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Some inequalities relative to convex and close-to-convex functions involving q–derivative

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Abstract. In this article, by using q–derivative, two subclasses of p–valent analytic functions are introduced. Some inequalities associated with convex and close-to-convex functions are obtained.

1 Introduction

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{+\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Gosper and Rahman [1], defined the q–derivative $(0 < q < 1)$ of a function f of the form $f(z) = z + \sum_{k=2}^{+\infty} a_k z^k$ by:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad (z \in \Delta). \tag{1.2}$$

From (1.1) and (1.2), we get:

$$D_q f(z) = [p]_q z^{p-1} + \sum_{k=p+1}^{\infty} a_k [k]_q z^{k-1}, \tag{1.3}$$

where

$$[x]_q = \frac{1-q^x}{1-q} = 1 + q + \ldots + q^{x-1}. \tag{1.4}$$

As $q \to 1^-$, then $[p]_q \to p$ and $[k]_q \to k$, so we conclude:

$$\lim_{q \to 1^-} D_q f(z) = f'(z), \qquad (z \in \Delta),$$

see also [3].

Now, we define two subclasses of $\mathcal{A}_p$ as follow.

Definition 1.1. A function $f(z) \in \mathcal{A}_p$ is said to be in the subclass $X_p(q)$ if it satisfies the inequality:

$$\left| \frac{1}{[p]_q} \frac{D_q f(z)}{z^{p-1}} - 1 \right| < 1, \tag{1.5}$$

where $z \in \Delta$ and $D_q f(z)$ is defined by (1.3).

A function $f(z) \in \mathcal{A}_p$ is said to be in the subclass $Y_p(q)$ if it is satisfies the inequality:

$$\left| \frac{z[D_q f(z)]'}{D_q f(z)} - p \right| < p.$$

To prove the main theorems, we need the following lemma, due to Jack [2], (see also [4]).

Lemma 1.2. *Let $w(z)$ be non-constant in Δ and $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then $z_0 w'(z_0) = tw(z_0)$, where $t \geqslant 1$ is a real number.*

2 Main Results

In this section, we will prove two theorems involving inequalities on p–valent functions.

Theorem 2.1. *If $f(z) \in \mathcal{A}_p$ satisfies the inequality*

$$\text{Re}\left\{\frac{z[D_q f(z)]'}{D_q f(z)} - (p-1)\right\} < \frac{1}{2}, \tag{2.1}$$

then $f(z) \in X_p(q)$.

Proof. Let $f(z) \in \mathcal{A}_p$, we define the function $w(z)$ by:

$$\frac{1}{[p]_q}\frac{D_q f(z)}{z^{p-1}} = 1 + w(z), \qquad (z \in \Delta). \tag{2.2}$$

With a sample calculation in Δ, we have $w(0) = 0$. From (2.2), we obtain:

$$\frac{1}{[p]_q} D_q f(z) = z^{p-1} + z^{p-1} w(z),$$

or

$$\frac{1}{[p]_q}[D_q f(z)]' = (p-1)z^{p-2} + (p-1)z^{p-2} z w(z) + z^{p-1} w'(z),$$

or

$$\frac{1}{[p]_q}\frac{[D_q f(z)]'}{z^{p-2}} = (p-1)(1 + w(z)) + z w'(z). \tag{2.3}$$

From (2.2) and (2.3), we get:

$$\frac{z w'(z)}{1 + w(z)} = \frac{z[D_q f(z)]'}{D_q f(z)} - (p-1). \tag{2.4}$$

Now, let for $z_0 \in \Delta$, $\max\limits_{|z| \leqslant |z_0|} |w(z)| = |w(z_0)| = 1$, then by using the Jack's lemma and putting $w(z_0) = e^{i\theta} \neq -1$ in (2.4) we have:

$$\begin{aligned}\text{Re}\left\{\frac{z[D_q f(z)]'}{D_q f(z)} - (p-1)\right\} &= \text{Re}\left\{\frac{z_0 w'(z_0)}{1 + w(z_0)}\right\} \\ &= \text{Re}\left\{\frac{t w(z_0)}{1 + w(z_0)}\right\} \\ &= \text{Re}\left\{\frac{t e^{i\theta}}{1 + e^{i\theta}}\right\} = \frac{t}{2} \geqslant \frac{1}{2},\end{aligned}$$

which is a contradiction with (2.1). Thus we have $|w(z)| < 1$ for all $z \in \Delta$. So from (2.2) we conclude:

$$\left|\frac{1}{[p]_q}\frac{D_q f(z)}{z^{p-1}} - 1\right| = |w(z)| < 1,$$

and this gives the result. □

By letting $q \to 1^-$, we have the following corollary that related to close-to-close functions.

Corollary 2.2. *If $f(z) \in \mathcal{A}_p$ satisfies the inequality:*

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - p\right\} < \frac{1}{2}, \qquad (z \in \Delta),$$

then $f(z)$ is $p-$valently close-to-close function with respect to the origin in Δ and $|\frac{f'(z)}{z^{p-1}} - p| < p$.

Theorem 2.3. *If $f(z) \in \mathcal{A}_p$ satisfies:*

$$\text{Re}\left\{1 + z\left\{\frac{[D_q f(z)]''}{[D_q f(z)]'} - \frac{[D_q f(z)]'}{D_q f(z)}\right\}\right\} < \frac{1}{2}, \tag{2.5}$$

where $f(z) \in Y_p(q)$.

Proof. Let the function $f(z) \in \mathcal{A}_p$, we define the function $w(z)$ by

$$\frac{z[D_q f(z)]'}{D_q f(z)} = p(1 + w(z)), \qquad (z \in \Delta). \tag{2.6}$$

It is easy to verify that $w(z)$ is analytic in Δ and $w(0) = 0$. By (2.6) we have:

$$z[D_q f(z)]' = pD_q f(z) + pD_q f(z)w(z),$$

or

$$[D_q f(z)]' + z[D_q f(z)]'' = p[D_q f(z)]' + p\{w'(z)D_q f(z) + w(z)[D_q f(z)]'\},$$

or

$$1 + \frac{z[D_q f(z)]''}{[D_q f(z)]'} = p(1 + w(z)) + pw'(z)\frac{D_q f(z)}{[D_q f(z)]'}.$$

By applying (2.6), we get:

$$1 + \frac{z[D_q f(z)]''}{[D_q f(z)]'} = p(1 + w(z) + \frac{zw'(z)}{1 + w(z)}.$$

Now, let for a point $z_0 \in \Delta$, $\max_{|z| \leqslant |z_0|} |w(z)| = |w(z_0)| = 1$.

By Jack's lemma and putting $w(z_0) = e^{i\theta}$, we have:

$$\begin{aligned}\text{Re}\left\{1 + z\left\{\frac{[D_q f(z)]''}{[D_q f(z)]'} - \frac{[D_q f(z)]'}{D_q f(z)}\right\}\right\} &= \text{Re}\left\{\frac{z_0 w'(z_0)}{1 + w(z_0)}\right\}\\ &= \text{Re}\left\{\frac{tw(z_0)}{1 + w(z_0)}\right\}\\ &= t\,\text{Re}\left\{\frac{e^{i\theta}}{1 + e^{i\theta}}\right\} = \frac{t}{2} \geqslant \frac{1}{2},\end{aligned}$$

which is a contradiction with (2.6). Thus for all $z \in \Delta$, $|w(z)| < 1$ and so from (2.6) we have:

$$\left|\frac{z[D_q f(z)]'}{D_q f(z)} - p\right| < p,$$

thus, the proof is complete. □

By letting $q \to 1^-$, we have the following corollary that related to ...

Corollary 2.4. *If $f(z) \in \mathcal{A}_p$, satisfies the inequality:*

$$\text{Re}\left\{1 + z\left[\frac{f'''}{f''} - \frac{f''}{f'}\right]\right\} < \frac{1}{2}, \qquad (z \in \Delta),$$

then $f(z)$ is $p-$valently convex function with respect to the origin in Δ and

$$|1 + \frac{zf''}{f'} - (p + 1)| < p.$$

References

[1] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia Math. Appl. **98(3)**, 282–285 (1991).

[2] I.S. Jack, Functions starlike and convex of order α, *J. Lond. Math. Soc.* **2(3)**, 469–474 (1971).

[3] Z. S. I. Mansour, *Linear sequential q–difference equations of fractional order*, Fract. Calc. Appl. Anal. **12(2)**, 159–178 (2009).

[4] S. S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65(2)**, 289–305 (1978).

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