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NOTE ON QUADRATIC MODULES IN FORMAL POWER SERIES RING

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Abstract Let R be an Euclidean field. In this note we characterise when the quadratic modules of $R[[X]]$ are Archimedean, which of them are orderings and we give the biggest element.

1 Introduction

Let A be a commutative ring with unit. A quadratic module in A is a subset M of A such that $1 \in M$, $M + M \subseteq M$ and $A^2M \subseteq M$ where $A^2 = \{x^2;\ x \in A\}$. Such a set is a preordering of A if $MM \subseteq M$. We do not exclude the case $-1 \in M$. The ring A contains a smallest quadratic module, namely, the set $\sum A^2$ consisting of all sums of squares of elements of A. The ring A itself is the biggest quadratic module of A. If $g_1, \cdots, g_r \in A$, the set $B = \sum A^2 + (\sum A^2)g_1 + \cdots + (\sum A^2)g_r$ is the smallest quadratic module containing $g_1, \cdots, g_r$. It is called the quadratic module generated by $g_1, \cdots, g_r$ and it is denoted by $QM(g_1, \cdots, g_r)$. The set $C = \left\{ \sum \sigma_\epsilon g_\epsilon; \epsilon = (\epsilon_1, \cdots, \epsilon_r) \in \{0,1\}^r,\ \sigma_\epsilon \in \sum A^2,\ g_\epsilon = \prod_{i=1}^r g_i^{\epsilon_i} \right\}$ is the smallest preordering of A containing $g_1, \cdots, g_r$. It is called the preordering generated by $g_1, \cdots, g_r$ and it is denoted by $PO(g_1, \cdots, g_r)$. It coincides with the quadratic module generated by the products $g_1^{\epsilon_1} \ldots g_r^{\epsilon_r}$, where $\epsilon_i = 0, 1$ and $(\epsilon_1, \ldots, \epsilon_r) \neq (0, \ldots, 0)$. We have $QM(g_1, \cdots, g_r) \subseteq PO(g_1, \cdots, g_r)$ and for all $g \in A$, $QM(g) = PO(g)$. For example $QM(1) = PO(1) = \sum A^2$ and if 2 is a unit in A then $QM(-1) = PO(-1) = A$. The notion of quadratic module has been studied extensively in [5] and a brief description of the quadratic modules in the ring $\mathbb{R}[[X]]$ appears in Chapter 9, page 128. In [1], the authors consider an Euclidean field R and classify all the quadratic modules in the ring $R[[X]]$. This is probably one of the very few nontrivial cases where an overview of all quadratic modules of a ring can be given. In [1, Theorem 4.1], they proved that every quadratic module of $R[[X]]$ is a preordering. They showed in [1, Theorem 2.3] that the partially ordered set of all monogenic (with one generator) quadratic modules has the following diagram.

$$
\begin{array}{ccccc}
 & & PO(-1) = A & & \\
 & / & | & \backslash & \\
PO(X) & & PO(-X^2) & & PO(-X) \\
| & / & | & \backslash & | \\
PO(X^3) & & PO(-X^4) & & PO(-X^3) \\
| & / & | & \backslash & | \\
PO(X^5) & & PO(-X^6) & & PO(-X^5) \\
| & / & | & \backslash & | \\
\vdots & & \vdots & & \vdots \\
 & & & & \\
 & & | & & \\
 & \backslash & & / & \\
 & & PO(1) = A^2 & &
\end{array}
$$

Following the terminology used in [1], we call these chains of preorderings the left, central and right columns of the diagram displayed above. We call $PO(-1) = A$ the top and $PO(1) = A^2$ the bottom of the diagram. Notice that the bottom is not a member of any column. In the present note, we continue the study of the quadratic modules in the ring $R[[X]]$. A field R is called Euclidean if it is formally real and for every $x \in R$ either x or $-x$ is a square. So it has a unique (total) ordering with positive cone R^2. Real closed fields are Euclidean. For example, $\mathbb{R}$ and the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{R}$ are Euclidean.

2 Results

Let A be a commutative ring with one and M a quadratic module of A. Then M is said to be Archimedean if for any element $f \in A$ there is $n \in \mathbb{N}$ such that $n - f \in M$. An ordered field R is said to be Archimedean if for any positive element $a \in R$ there is $n \in \mathbb{N}$ such that $a \leq n$.

Theorem 2.1. *Let R be an Euclidean field and $A = R[[X]]$. The following are equivalent*
1. The field R is Archimedean.
2. All the quadratic modules of A are Archimedean.
3. The quadratic module A^2 is Archimedean.

Proof. $''(1) \implies (2)''$ Let $f = a_0 + a_1X + \ldots \in R[[X]]$ and M a quadratic module of A. If $a_0 < 0$ then $-a_0 = b^2$ with $b \in R^*$. By [1, Proposition 2.2], $0 - f = b^2 - a_1X - \ldots \in A^2 \subseteq M$. If $a_0 \geq 0$ there is $n \in \mathbb{N}$ such that $a_0 < n$, so $n - a_0 = c^2$ with $c \in R^*$. By [1, Proposition 2.2], $n - f = c^2 - a_1X - \ldots \in A^2 \subseteq M$.
$''(2) \implies (3)''$ Clear. $''(3) \implies (1)''$ Let $a \in R$ with $a \geq 0$. Since R is Euclidean there is $b \in R$ such that $a = b^2 \in R^2 \subset A^2$. But A^2 is an archimedean quadratic module. Then there is $n \in \mathbb{N}$ such that $n - a \in A^2$. So $n - a \in R^2$ and $a \leq n$. □
Example. All the quadratic modules of $\mathbb{R}[[X]]$ and $\overline{\mathbb{Q}}[[X]]$ are Archimedean.

Remark 2.2. If R is an Euclidean non Archimedean field, then the only Archimedean quadratic module of $A = R[[X]]$ is A itself. Indeed, there is $r \in R$ such that for all $n \in \mathbb{N}$, $n < r$. So $n - r = -a_n^2$ with $a_n \in R^*$. Let M be a quadratic module of $R[[X]]$ and suppose that $n - r \in M$ for some $n \in \mathbb{N}$. Then $-1 = (\frac{1}{a_n})^2(-a_n^2) \in A^2M \subseteq M$ and $R[[X]] = PO(-1) \subseteq M$.

Example. By [3, Theorem], the field $R = \mathbb{R}((T^{\frac{1}{\infty}})) = \bigcup_{n:1}^{\infty} \mathbb{R}((T^{\frac{1}{n}}))$ of Puiseux series with coefficients in $\mathbb{R}$ is real closed and non Archimedean. All the proper quadratic modules of $R[[X]]$ are non Archimedean.

Let Q be a quadratic module of $R[[X]]$. If Q contains a monogenic quadratic module from the left column of the diagram in the introduction, we denote the union of all these submodules by Q_l. Otherwise we put $Q_l = A^2$. We call Q_l the left component of Q. In the same way we define the central component Q_c and the right component Q_r of Q.

Theorem 2.3. *Let R be an Euclidean field. The biggest proper quadratic module (preordering) of $R[[X]]$ is $Q = R^2 + XR[[X]]$.*

Proof. First, note that $Q = PO(X) + PO(-X) + PO(-X^2) = QM(X, -X, -X^2)$. Indeed, by [1, Lemma 2.4] with $A = R[[X]]$, we have $\big(PO(X) + PO(-X)\big) + PO(-X^2) = (A^2 + AX) + (A^2 + AX^2) = A^2 + AX = R[[X]]^2 + XR[[X]] = R^2 + XR[[X]] = Q$. Now, let Q' be any proper quadratic module of $R[[X]]$. Then $Q'_g \subseteq PO(X)$, $Q'_d \subseteq PO(-X)$ and $Q'_c \subseteq PO(-X^2)$. By [1, Proposition 3.2], $Q' = Q'_g + Q'_d + Q'_c \subseteq Q$. □

Let A be a commutative ring with one. An ordering of A is a subset P of A such that $P + P \subseteq P$, $PP \subseteq P$, $P \cup -P = A$ and $P \cap -P$ is a prime ideal of A.

Theorem 2.4. *Let R be an Euclidean field. The only orderings of $A = R[[X]]$ are $PO(X) = A^2 + A^2X$, $PO(-X) = A^2 - A^2X$ and $Q = R^2 + XR[[X]]$. Moreover $PO(X) \cap -PO(X) = PO(-X) \cap -PO(-X) = (0)$ and $Q \cap -Q = XR[[X]]$ is the maximal ideal of $R[[X]]$.*

Proof. 1) Since $-Q = -R^2 + XR[[X]]$ and $R = R^2 \cup -R^2$, it is clear that $Q \cup -Q = R[[X]]$ and $Q \cap -Q = XR[[X]]$. Then Q is an ordering. We can also note that by [4, Theorem 5.1.3, page 85], every proper preordering is contained in an ordering. By Theorem 2.3, Q must be an ordering.
Now, let $P = PO(X) = A^2 + A^2X$. An element of $P \cap -P$ has the form $f^2 + g^2X = -h^2 - k^2X$ with $f, g, h, k \in R[[X]]$. Then $f^2 + h^2 = -(g^2 + k^2)X$. The left hand if it is nonzero, its order is even and its first coefficient is positive. But the right hand if it is nonzero, its order is odd and its first coefficient is negative. So $f^2 + h^2 = g^2 + k^2 = 0$ and since $R((X))$ is ordered then $f = g = h = k = 0$ and $P \cap -P = (0)$ a prime ideal of $R[[X]]$. Let $0 \neq f \in R[[X]]$, $f = aX^dg^2$ with $a \in R^*$ and $g \in R[[X]]$. If $a > 0$, so a square and d is even then $f \in A^2 \subseteq P$. If $a > 0$ and d is odd then $f \in A^2X \subseteq P$. If $a < 0$ and d is even then $f \in -A^2 \subseteq -P$. If $a < 0$ and d is odd then $f \in -A^2X \subseteq -P$. We conclude that $A = P \cup -P$ and P is an ordering on A.
Let $i : R[[X]] \longrightarrow R[[X]]$ be the automorphism introduced in [1, p. 76] and defined by $i(f(X)) = f(-X)$. Then $PO(-X)) = i(PO(X))$ is an ordering on $R[[X]]$.
2) We will prove that $PO(X)$, $PO(-X)$ and Q are the only orderings of A. The prime spectrum $Spec(R[[X]]) = \{(0), XR[[X]]\}$. The quotient ring $R[[X]]/(0) \simeq R[[X]]$ has a quotient field $R((X))$ with exactly two orderings, extending the unique ordering of R. One for which $X > 0$ and one for which $X < 0$. See [2, p. 11]. The quotient ring $R[[X]]/XR[[X]] \simeq R$ is an Euclidean field, with a single ordering. By [4, Proposition 5.1.1 page 83], the set of orderings of $R[[X]]$ is one to one with the set $\{(\mathcal{P}, \bar{P})\}$ where $\mathcal{P} \in Spec(R[[X]])$ and $\bar{P}$ is an ordering on the quotient field of $R[[X]]/\mathcal{P}$. The second set is of cardinality 3. Then the only orderings of $R[[X]]$ are $PO(X)$, $PO(-X)$ and Q. □

Remark 2.5. 1) It is well known that orderings on fields are maximal, which is not the case on rings. See [4, Example 5.2 (2) page 86]. Theorem 2.3 gives another example. Indeed, $PO(X)$ and $PO(-X)$ are strictly contained in $R^2 + XR[[X]]$.
2) The ordering $R^2 + XR[[X]]$ is the natural extension of the unique ordering of R to $R[[X]]$, defined by $f(X) = a_0 + a_1X + \ldots \geq 0$ if and only if $a_0 \geq 0$.
3) The domain $R[[X]]$ has three orderings but its quotient field $R((X))$ only two orderings.

We close this note by the following

Proposition 2.6. *Let K be a field which is not formally real with characteristic different from 2. Then the only quadratic module of $K[[X]]$ is $K[[X]]$.*

Proof. Let M be a quadratic module of $K[[X]]$. Then $-1 \in \sum K^2 \subseteq \sum K[[X]]^2 \subseteq M$. So $K[[X]] = QM(-1) \subseteq M$. □

Remark 2.7. Let K be a field of characteristic 2. Then $K[[X]]$ has at least two quadratic modules $K[[X]]$ and $\sum K[[X]]^2 = K[[X]]^2 = \{\sum_{i:0}^{\infty} a_i^2X^{2i},\ a_i \in K\}$.

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