear operator D_ς^n by the following form

$$\begin{aligned} D_\varsigma^0\vartheta(z) &= \vartheta(z) \\ D_\varsigma^1\vartheta(z) &= (1-\varsigma)\vartheta(z) + \varsigma\frac{(z^2\vartheta(z))'}{z},\ \varsigma \geq 0 \\ &= (1+\varsigma)\vartheta(z) + \varsigma z\vartheta'(z) = D_\varsigma\vartheta(z) \\ D_\varsigma^2\vartheta(z) &= D_\varsigma(D^1\vartheta(z)) \\ &\vdots \end{aligned}$$

$$D_\varsigma^n\vartheta(z) = D_\varsigma(D_\varsigma^{n-1}\vartheta(z)) = \frac{1}{z} + \sum_{m=1}^{\infty}[1+\varsigma(m+1)]^n a_m z^m,\ \text{for } n \in \mathbb{N}_0 = 0,1,2,\cdots. \quad (1.5)$$

The classes Σ_p^* and various other subclasses of Σ were studied rather extensively by Clunie [1] and also see [9, 12, 14]. Motivated by works of Madhavi et al. [8], we define the following a new subclass $\sigma_p(\varrho, \upsilon, \varsigma)$ of meromorphically uniformly convex functions in Σ_p by making use of generalized the differential operator.

Definition 1.1. For $-1 \leq \varrho < 1, \varsigma > 0$ and $\upsilon \geq 1$, we let $\sigma_p(\varrho, \upsilon, \varsigma)$ be the subclass of Σ_p consisting of the form (1.4) and satisfying the analytic criterion

$$-Re\left\{\frac{z(D_\varsigma^n\vartheta(z))'}{D_\varsigma^n\vartheta(z)} + \varrho\right\} > \upsilon\left|\frac{z(D_\varsigma^n\vartheta(z))'}{D_\varsigma^n\vartheta(z)} + 1\right|, \quad (1.6)$$

$D_\varsigma^n\vartheta(z)$ is given by (1.5) .

The function class $\sigma_p(\varrho, \upsilon, \varsigma)$ unifies well known classes of meromorphic uniformly convex function with positive coefficients. To illustrate, we observe that the class $\sigma_p(\varrho, \upsilon, 1) = \sigma_p(\varrho, \upsilon)$ was studied by Madhavi et al. [8] .

The main object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds, growth and distortion properties, radius of convexity, convex linear combination and convolution properties, integral operators and $\delta-$neighbourhoods for the class $\sigma_p(\varrho, \upsilon, \varsigma)$.

2 Coefficient inequality

In this section, we obtain the coefficient bounds of function $\vartheta(z)$ for the class $\sigma_p(\varrho, \upsilon, \varsigma)$.

Theorem 2.1. *A function $\vartheta(z)$ of the form (1.4) is in $\sigma_p(\varrho, \upsilon, \varsigma)$ if*

$$\sum_{m=1}^{\infty}[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]\,|a_m| \;\leq\; (1-\varrho),\ -1 \leq \varrho < 1 \text{ and } \upsilon \geq 1. \quad (2.1)$$

Proof. It sufficient to show that

$$\upsilon\left|\frac{z(D_\varsigma^n\vartheta(z))'}{D_\varsigma^n\vartheta(z)} + 1\right| + Re\left\{\frac{z(D_\varsigma^n\vartheta(z))'}{D_\varsigma^n\vartheta(z)} + 1\right\} \leq (1-\varrho).$$

$$\text{We have } \upsilon\left|\frac{z(D_\varsigma^n\vartheta(z))'}{D_\varsigma^n\vartheta(z)}+1\right|+Re\left\{\frac{z(D_\varsigma^n\vartheta(z))'}{D_\varsigma^n\vartheta(z)}+1\right\}$$

$$\leq(1+\upsilon)\left|\frac{z(D_\varsigma^n\vartheta(z))'}{D_\varsigma^n\vartheta(z)}+1\right|$$

$$\leq\frac{(1+\upsilon)\sum\limits_{m=1}^{\infty}[1+\varsigma(m+1)]^n(m+1)\varsigma|a_m||z^m|}{\frac{1}{|z|}-\sum\limits_{m=1}^{\infty}[1+\varsigma(m+1)]^n|a_m||z^m|}$$

Letting $z\to 1$ along the real axis, we obtain

$$\leq\frac{(1+\upsilon)\sum\limits_{m=1}^{\infty}[1+\varsigma(m+1)]^n(m+1)\varsigma|a_m|}{\frac{1}{|z|}-\sum\limits_{m=1}^{\infty}[1+\varsigma(m+1)]^n|a_m|}.$$

The above expression is bounded by $(1-\varrho)$ if

$$\sum_{m=1}^{\infty}[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]\,|a_m|\ \leq\ (1-\varrho).$$

Hence the theorem is completed. □

Corollary 2.2. *Let the function $\vartheta(z)$ defined by (1.4) be in the class $\sigma_p(\varrho,\upsilon,\varsigma)$. Then*

$$a_m\leq\frac{(1-\varrho)}{\sum\limits_{m=1}^{\infty}[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]},\quad(m\geq 1). \tag{2.2}$$

Equality holds for the function of the form

$$\vartheta_m(z)=\frac{1}{z}+\frac{(1-\varrho)}{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}\,z^m. \tag{2.3}$$

Remark 2.3.

(i) For the choice of $\varsigma=1$ in Theorem 2.1 and Corollary 2.2, we observed that the coefficient estimates for the functions of the class,

$$|a_m|\ \leq\ \frac{(1-\varrho)}{(m+2)^n[(1+\upsilon)(m+1)+1-\varrho]}$$

is coincide with Madhavi et al. [8] .

3 Distortion Theorems

In this section, we obtain the sharp for the distortion theorems of the form (1.4) .

Theorem 3.1. *Let the function $\vartheta(z)$ defined by (1.4) be in the class $\sigma_p(\varrho,\upsilon,\varsigma)$. Then for $0<|z|=r<1$,*

$$\frac{1}{r}-\frac{(1-\varrho)}{(1+2\varsigma)^n[2\varsigma(1+\upsilon)+1-\varrho]}r\leq|\vartheta(z)|\leq\frac{1}{r}+\frac{(1-\varrho)}{(1+2\varsigma)^n[2\varsigma(1+\upsilon)+1-\varrho]}r \tag{3.1}$$

with equality for the function

$$\vartheta(z)=\frac{1}{z}+\frac{(1-\varrho)}{(1+2\varsigma)^n[2\varsigma(1+\upsilon)+1-\varrho]}\,z,\ \text{at}\ z=r,ir. \tag{3.2}$$

Proof. Suppose $\vartheta(z)$ is in $\sigma_p(\varrho, \upsilon, \varsigma)$. In view of Theorem 2.1, we have

$$(1+2\varsigma)^n[2\varsigma(1+\upsilon)+1-\varrho]\sum_{m=1}^{\infty} a_m \leq \sum_{m=1}^{\infty}[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho] \leq (1-\varrho)$$

which evidently yields $\sum\limits_{m=1}^{\infty} a_m \leq \frac{1-\varrho}{(1+2\varsigma)^n[2\varsigma(1+\upsilon)+1-\varrho]}$.
Consequently, we obtain

$$\begin{aligned}|\vartheta(z)| &= \left|\frac{1}{z}+\sum_{m=1}^{\infty} a_m z^m\right| \leq \left|\frac{1}{z}\right| + \sum_{m=1}^{\infty} a_m |z|^m \\ &\leq \frac{1}{r} + r\sum_{m=1}^{\infty} a_m \\ &\leq \frac{1}{r} + \frac{1-\varrho}{(1+2\varsigma)^n[2\varsigma(1+\upsilon)+1-\varrho]}\, r.\end{aligned}$$

$$\begin{aligned}\text{Also, } |\vartheta(z)| &= \left|\frac{1}{z}+\sum_{m=1}^{\infty} a_m z^m\right| \geq \left|\frac{1}{z}\right| - \sum_{m=1}^{\infty} a_m |z|^m \\ &\geq \frac{1}{r} - r\sum_{m=1}^{\infty} a_m \\ &\geq \frac{1}{r} - \frac{1-\varrho}{(1+2\varsigma)^n[2\varsigma(1+\upsilon)+1-\varrho]}\, r.\end{aligned}$$

Hence the result (3.1) follows. □

Theorem 3.2. *Let the function* $\vartheta(z)$ *defined by* (1.4) *be in the class* $\sigma_p(\varrho, \upsilon, \varsigma)$. *Then for* $0 < |z| = r < 1$,

$$\frac{1}{r^2} - \frac{1-\varrho}{(1+2\varsigma)^n[2\varsigma(1+\upsilon)+1-\varrho]} \leq |\vartheta'(z)| \leq \frac{1}{r^2} + \frac{1-\varrho}{(1+2\varsigma)^n[2\varsigma(1+\upsilon)+1-\varrho]}.$$

The result is sharp, the extremal function being of the form (2.3)

Proof. From Theorem 2.1, we have

$$\begin{aligned}&(1+2\varsigma)^n[2\varsigma(1+\upsilon)+1-\varrho]\sum_{m=1}^{\infty} m a_m \\ &\leq \sum_{m=1}^{\infty}[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho] \\ &\leq (1-\varrho)\end{aligned}$$

which evidently yields $\sum\limits_{m=1}^{\infty} m a_m \leq \frac{1-\varrho}{[1+2\varsigma]^n[2\varsigma(1+\upsilon)+1-\varrho]}$.

Consequently, we obtain

$$
\begin{aligned}
|\vartheta'(z)| &\leq \left|\frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m r^{m-1}\right| \\
&\leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m \\
&\leq \frac{1}{r^2} + \frac{(1-\varrho)}{(1+2\varsigma)^n[2\varsigma(1+\upsilon)+1-\varrho]}. \\
\text{Also, } |\vartheta'(z)| &\geq \left|\frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m r^{m-1}\right| \\
&\geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m \\
&\geq \frac{1}{r^2} + \frac{(1-\varrho)}{(1+2\varsigma)^n[2\varsigma(1+\upsilon)+1-\varrho]}.
\end{aligned}
$$

This completes the proof. □

Remark 3.3.

(i) For the choice of $\varsigma = 1$ in Theorems 3.1 and 3.2, we observed that the sharp for the distortion theorems for the functions of the class are coincide with Madhavi et al. [8] .

4 Class preserving integral operators

In this section, we consider the class preserving integral operator of the form (1.4) .

Theorem 4.1. *Let the function $\vartheta(z)$ defined by (1.4) be in the class $\sigma_p(\varrho, \upsilon, \varsigma)$. Then*

$$\vartheta(z) = cz^{-c-1} \int_0^z t^c f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{c}{c+m+1} a_m z^m, \quad c > 0 \tag{4.1}$$

is in $\sigma_p(\delta, \upsilon, \varsigma)$, where

$$\delta(\varrho, \upsilon, c, \varsigma) = \frac{[2\varsigma(1+\upsilon)+(1-\varrho)](c+2) - c(1-\varrho)[2\varsigma(1+\upsilon)+1]}{[2\varsigma(1+\upsilon)(1-\varrho)](c+2) - (1-\varrho)c}. \tag{4.2}$$

The result is sharp for $\vartheta(z) = \frac{1}{z} + \frac{(1-\varrho)}{(1+2\varsigma)^n[2\varsigma(1+\upsilon)+(1-\varrho)]} z$.

Proof. Suppose $\vartheta(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ is in $\sigma_p(\varrho, \upsilon, \varsigma)$. We have
$\vartheta(z) = cz^{-c-1} \int\limits_0^z t^c f(t)dt = \frac{1}{z} + \sum\limits_{m=1}^{\infty} \frac{c}{c+m+1} a_m z^m, \quad c > 0.$
It is sufficient to show that

$$\sum_{m=1}^{\infty} \frac{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\delta]}{1-\delta} \frac{c}{c+m+1} a_m \leq 1. \tag{4.3}$$

Since $\vartheta(z)$ is in $\sigma_p(\varrho, \upsilon, \varsigma)$, we have

$$\sum_{m=1}^{\infty} \frac{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}{1-\varrho} |a_m| \leq 1. \tag{4.4}$$

Thus (4.3) will be satisfied if

$$\sum_{m=1}^{\infty}\frac{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\delta]}{1-\delta}\frac{c}{c+m+1}$$
$$\leq \sum_{m=1}^{\infty}\frac{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}{1-\varrho}.$$

Solving for δ, we obtain

$$\delta \leq \frac{[(1+\upsilon)(m+1)\varsigma+1-\varrho](c+m+1)-c[(1+\upsilon)(m+1)\varsigma+1](1-\varrho)}{[(1+\upsilon)(m+1)\varsigma+1-\varrho](c+m+1)-c(1-\varrho)} = G(m) \quad (4.5)$$

A simple computation will show that $G(m)$ is increasing and $G(m) \geq G(1)$. Using this, the result follows. □

5 Convex linear combinations and convolution properties

In this section, we obtain sharp for $\vartheta(z)$ is meromorphically convex of order δ and necessary and sufficient condition for $\vartheta(z)$ is in the class $\sigma_p(\varrho,\upsilon,\varsigma)$. And also proved that convolution is in the class $\sigma_p(\varrho,\upsilon,\varsigma)$.

Theorem 5.1. *If the function* $\vartheta(z)=\frac{1}{z}+\sum\limits_{m=1}^{\infty}a_m z^m$ *is in* $\sigma_p(\varrho,\upsilon,\varsigma)$ *then* $\vartheta(z)$ *is meromorphically convex of order* $\delta(0\leq\delta<1)$ *in* $|z|<r=r(\varrho,\upsilon,\delta)$, *where*

$$r(\varrho,\upsilon,\delta)=\inf_{n\geq1}\left\{\frac{(1-\delta)(m+2)^n[(1+\upsilon)(1+m)+1-\varrho]}{(1-\varrho)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}.$$

The result is sharp.

Proof. Let $\vartheta(z)$ be in $\sigma_p(\varrho,\upsilon,\varsigma)$. Then, by Theorem 2.1, we have

$$\sum_{m=1}^{\infty}[1+\varsigma(m+1)]^n[(1+\upsilon)(1+m)\varsigma+1-\varrho]|a_m| \leq (1-\varrho). \quad (5.1)$$

It is sufficient to show that $\left|2+\frac{z\vartheta''(z)}{\vartheta'(z)}\right| \leq (1-\delta)$ for $|z|<r=r(\varrho,\upsilon,\delta,\varsigma)$, where $r(\varrho,\upsilon,\delta,\varsigma)$ is specified in the statement of the theorem. Then

$$\left|2+\frac{z\vartheta''(z)}{\vartheta'(z)}\right| = \left|\frac{\sum\limits_{m=1}^{\infty}m(m+1)a_m z^{m-1}}{\frac{-1}{z^2}+\sum\limits_{m=1}^{\infty}m a_m z^{m-1}}\right| \leq \frac{\sum\limits_{m=1}^{\infty}m(m+1)a_m|z|^{m+1}}{1-\sum\limits_{m=1}^{\infty}m a_m|z|^{m+1}}.$$

This will be bounded by $(1-\delta)$ if

$$\sum_{m=1}^{\infty}\frac{m(m+2-\delta)}{1-\delta}a_m|z|^{m+1} \leq 1. \quad (5.2)$$

By (5.1), it follows that (5.2) is true if

$$\frac{m(m+2-\delta)}{1-\delta}|z|^{m+1} \leq \frac{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}{1-\varrho}|a_m|, \quad m\geq 1$$

$$\text{or} \quad |z| \leq \left\{\frac{(1-\delta)[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}{(1-\varrho)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}. \quad (5.3)$$

Setting $|z|=r(\varrho,\upsilon,\delta,\varsigma)$ in (5.3), the result follows. The result is sharp for the function.

$$\vartheta_m(z)=\frac{1}{z}+\frac{(1-\varrho)}{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}z^m, \quad m\geq 1.$$

□

Theorem 5.2. *Let* $\vartheta_0(z) = \frac{1}{z}$ *and* $\vartheta_m(z) = \frac{1}{z} + \frac{(1-\varrho)}{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]} z^m, \quad m \geq 1$. *Then* $\vartheta(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ *is in the class* $\sigma_p(\varrho, \upsilon, \varsigma)$ *if and only if it can be expressed in the form* $\vartheta(z) = \omega_0 f_0(z) + \sum\limits_{m=1}^{\infty} \omega_m \vartheta_m(z)$, *where* $\omega_0 \geq 0, \omega_m \geq 0, m \geq 1$ *and* $\omega_0 + \sum\limits_{m=1}^{\infty} \omega_m = 1$.

Proof. Let $\vartheta(z) = \omega_0\vartheta_0(z) + \sum\limits_{m=1}^{\infty} \omega_m \vartheta_m(z)$ with $\omega_0 \geq 0, \omega_m \geq 0, m \geq 1$ and $\omega_0 + \sum\limits_{m=1}^{\infty} \omega_m = 1$. Then

$$\begin{aligned}\vartheta(z) &= \omega_0\vartheta_0(z) + \sum_{m=1}^{\infty} \omega_m \vartheta_m(z) \\ &= \frac{1}{z} + \sum_{m=1}^{\infty} \omega_m \frac{(1-\varrho)}{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]} z^m\end{aligned}$$

Since $\sum\limits_{m=1}^{\infty} \frac{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}{(1-\varrho)} \omega_m \frac{(1-\varrho)}{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}$

$$= \sum_{m=1}^{\infty} \omega_m = 1 - \omega_0 \leq 1.$$

By Theorem 2.1, $\vartheta(z)$ is in the class $\sigma_p(\varrho, \upsilon, \varsigma)$.

Conversely suppose that the function $\vartheta(z)$ is in the class $\sigma_p(\varrho, \upsilon, \varsigma)$, since

$$a_m \leq \frac{(1-\varrho)}{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]} z^m, m \geq 1.$$

$$\omega_m = \sum_{m=1}^{\infty} \frac{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}{(1-\varrho)} a_m \quad \text{and} \quad \omega_0 = 1 - \sum_{m=1}^{\infty} \omega_m.$$

It follows that $\vartheta(z) = \omega_0\vartheta_0(z) + \sum\limits_{m=1}^{\infty} \omega_m \vartheta_m(z)$.
This completes the proof of the theorem. □

For the functions $\vartheta(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ and $g(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} b_m z^m$ belongs to Σ_p, we denoted by $(\vartheta * g)(z)$ the convolution of $\vartheta(z)$ and $g(z)$ and defined as

$$(\vartheta * g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m$$

Theorem 5.3. *If the function* $\vartheta(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ *and* $g(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} b_m z^m$ *are in the class* $\sigma_p(\varrho, \upsilon, \varsigma)$ *then* $(\vartheta * g)(z)$ *is in the class* $\sigma_p(\varrho, \upsilon, \varsigma)$.

Proof. Suppose $\vartheta(z)$ and $g(z)$ are in $\sigma_p(\varrho, \upsilon, \varsigma)$. By Theorem 2.1, we have

$$\sum_{m=1}^{\infty} \frac{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}{(1-\varrho)} a_m \leq 1$$

$$\text{and} \sum_{m=1}^{\infty} \frac{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}{(1-\varrho)} b_m \leq 1 .$$

Since $\vartheta(z)$ and $g(z)$ are regular are in E, so is $(\vartheta * g)(z)$. Furthermore

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}{(1-\varrho)} a_m b_m \\
&\leq \sum_{m=1}^{\infty} \left\{ \frac{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}{(1-\varrho)} \right\}^2 a_m b_m \\
&\leq \left(\sum_{m=1}^{\infty} \frac{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}{(1-\varrho)} a_m \right) \\
&\qquad \left(\sum_{m=1}^{\infty} \frac{[1+\varsigma(m+1)]^n[(1+\upsilon)(m+1)\varsigma+1-\varrho]}{(1-\varrho)} b_m \right) \\
&\leq 1.
\end{aligned}$$

Hence, by Theorem 2.1, $(\vartheta * g)(z)$ is in the class $\sigma_p(\varrho, \upsilon, \varsigma)$. □

Remark 5.4.

(i) For the choice of $\varsigma = 1$ in Theorems 5.1, 5.2 and 5.3, we observed that the the results are coincide with Madhavi et al. [8] .

6 Neighborhoods for the class $\sigma_p(\varrho, \upsilon, \gamma, \varsigma)$

In this section, we define the $\delta-$neighborhood of a function $\vartheta(z)$ and establish a relation between $\delta-$neighborhood and $\sigma_p(\varrho, \upsilon, \gamma, \varsigma)$ class of a function.

Definition 6.1. A function $\vartheta \in \Sigma_p$ is said to in the class $\sigma_p(\varrho, \upsilon, \gamma, \varsigma)$ if there exists a function $g \in \sigma_p(\varrho, \upsilon, \varsigma))$ such that

$$\left| \frac{\vartheta(z)}{g(z)} - 1 \right| < (1-\gamma), \quad z \in E,\ 0 \leq \gamma < 1. \tag{6.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [2] and Ruschweyh [13]. We defined the $\delta-$neighborhood of a function $\vartheta \in \Sigma_p$ by

$$N_\delta(\vartheta) = \left\{ g \in \Sigma_p \ : \ g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m \ \text{and} \ \sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta \right\} \tag{6.2}$$

Theorem 6.2. *If $g \in \sigma_p(\varrho, \upsilon, \varsigma)$ and*

$$\gamma = 1 - \frac{\delta[2\varsigma(1+\upsilon)+1-\varrho]}{2\varsigma(1+\upsilon)} \tag{6.3}$$

then $N_\delta(g) \subset \sigma_p(\varrho, \upsilon, \gamma, \varsigma)$.

Proof. Let $\vartheta \in N_\delta(g)$. Then we find from (6.2) that

$$\sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta \tag{6.4}$$

which implies the coefficient of inequality $\sum\limits_{m=1}^{\infty} |a_m - b_m| \leq \frac{\delta}{m}, \ \ m \in \mathbb{N}$.

Since $g \in \sigma_p(\varrho, \upsilon, \varsigma)$, we have $\sum\limits_{m=1}^{\infty} b_m = \frac{1-\varrho}{2\varsigma(1+\upsilon)+1-\varrho}$.

So that $\left|\frac{\vartheta(z)}{g(z)} - 1\right| < \frac{\sum_{m=1}^{\infty} |a_m - b_m|}{1 - \sum_{m=1}^{\infty} b_m} \leq \frac{\delta[2\varsigma(1+\upsilon)+1-\varrho]}{2\varsigma(1+\upsilon)} = 1-\gamma$, provided γ is given by (6.3).

Hence, by Definition 6.1, $\vartheta \in \sigma_p(\varrho, \upsilon, \gamma)$ for γ given by (6.3), which completes the proof of theorem. □

Remark 6.3.

(i) For the choice of $\varsigma = 1$ in Theorem 6.2, we observed that the the result is coincide with Madhavi et al. [8] .

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