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A DIFFERENTIAL SUBORDINATION AND UNIVALENCE OF ANALYTIC FUNCTIONS

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Abstract. In the present paper, we study a differential subordination and obtain certain results on close-to-convexity and univalence of normalized analytic functions.

1 Introduction

Let $\mathcal{A}$ be the class of analytic functions f, normalized by the conditions $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbb{E} = \{z \in \mathcal{C} : |z| < 1\}$. The members of the class $\mathcal{A}$ have the Taylor series expansion of the following form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let $\mathcal{S}$ denote the class of normalized analytic functions which are univalent in $\mathbb{E}$. A function $f \in \mathcal{A}$ is said to be starlike if it is univalent in $\mathbb{E}$ and $f(\mathbb{E})$ is a starlike domain. It is well-known that a function $f \in \mathcal{A}$ is starlike if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in \mathbb{E}. \tag{1.1}$$

Let $\mathcal{S}^*$ denote the class of starlike functions in $\mathcal{A}$. A function $f \in \mathcal{A}$ is said to be convex if it is univalent in $\mathbb{E}$ and $f(\mathbb{E})$ is a convex domain. A necessary and sufficient condition for a function $f \in \mathcal{A}$ to be convex is that

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ z \in \mathbb{E}. \tag{1.2}$$

The class of convex functions is denoted by $\mathcal{K}$. It is well known that

$$f \in \mathcal{K} \Leftrightarrow zf' \in \mathcal{S}^*. \tag{1.3}$$

A function $f \in \mathcal{A}$ is said to be close-to-convex if there is a real number α, $\frac{-\pi}{2} < \alpha < \frac{\pi}{2}$ and a convex function g (not necessarily normalized) such that

$$\Re\left(e^{\iota\alpha}\,\frac{f'(z)}{g'(z)}\right) > 0, z \in \mathbb{E}. \tag{1.4}$$

In case g is normalized, the class of close-to-convex functions is denoted by $\mathcal{C}$. Moreover, in view of (1.3), we have: A function $f \in \mathcal{A}$ is said to be close-to-convex if there is a real number α, $\frac{-\pi}{2} < \alpha < \frac{\pi}{2}$ and a starlike function ϕ such that

$$\Re\left(e^{\iota\alpha}\,\frac{zf'(z)}{\phi(z)}\right) > 0, z \in \mathbb{E}. \tag{1.5}$$

Let $\mathcal{M}$ be the class of analytic functions $q(z)$ in $\mathbb{E}$, normalized by the condition $q(0) = 1$. Let $\mathcal{N}$ be the subclass of $\mathcal{M}$ consisting of all univalent functions q for which $q(\mathbb{E})$ is a convex domain. Let $\mathcal{P}$ denote the well-known class of analytic functions $p(z)$ with $p(0) = 1$ and $\Re(p(z)) > 0$, $z \in \mathbb{E}$.
An analytic function g said to be subordinate to an analytic function f in $|z| < 1$, written as $g \prec f$, if $g(z) = f(w(z))$, where $|w(z)| < 1$ in $|z| < 1$.
Let $\Phi : \mathcal{C}^2 \times \mathbb{E} \to \mathcal{C}$ and let p be an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathcal{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and let h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0). \tag{1.6}$$

A univalent function q is called a dominant of the differential subordination (1.6) if $p(0) = q(0)$ and $p \prec q$ and for all p satisfying (1.6). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.6), is said to be the best dominant of the differential subordination(1.6). The best dominant is unique up to a rotation of $\mathbb{E}$.

Recently, Srivastava et al. [3] considered the class $R_\alpha(q)$, the class of normalized analytic functions in the open unit disk $\mathbb{E}$ satisfying the differential subordination

$$f'(z) + \frac{1}{2}(1 + e^{\iota\alpha})zf''(z) \prec q(z), z \in \mathbb{E}, \alpha \in (-\pi, \pi], \tag{1.7}$$

where the function $q(z)$ is analytic in the open disk $\mathbb{E}$ such that $q(0) = 1$. They studied the above differential subordination for $\alpha \in (-\pi, \pi)$ and proved the following result:

Theorem 1.1. *Let $\alpha \in (-\pi, \pi)$ and let $q \in \mathcal{N}$. If $f \in R_\alpha(q)$, then*

$$f'(z) \prec \gamma z^{-\gamma} \int_0^z q(\xi)\xi^{\gamma-1} d\xi \prec q(z),$$

where $\gamma = \dfrac{2}{1 + e^{\iota\alpha}}$. The result is sharp.

In the present article, we generalize the subordination given in (1.7) and obtain sufficient conditions for close-to-convexity and hence univalence for a function $f \in \mathcal{A}$, by selecting the starlike function ϕ and the dominant q. We also study the coefficient estimate and Fekete-Szegö problem for the class under consideration.
To prove the main results, we shall use the following lemmas:

Lemma 1.2. *[1] Let h be convex in $\mathbb{E}$ with $h(0) = a$ and $\gamma \in \mathcal{C}$ with $\Re\gamma \geq 0$. If the function $p(z) = a + p_n z^n + p_{n+1} z^{n+1} + ...$ is analytic in $\mathbb{E}$ and $p(z) + \frac{1}{\gamma} zp'(z) \prec h(z)$, then $p(z) \prec q(z) \prec h(z), z \in \mathbb{E}$, where $q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(\xi)\xi^{\frac{\gamma}{n}-1} d\xi, z \in \mathbb{E}$. The result is sharp.*

Lemma 1.3. *[10] Let $w(z) = \sum_{n=1}^{\infty} c_n z^n$ be an analytic function in $\mathbb{E}$ such that, $w(0) = 0, |w(z)| < 1$. It is known that $|c_1| \leq 1$, $|c_2| \leq 1 - |c_1|^2$.*

Lemma 1.4. *[5] For every $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*$, $|a_n| \leq n$, $n = 2, 3, ...$*

Lemma 1.5. *[9] Let $\phi \in S^*$ with $\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then for μ real,*

$$|b_3 - \mu b_2^2| \leq \begin{cases} 3 - 4\mu, & \text{for } \mu \leq \frac{1}{2} \\ 1, & \text{for } \frac{1}{2} \leq \mu \leq 1, \\ 4\mu - 3, & \text{for } \mu \geq 1. \end{cases}$$

2 Main results

Theorem 2.1. *Let $\alpha \in (-\pi, \pi)$, $q \in \mathcal{N}$ and ϕ be a starlike function. Let $f \in \mathcal{A}$ satisfy*

$$\left(1 + \frac{1}{\gamma}\right) \frac{zf'(z)}{\phi(z)} + \frac{1}{\gamma} \left[\frac{z^2 f''(z)}{\phi(z)} - \frac{z^2 f'(z)\phi'(z)}{\phi^2(z)}\right] \prec q(z), z \in \mathbb{E},$$

or, equivalently

$$\frac{zf'(z)}{\phi(z)} + \left(\frac{z}{\gamma}\right)\left(\frac{zf'(z)}{\phi(z)}\right)' \prec q(z), z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{\phi(z)} \prec \frac{\gamma}{z^\gamma}\int_0^z q(\xi)\xi^{\gamma-1}d\xi \prec q(z), z \in \mathbb{E},$$

where $\gamma = \dfrac{2}{1+e^{\iota\alpha}}$.

Proof. Write $p(z) = \dfrac{zf'(z)}{\phi(z)}$ and $\gamma = \dfrac{2}{1+e^{\iota\alpha}}$, then

$$p(z) = 1 + p_1 z + p_2 z^2 + ...$$

is analytic in $\mathbb{E}$ and $\Re\gamma \geq 0$ for $\alpha \in (-\pi, \pi)$. Thus

$$p(z) + \frac{1}{\gamma}zp'(z) = \left(1+\frac{1}{\gamma}\right)\frac{zf'(z)}{\phi(z)} + \frac{1}{\gamma}\left[\frac{z^2 f''(z)}{\phi(z)} - \frac{z^2 f'(z)\phi'(z)}{\phi^2(z)}\right] \prec q(z).$$

As $q \in \mathcal{N}$, by Lemma 1.2, we have

$$\frac{zf'(z)}{\phi(z)} \prec \frac{\gamma}{z^\gamma}\int_0^z q(\xi)\xi^{\gamma-1}d\xi \prec q(z), z \in \mathbb{E}. \tag{2.1}$$

Remark 2.2. If we choose $q \in \mathcal{N}$ such that $\Re(q(z)) > 0$, then in view of (2.1), $\Re\left(\dfrac{zf'(z)}{\phi(z)}\right) > 0$. Since ϕ is starlike. Therefore f is close-to-convex and hence univalent.

Setting $\phi(z) = z$ in Theorem 2.1, we obtain Theorem 1.1 proved by Srivastava et al. in [3].

We derive the following results by selecting the starlike functions

$$\phi(z) = 1 + z, ze^z, \frac{z}{(1-z)^2}, \frac{z}{1-z^2}$$

respectively.

Corollary 2.3. *For* $\alpha \in (-\pi, \pi)$ *and* $q \in \mathcal{N}$, *if* $f \in \mathcal{A}$ *satisfies*

$$\left(1+\frac{1}{\gamma}\right)\frac{z}{1+z}f'(z) + \frac{1}{\gamma}\left[\frac{z^2}{1+z}f''(z) - \left(\frac{z}{1+z}\right)^2 f'(z)\right] \prec q(z), z \in \mathbb{E},$$

or, equivalently,

$$\frac{z}{1+z}f'(z) + \left(\frac{z}{\gamma}\right)\left(\frac{zf'(z)}{1+z}\right)' \prec q(z), z \in \mathbb{E},$$

then

$$\frac{z}{1+z}f'(z) \prec \frac{\gamma}{z^\gamma}\int_0^z q(\xi)\xi^{\gamma-1}d\xi \prec q(z), z \in \mathbb{E},$$

where $\gamma = \dfrac{2}{1+e^{\iota\alpha}}$.

Corollary 2.4. *For* $\alpha \in (-\pi, \pi)$ *and* $q \in \mathcal{N}$, *if* $f \in \mathcal{A}$ *satisfies*

$$e^{-z}\left(1-\frac{z}{\gamma}\right)f'(z) + \frac{e^{-z}}{\gamma}zf''(z) \prec q(z), z \in \mathbb{E},$$

or, equivalently

$$e^{-z}f'(z) + \left(\frac{z}{\gamma}\right)\left(e^{-z}f'(z)\right)' \prec q(z), z \in \mathbb{E},$$

then

$$e^{-z}f'(z) \prec \frac{\gamma}{z^\gamma}\int_0^z q(\xi)\xi^{\gamma-1}d\xi \prec q(z), z \in \mathbb{E},$$

where $\gamma = \dfrac{2}{1+e^{\iota\alpha}}$.

Corollary 2.5. *For $\alpha \in (-\pi, \pi)$ and $q \in \mathcal{N}$, if $f \in \mathcal{A}$ satisfies*

$$\left[\left(1+\frac{1}{\gamma}\right)(1-z)^2 - \frac{1}{\gamma}(1-z^2)\right] f'(z) + \frac{1}{\gamma} z(1-z)^2 f''(z) \prec q(z), z \in \mathbb{E},$$

or, equivalently,

$$(1-z)^2 f'(z) + \frac{z}{\gamma}\left[(1-z)^2 f'(z)\right]' \prec q(z), z \in \mathbb{E},$$

then

$$(1-z)^2 f'(z) \prec \frac{\gamma}{z^\gamma}\int_0^z q(\xi)\xi^{\gamma-1} d\xi \prec q(z), z \in \mathbb{E},$$

where $\gamma = \dfrac{2}{1+e^{\iota\alpha}}$.

Corollary 2.6. *For $\alpha \in (-\pi, \pi)$ and $q \in \mathcal{N}$, if $f \in \mathcal{A}$ satisfies*

$$\left[\left(1+\frac{1}{\gamma}\right)(1-z^2) - \frac{1}{\gamma}(1+z^2)\right] f'(z) + \frac{1}{\gamma} z(1-z^2) f''(z) \prec q(z), z \in \mathbb{E},$$

or, equivalently

$$(1-z^2) f'(z) + \frac{z}{\gamma}\left[(1-z^2) f'(z)\right]' \prec q(z), z \in \mathbb{E},$$

then

$$(1-z^2) f'(z) \prec \frac{\gamma}{z^\gamma}\int_0^z q(\xi)\xi^{\gamma-1} d\xi \prec q(z), z \in \mathbb{E},$$

where $\gamma = \dfrac{2}{1+e^{\iota\alpha}}$.

Theorem 2.7. *For $\alpha \in (-\pi, \pi)$, $q \in \mathcal{N}$ and $\phi \in \mathcal{S}^*$. If $f \in \mathcal{A}$ satisfies*

$$\Re\left[\frac{zf'(z)}{\phi(z)} + \frac{1+e^{\iota\alpha}}{2} z\left(\frac{zf'(z)}{\phi(z)}\right)'\right] > 0, \tag{2.2}$$

then there exists $p \in \mathcal{P}$ such that,

$$f(z) = \gamma\int_0^z \frac{\phi(\eta)}{\eta^{\gamma+1}}\left[\int_0^\eta \xi^{\gamma-1} p(\xi) d\xi\right] d\eta,$$

where $\gamma = \dfrac{2}{1+e^{\iota\alpha}}$.

Proof. Let $f \in \mathcal{A}$ such that

$$\Re\left[\frac{zf'(z)}{\phi(z)} + \left(\frac{1+e^{\iota\alpha}}{2}\right) z\left(\frac{zf'(z)}{\phi(z)}\right)'\right] > 0.$$

Write

$$\frac{zf'(z)}{\phi(z)} + \frac{1+e^{\iota\alpha}}{2} z\left(\frac{zf'(z)}{\phi(z)}\right)' = p(z), where\ \ p \in \mathcal{P}.$$

or

$$\left[z^{\left(\frac{2}{1+e^{\iota\alpha}}\right)}\frac{zf'(z)}{\phi(z)}\right]' = \left(\frac{2}{1+e^{\iota\alpha}}\right) z^{\left(\frac{2}{1+e^{\iota\alpha}}\right)}\left(\frac{p(z)}{z}\right).$$

Writing $\dfrac{2}{1+e^{\iota\alpha}} = \gamma$, we obtain

$$\left[z^{\gamma+1}\frac{f'(z)}{\phi(z)}\right]' = \gamma z^{\gamma-1} p(z)$$

or

$$z^{\gamma+1}\frac{f'(z)}{\phi(z)} = \gamma\int_0^z \xi^{\gamma-1} p(\xi) d\xi$$

which is equivalent to

$$f(z) = \gamma\int_0^z \frac{\phi(\eta)}{\eta^{\gamma+1}}\left[\int_0^\eta \xi^{\gamma-1} p(\xi) d\xi\right] d\eta.$$

Remark 2.8. In view of Remark 2.2, Condition (2.2) implies that f is close-to-convex and hence univalent in $\mathbb{E}$. In the next result, we study Fekete-Szegö problem for the functions satisfying condition (2.2).

3 Deductions

Selecting $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ as a dominant in Theorem 2.1, we have $1 + \frac{zq''(z)}{q'(z)} = \frac{1-Bz}{1+Bz}$. Therefore $\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > 0$ for $-1 \le B < A \le 1$. Thus we immediately arrive at the following result:
Let ϕ be a starlike function and let $f \in \mathcal{A}$ satisfy

$$\frac{zf'(z)}{\phi(z)} + \frac{z}{\gamma}\left(\frac{zf'(z)}{\phi(z)}\right)' \prec \frac{1+Az}{1+Bz}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{\phi(z)} \prec \frac{\gamma}{z^\gamma}\int_0^z \frac{1+A\xi}{1+B\xi}\xi^{\gamma-1}d\xi \prec \frac{1+Az}{1+Bz}, z \in \mathbb{E},$$

where $\gamma = \frac{2}{1+e^{\iota\alpha}}$ for $\alpha \in (-\pi, \pi)$. Thus f is close-to-convex and hence univalent in $\mathbb{E}$.
Selecting $q(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \le \beta < 1)$ as a dominant in Theorem 2.1, we have $1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{1-z}$. Therefore $\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > 0$. We notice that $q(z)$ is a convex function and we obtain the following result:
Let ϕ be a starlike function and let $f \in \mathcal{A}$ satisfy

$$\frac{zf'(z)}{\phi(z)} + \frac{z}{\gamma}\left(\frac{zf'(z)}{\phi(z)}\right)' \prec \frac{1+(1-2\beta)z}{1-z}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{\phi(z)} \prec \frac{\gamma}{z^\gamma}\int_0^z \frac{1+(1-2\beta)\xi}{1-\xi}\xi^{\gamma-1}d\xi \prec \frac{1+(1-2\beta)z}{1-z}, z \in \mathbb{E},$$

where $\gamma = \frac{2}{1+e^{\iota\alpha}}$ for $\alpha \in (-\pi, \pi)$. Therefore f is close-to-convex and hence univalent in $\mathbb{E}$.
Writing $q(z) = e^z$ as a dominant in Theorem 2.1 , we obtain $1 + \frac{zq''(z)}{q'(z)} = 1 + z$ and $\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > 0$. Therefore $q(z)$ is a convex function and we obtain the following result:
For a starlike function ϕ if $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\phi(z)} + \frac{z}{\gamma}\left(\frac{zf'(z)}{\phi(z)}\right)' \prec e^z, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{\phi(z)} \prec \frac{\gamma}{z^\gamma}\int_0^z e^\xi \xi^{\gamma-1}d\xi \prec e^z, z \in \mathbb{E},$$

where $\gamma = \frac{2}{1+e^{\iota\alpha}}$ for $\alpha \in (-\pi, \pi)$. Thus f is close-to-convex and hence univalent in $\mathbb{E}$.
Setting $q(z) = 1 + \frac{2}{\pi^2}log^2\frac{1+\sqrt{z}}{1-\sqrt{z}}$ as a dominant in Theorem 2.1, a little calculation yields that $1 + \frac{zq''(z)}{q'(z)} = 1 + \frac{\frac{1}{(1-z)^2} + \frac{3z-1}{2\sqrt{z}(1-z)^2}\log(\frac{1+\sqrt{z}}{1-\sqrt{z}})}{\frac{1}{\sqrt{z}(1-z)}\log(\frac{1+\sqrt{z}}{1-\sqrt{z}})}$ and $\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > 0$. Thus $q(z)$ is a convex function and we obtain the following result.

Suppose that ϕ be a starlike function and $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\phi(z)} + \frac{z}{\gamma}\left(\frac{zf'(z)}{\phi(z)}\right)' \prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}},$$

then

$$\frac{zf'(z)}{\phi(z)} \prec \frac{\gamma}{z^\gamma}\int_0^z \left(1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{\xi}}{1-\sqrt{\xi}}\xi^{\gamma-1}\right) d\xi \prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}, z \in \mathbb{E},$$

where $\gamma = \dfrac{2}{1+e^{\iota\alpha}}$ for $\alpha \in (-\pi, \pi)$.

Selecting $\phi(z) = z$, $f \in \mathcal{A}$ in above result , we obtain the following subordination result for the functions in class $\mathcal{L}P_\alpha$, the class of all normalized analytic functions in the unit disk $\mathbb{E}$ for which

$$f'(z) + \frac{1+e^{\iota\alpha}}{2} zf''(z) \prec Q(z) = 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}, Q(0) = 1, z \in \mathbb{E}.$$

This class was studied by L. Trojnar Spelina in [4].
If $f \in \mathcal{A}$ satisfies

$$f'(z) + \frac{1}{\gamma} zf''(z) \prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}},$$

then

$$f'(z) \prec \frac{\gamma}{z^\gamma}\int_0^z \left(1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{\xi}}{1-\sqrt{\xi}}\xi^{\gamma-1}\right) d\xi \prec 1 + \frac{2}{\pi^2} log^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}.$$

Theorem 3.1. *Suppose $\alpha \in (-\pi, \pi)$, $p \in \mathcal{P}$ and ϕ be a starlike function. If $f \in \mathcal{A}$ satisfies*

$$\left(1 + \frac{1}{\gamma}\right)\frac{zf'(z)}{\phi(z)} + \frac{1}{\gamma}\left[\frac{z^2 f''(z)}{\phi(z)} - \frac{z^2 f'(z)\phi'(z)}{\phi^2(z)}\right] \prec p(z), z \in \mathbb{E},$$

or, equivalently,

$$\frac{zf'(z)}{\phi(z)} + \left(\frac{z}{\gamma}\right)\left(\frac{zf'(z)}{\phi(z)}\right)' \prec p(z), z \in \mathbb{E},$$

then

$$|a_3 - \mu a_2^2| \leq \begin{cases} (1-\mu) + \frac{2}{3}|(2-3\mu)\frac{\gamma}{\gamma+1}| + |\mu(\frac{\gamma}{\gamma+1})^2|, & for \quad \mu \leq \frac{2}{3}, \\ \frac{1}{3} + \frac{2}{3}|(2-3\mu)||\frac{\gamma}{\gamma+1}| + |\mu(\frac{\gamma}{\gamma+1})^2|, & for \quad \frac{2}{3} \leq \mu \leq \frac{4}{3}, \\ (\mu - 1) + \frac{2}{3}|(2-3\mu)\frac{\gamma}{\gamma+1}| + |\mu(\frac{\gamma}{\gamma+1})^2|, & for \quad \mu \geq \frac{4}{3}, \end{cases} \tag{3.1}$$

where $\gamma = \dfrac{2}{1+e^{\iota\alpha}}$.

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $\phi(z) = z + \sum_{k=2}^{\infty} b_k z^k$.
Writting

$$p_1(z) = \frac{zf'(z)}{\phi(z)} + \left(\frac{z}{\gamma}\right)\left(\frac{zf'(z)}{\phi(z)}\right)'$$

$$= 1 + \left(1 + \frac{1}{\gamma}\right)(2a_2 - b_2)z + \left(1 + \frac{2}{\gamma}\right)(b_2^2 - b_3 + 3a_3 - 2a_2 b_2)z^2$$

$+\left(1 + \frac{3}{\gamma}\right)(4a_4 - b_4 + 2b_2 b_3 - 3a_3 b_2 + 2a_2 b_2^2 - 2a_2 b_3)z^3 + ...$

As $p_1 \prec p$, there exists $w(z) = \sum_{k=1}^{\infty} c_k z^k$ such that $w(0) = 0$, $|w(z)| < 1$, such that

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + 2c_1 z + ...$$

Then $(2a_2 - b_2)(1 + \frac{1}{\gamma}) = 2c_1 \Rightarrow a_2 = \frac{\gamma}{\gamma+1}c_1 + \frac{b_2}{2}$

and $(1+\frac{2}{\gamma})(b_2^2-b_3+3a_3-2a_2b_2)=2c_2 \Rightarrow a_3=\frac{1}{3}\left[\frac{2\gamma}{\gamma+2}c_2+\frac{2\gamma}{\gamma+1}b_2c_1+b_3\right]$
Now,

$$|a_3-\mu a_2^2|=\left|-\mu(\frac{\gamma}{\gamma+1})^2c_1^2+\frac{2\gamma}{3(\gamma+2)}c_2+\left[\frac{2\gamma}{3(\gamma+1)}-\mu\frac{\gamma}{\gamma+1}\right]b_2c_1+\left[\frac{b_3}{3}-\mu\frac{b_2^2}{4}\right]\right|$$

$$|a_3-\mu a_2^2|\leq\frac{1}{3}\left|b_3-\kappa b_2^2\right|+\left|\frac{2\gamma}{3(\gamma+1)}-\mu\frac{\gamma}{\gamma+1}\right||b_2||c_1|+\left|-\mu(\frac{\gamma}{\gamma+1})^2\right|c_1|^2|+\left|\frac{2\gamma}{3(\gamma+2)}\right||c_2|$$

where $\kappa=\frac{3\,\mu}{4}$.
By applying Lemma 1.4 for upper bound of $|b_2|$,

$$|a_3-\mu a_2^2|\leq\frac{1}{3}\left|b_3-\kappa b_2^2\right|+2\left|\frac{2\gamma}{3(\gamma+1)}-\mu\frac{\gamma}{\gamma+1}\right||c_1|+\frac{2}{3}\left|\frac{\gamma}{\gamma+2}\right||c_2|+\left|-\mu\left(\frac{\gamma}{\gamma+1}\right)^2\right||c_1|^2$$

and now by Lemma 1.3, for upper bound of $|c_2|$,

$$\begin{aligned}&|a_3-\mu a_2^2|\\&\leq\frac{1}{3}\left|b_3-\kappa b_2^2\right|+2\left|\frac{2\gamma}{3(\gamma+1)}-\mu\frac{\gamma}{\gamma+1}\right||c_1|+\frac{2}{3}\left|\frac{\gamma}{\gamma+2}\right|\left|(1-|c_1|^2)\right|+\left|-\mu\left(\frac{\gamma}{\gamma+1}\right)^2\right||c_1|^2\\&=\frac{1}{3}\left|b_3-\kappa b_2^2\right|+2\left|\frac{2\gamma}{3(\gamma+1)}-\mu\frac{\gamma}{\gamma+1}\right||c_1|+\frac{2}{3}\left|\frac{\gamma}{\gamma+2}\right|-\frac{2}{3}\left|\frac{\gamma}{\gamma+2}\right||c_1|^2+\left|-\mu\left(\frac{\gamma}{\gamma+1}\right)^2\right||c_1|^2.\end{aligned}$$

Again by applying Lemma 1.3, for upper bound of $|c_1|$,

$$|a_3-\mu a_2^2|\leq\frac{1}{3}\left|b_3-\kappa b_2^2\right|+2\left|\left(\frac{2}{3}-\mu\right)\frac{\gamma}{\gamma+1}\right|-\left|\mu\left(\frac{\gamma}{\gamma+1}\right)^2\right|. \tag{3.2}$$

Now, we consider the following cases to apply Lemma 1.5 for the upper bound of $\left|b_3-\kappa b_2^2\right|$,

Case 1 $\mu\leq\frac{2}{3}$

In this case, we apply Lemma 1.5 on the first term on R.H.S. of (3.2), we obtain

$|a_3-\mu a_2^2|\leq\frac{1}{3}(3-4\kappa)+\frac{2}{3}\left|(2-3\mu)\frac{\gamma}{\gamma+1}\right|+\left|-\mu(\frac{\gamma}{\gamma+1})^2\right|$

or

$|a_3-\mu a_2^2|\leq(1-\mu)+\frac{2}{3}\left|(2-3\mu)\frac{\gamma}{\gamma+1}\right|+\left|-\mu(\frac{\gamma}{\gamma+1})^2\right|.$

Case 2 when $\frac{2}{3}\leq\mu\leq\frac{4}{3}$

By applying Lemma 1.5, we obtain

$$|a_3-\mu a_2^2|\leq\frac{1}{3}+\frac{2}{3}|(2-3\mu)|\left|\frac{\gamma}{\gamma+1}\right|+\left|-\mu(\frac{\gamma}{\gamma+1})^2\right|.$$

Case 3 when $\mu\geq\frac{4}{3}$ In this case, (3.2) with Lemma 1.5 gives

$$\begin{aligned}|a_3-\mu a_2^2|&\leq\frac{1}{3}(4\kappa-3)+\frac{2}{3}\left|(2-3\mu)\frac{\gamma}{\gamma+1}\right|+\left|-\mu(\frac{\gamma}{\gamma+1})^2\right|\\&=(\mu-1)+\frac{2}{3}\left|(2-3\mu)\frac{\gamma}{\gamma+1}\right|+\left|-\mu(\frac{\gamma}{\gamma+1})^2\right|.\end{aligned}$$

Compiling all the above cases, we obtain

$$|a_3-\mu a_2^2|\leq\begin{cases}(1-\mu)+\frac{2}{3}|(2-3\mu)\frac{\gamma}{\gamma+1}|+|\mu(\frac{\gamma}{\gamma+1})^2|, & \text{for}\quad \mu\leq\frac{2}{3};\\ \frac{1}{3}+\frac{2}{3}|(2-3\mu)||\frac{\gamma}{\gamma+1}|+|\mu(\frac{\gamma}{\gamma+1})^2|, & \text{for}\quad \frac{2}{3}\leq\mu\leq\frac{4}{3};\\ (\mu-1)+\frac{2}{3}|(2-3\mu)\frac{\gamma}{\gamma+1}|+|\mu(\frac{\gamma}{\gamma+1})^2|, & \text{for}\quad \mu\geq\frac{4}{3}.\end{cases} \tag{3.3}$$

□

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