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ON DERIVATION IN NEAR-RINGS AND ITS GENERALIZATIONS: A SURVEY

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Abstract We present an historical account of the study of derivations, generalized derivations, n-derivations, generalized n-derivation and other kinds of derivations in near-rings, based on the work of several authors. Moreover, recent results on semigroup ideals and generalized n-derivations on these topics have been discussed in details. Examples of various notions have also been included.

1 Introduction

The present paper is an attempt to present an up-to-date account of work on derivations and its various invariants in the setting of near-rings. The work has been presented in a manner suitable for everybody who have some basic knowledge in near-ring theory. In order to make the treatment as self-contained as possible, and to bring together all the relevant material in a single paper, we have included several references. Some times, many results have been unified in a single theorem. Proper references of almost all the results are given. Let $\mathcal{N}$ be non empty set, equipped with two binary operations say '+' and '.'. $\mathcal{N}$ is called a left near-ring if $(i)(\mathcal{N},+)$ is a group (not necessarily abelian) $(ii)(\mathcal{N},\,.)$ is a semigroup and $(iii)x.(y+z)=x.y+x.z$ for all $x,y,z\in\mathcal{N}$. Similarly a right near-ring can also be defined. A left near-ring $\mathcal{N}$ is called zero-symmetric if $0.x=0$ for all $x\in\mathcal{N}$(recall that in a left near ring $x.0=0$ for all $x\in\mathcal{N}$). Similar remarks hold for a right near-ring also. For a natural example of a near-ring, let $(G,+)$ be a group (not necessarily abelian). Consider S, the set of all mappings from G to G. Then S is a zero-symmetric right near-ring with regard to the operations $'+'$ and $'.'$ defined as below:

$$(f+g)(x)=f(x)+g(x) \text{ for all } x\in G,$$
$$(fg)(x)=f(g(x))\text{for all } x\in G.$$

where $f,g\in EndG$. It is to be noted that it is not a left near-ring.

Example 1.1. Let $(\mathcal{C},+)$ be the usual group of complex numbers with regard to ordinary addition of complex numbers. Let us define $'*'$ in $\mathcal{C}$ as following $a*b=|a|b$ for all $a,b\in\mathcal{C}$. Then $(\mathcal{C},+,*)$ is a zero-symmetric left near-ring which is not a right near-ring. If $'*'$ is defined as $a*b=a|b|$ for all $a,b\in\mathcal{C}$. Then it can be easily seen that $(\mathcal{C},+,*)$ is a zero-symmetric right near-ring which is not a left near- ring.

Throughout the paper unless otherwise stated $\mathcal{N}$ will denote a zero-symmetric left near-ring. $\mathcal{N}$ is called 3-prime near-ring if $x\mathcal{N}y=\{0\}$ implies $x=0$ or $y=0$. It is called semiprime near-ring if $x\mathcal{N}x=\{0\}$ implies $x=0$. Near-ring $\mathcal{N}$ is called n (being an integer greater than 1) torsion free if $nx=0$ implies $x=0$. The symbol Z will represent the multiplicative center of $\mathcal{N}$ i.e., $Z=\{x\in\mathcal{N}\mid xy=yx \text{ for all } y\in\mathcal{N}\}$. As usual, for any $x,y\in\mathcal{N}$, the symbol $[x,y]$ will denote the commutator $xy-yx$. (x,y) will indicate the additive commutator $x+y-x-y$ and $x\circ y$ will represent the anti-commutator $xy+yx$. The symbol C will represent the set of all additive commutators of near- ring $\mathcal{N}$, that is $C=\{x+y-x-y\mid x,y\in\mathcal{N}\}$. For terminology one can see Pilz [38].

2 Derivations in Near-Rings

The notion of derivation in rings is quite old and plays a significant role in various branches of mathematics. It has got a tremendous development when in 1957, Posner [39] established two very striking results on derivations in prime rings. Also there has been considerable interest in investigating commutativity of rings, more often that of prime ring and semiprime rings admitting suitable constrained derivations. Derivations in prime rings and semiprime rings have been studied by several algebraists in various directions. Motivated by the concept of derivation in rings Bell and Mason [24] introduced the concept of derivation in near-rings as following.

Definition 2.1. A derivation $'d'$ on $\mathcal{N}$ is defined to be an additive mapping $d : \mathcal{N} \to \mathcal{N}$ satisfying the product rule $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$.

Example 2.2. Let $\mathcal{N} = \mathcal{N}_1 \bigoplus \mathcal{N}_2$, where $\mathcal{N}_1$ is a zero symmetric left near-ring and $\mathcal{N}_2$ is a ring having derivation δ. Then $d : \mathcal{N} \longrightarrow \mathcal{N}$ defined by $d(x, y) = (0, \delta(y))$ for all $x, y \in \mathcal{N}$ is a nonzero derivation of $\mathcal{N}$, where $\mathcal{N}$ is a zero-symmetric left near- ring.

For an example of a derivation on noncommutative near-ring one can consider the following:

Example 2.3. Let us consider $(\mathcal{C}, +, *)$ where $'*'$ is defined as $x * y = |x|y$ for all $x, y \in \mathcal{C}$, then it can be easily seen that $(\mathcal{C}, +, *)$ is a zero-symmetric left near-ring which is not a right near -ring. Assume $\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b \in \mathcal{C} \right\}$, then $\mathcal{N}$ is a zero-symmetric left near-ring which is not a right near-ring. Define $d : \mathcal{N} \longrightarrow \mathcal{N}$ as $d\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. Then d is a non-zero derivation on $\mathcal{N}$.

In a left near-ring, right distributive property does not hold in general, the following lemmas plays a vital role in further study. For any $a, b, c \in \mathcal{N}$ expanding $d(a(bc))$ and $d((ab)c)$ and comparing the relations so obtained we get the following (for reference see ([24], Lemma 1)).

Lemma 2.4. *Let d be an arbitrary derivation on a near-ring $\mathcal{N}$. Then $\mathcal{N}$ satisfies the following partial distributive law:*

$$(ad(b) + d(a)b)c = ad(b)c + d(a)bc \textit{ for all } a, b, c \in \mathcal{N}.$$

The study of derivation was initiated by H. E. Bell and G. Mason [24], pertaining to the 3-prime near-rings and semiprime near-rings. Some basic properties of 3-prime near-rings are given below which are helpful in the study of derivations in 3-prime near-rings:

- If $z \in Z \backslash \{0\}$, then z is not a zero divisor.
- If Z contains a nonzero element z for which $z + z \in Z$, then $(\mathcal{N}, +)$ is abelian.
- Let d be a nonzero derivation on $\mathcal{N}$. Then $xd(\mathcal{N}) = \{0\}$ implies $x = 0$ and $d(\mathcal{N})x = \{0\}$ implies $x = 0$.
- If $\mathcal{N}$ is 2-torsion free and d is a derivation on $\mathcal{N}$ such that $d^2 = 0$, then $d = 0$.

In the year 1984 X.K.Wang ([41], Proposition 1) gave an equivalent definition of derivation on a near-ring $\mathcal{N}$ as below and also obtained partial commutativity of addition and partial distributive law in the near-ring $\mathcal{N}$.

Definition 2.5. Let d be an arbitrary additive endomorphism of $\mathcal{N}$. Then d is a derivation on $\mathcal{N}$ if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$.

Lemma 2.6. *Let d be a derivation on $\mathcal{N}$. Then $\mathcal{N}$ satisfies the following partial distributive law:*

$$(d(x)y + xd(y))z = d(x)yz + xd(y)z \ \textit{ for all } x, y, z \in \mathcal{N}.$$

Lemma 2.7. *Let $\mathcal{N}$ be a near-ring with center Z, and let d be a derivation on $\mathcal{N}$. Then $d(Z) \subseteq Z$.*

Major study in this area was carried out by Bell and Mason [24], Beidar, Fong and Wang [16] etc. which consists of commutativity of addition and multiplication of 3-prime near-ring and semiprime near-ring with constrained derivations. It has been also studied that under suitable constrained derivations, 3-prime near-rings behave like rings.

Now we list several commutativity theorems, obtained by above authors for 3-prime near-rings, admitting suitable constrained derivations as below.

Results given below have been proved by Bell and Mason [24].

Theorem 2.8. *If a* 3*-prime near-ring* $\mathcal{N}$*, admits a non trivial derivation satisfying either of the following properties*

(i) $d(\mathcal{N}) \subseteq Z$,

(ii) $[d(x), d(y)] = 0$ *for all* $x, y \in \mathcal{N}$,

then $(\mathcal{N}, +)$ *is abelian and if* $\mathcal{N}$ *is* 2*-torsion free as well, then* $\mathcal{N}$ *is a commutative ring.*

Following results concerning commutativity of near-ring have been proved by Beidar, Fong and Wang [16]

Theorem 2.9. *Let* $\mathcal{N}$ *be* 3*-prime near-ring which admits derivations* d_1 *and* d_2*. Suppose* $\mathcal{N}$ *satisfies any one of the following properties:*

(i) $d_1^2 \neq 0 \neq d_2^2$ *and* $d_1(x)d_2(y) = d_2(y)d_1(x)$ *for all* $x, y \in \mathcal{N}$,

(ii) $2\mathcal{N} \neq 0, d_1 \neq 0, d_2 \neq 0$ *and* $d_1(x)d_2(y) = d_2(y)d_1(x)$ *for all* $x, y \in \mathcal{N}$.

Then $\mathcal{N}$ *is a commutative ring.*

Theorem 2.10. *Let* $\mathcal{N}$ *be* 3*-prime near-ring with nonzero derivations* d_1 *and* d_2 *such that* $d_1(x)d_2(y) = -d_2(x)d_1(y)$ *for all* $x, y \in \mathcal{N}$*. Then* $(\mathcal{N}, +)$ *is abelian.*

Very recently Boua and Oukhtite [25] investigated some differential identities which force a 3-prime near-ring to be a commutative ring and also gave the suitable examples, proving the necessity of the 3-primeness condition.

Theorem 2.11. *([25], Theorem 2.2-2.3). Let* $\mathcal{N}$ *be a* 3*-prime near-ring. Suppose that* $\mathcal{N}$ *admits a nonzero derivation* d *satisfying the following property, i.e;* $d([x, y]) = \pm[x, y]$ *for all* $x, y \in \mathcal{N}$*. Then* $\mathcal{N}$ *is a commutative ring.*

Remark 2.12. The following example shows that the 3-primeness in the hypothesis of the above theorem is essential even in the case of arbitrary rings.

Example 2.13. Let R be a commutative ring, which is not a zero ring and consider $\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} |0, x, y \in R \right\}$. If we define $d : \mathcal{N} \longrightarrow \mathcal{N}$ by $d\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$, then it is straightforward to check that d is a nonzero derivation of $\mathcal{N}$. On the other hand, if $a = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}$, where $0 \neq r$, then $a\mathcal{N}a = \{0\}$ which proves that N is not 3-prime. Moreover, d satisfies the condition $d[A, B]) = [A, B]$ for all $A, B \in N$ but $\mathcal{N}$ is not a commutative ring.

H. E. Bell, A. Boua, L. Oukhtite [22] studied the commutativity of 3-prime near-rings with derivations, satisfying certain differential identities on 3-prime near-rings.

Theorem 2.14. *([22],Theorem 2.2-2.3). Let* $\mathcal{N}$ *be a* 2*-torsion free* 3*-prime near-ring. If* $\mathcal{N}$ *admits a nonzero derivation* d *satisfying any one of the following properties:*

(i) $[d(x), y] = [x, d(y)]$ *for all* $x, y \in \mathcal{N}$,

(ii) $[d(x), y] = \pm[d(x), d(y)]$ *for all* $x, y \in \mathcal{N}$,

(iii) $[x, d(y)] = [d(x), d(y)]$ *for all* $x, y \in \mathcal{N}$,

(iv) $[x, d(y)] = -[d(x), y]$ *for all* $x, y \in \mathcal{N}$,

then $\mathcal{N}$ *is a commutative ring.*

3 Generalized Derivations in Near-Rings

Matej Bresar [27] introduced the concept of generalized derivation in associative rings. This concept covers the concept of derivation already known to us for ring theory. Later a lot of study was done by Hvala, Golbasi, T. K. Lee etc. about generalized derivations in the setting of prime rings and semiprime rings and several known results for derivation in prime and semiprime rings were extended in the setting of generalized derivations in rings by above authors.

Motivated by the above concept, Golbasi[28] introduced the concept of generalized derivations in near-rings as given below and studied this in the setting of 3-prime and semi prime near-rings. Later in 2008, H. E. Bell[19] also studied this notion and derived some commutativity theorems of 3-prime near-rings equipped with generalized derivation. The above authors also generalized the several known results of derivations in 3-prime and semiprime near-rings.

Definition 3.1. Let $\mathcal{N}$ be a near-ring. An additive mapping $f : \mathcal{N} \longrightarrow \mathcal{N}$ is called

(i) a *right generalized derivation* of $\mathcal{N}$ if there exists a derivation d of $\mathcal{N}$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in \mathcal{N}$.

(ii) a *left generalized derivation* of $\mathcal{N}$ if there exists a derivation d of $\mathcal{N}$ such that $f(xy) = d(x)y + xf(y)$ for all $x, y \in \mathcal{N}$.

(iii) a *generalized derivation* of $\mathcal{N}$ if there exists a derivation d of $\mathcal{N}$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in \mathcal{N}$ and $f(xy) = d(x)y + xf(y)$ hold for all $x, y \in \mathcal{N}$.

Example 3.2. Let S be any zero-symmetric left near-ring. Consider $\mathcal{N}_1 = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} | 0, a, b \in S \right\}$. Then $\mathcal{N}_1$ is a zero-symmetric left near-ring with regard to the matrix addition and multiplication. Define $d, f : \mathcal{N}_1 \longrightarrow \mathcal{N}_1$ as following $d\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ and $f\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$. It can be easily seen that f is a *right generalized derivation* of $\mathcal{N}_1$ with associated derivation d of $\mathcal{N}_1$ but it is not a left generalized derivation of $\mathcal{N}_1$ with associated derivation d of $\mathcal{N}_1$.

Example 3.3. Consider $\mathcal{N}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | 0, a, b \in S \right\}$. Then $\mathcal{N}_2$ is a zero-symmetric left near-ring with regard to the matrix addition and multiplication.. Define $d, f : \mathcal{N}_2 \longrightarrow \mathcal{N}_2$ as following $d\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ and $f\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. It can be noted that f is a *left generalized derivation* of $\mathcal{N}_2$ with associated derivation d of $\mathcal{N}_2$ but f is not a right generalized derivation of $\mathcal{N}_2$ with associated derivation d of $\mathcal{N}_2$.

Example 3.4. Consider $\mathcal{N}_3 = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} | x, y, z \in S \right\}$. Here $\mathcal{N}_3$ is a zero-symmetric left near-ring with regard to the matrix addition and multiplication. Define $d, f : \mathcal{N}_3 \longrightarrow \mathcal{N}_3$ as below,

$d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $f\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then it can be shown that f is a *generalized derivation* of $\mathcal{N}_3$ with associated derivation d of $\mathcal{N}_3$.

Since in a left near-ring addition need not be commutative in general and right distributive property does not hold. Gölbasi [28] proved some partial commutative law of addition and partial right distributive law in the setting of left near-rings equipped with generalized derivation which are given below ([28], Lemma 2.2-2.3):

- If f is a right generalized derivation of $\mathcal{N}$ with associated derivation d, then $f(xy) = xd(y) + f(x)y$ for all $x, y \in \mathcal{N}$.
- If f is a left generalized derivation of $\mathcal{N}$ with associated derivation d, then $f(xy) = xf(y) + d(x)y$ for all $x, y \in \mathcal{N}$.
- If f is a right generalized derivation of $\mathcal{N}$ with associated derivation d, then $(f(x)y + xd(y))z = f(x)yz + xd(y)z$ for all $x, y, z \in \mathcal{N}$.
- If f is a generalized derivation of $\mathcal{N}$ with associated derivation d, then $(d(x)y + xf(y))z = d(x)yz + xf(y)z$ for all $x, y, z \in \mathcal{N}$.

4 On n- derivations in near-rings

Recently K. H. Park [36] introduced the notion of an n-derivation and symmetric n-derivation, where n is any positive integer in rings and extended several known results, earlier in the setting of derivations in prime rings and semiprime rings. Motivated by the above notion in rings the authors [5] introduced the notion of n-derivations in the setting of near-rings and generalized several known results obtained earlier in the setting of 3-prime near-rings and semiprime near-rings.

Definition 4.1. A map $D : \underbrace{\mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N}}_{n-\text{times}} \longrightarrow \mathcal{N}$ is said to be *permuting* if the equation $D(x_1, x_2, \cdots, x_n) = D(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)})$ holds for all $x_1, x_2, \cdots, x_n \in \mathcal{N}$ and for every permutation $\pi \in S_n$, where S_n is the permutation group on $\{1, 2, \cdots, n\}$. A map $d : \mathcal{N} \to \mathcal{N}$ defined by $d(x) = D(x, x, \cdots, x)$ for all $x \in \mathcal{N}$ where $D : \underbrace{\mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N}}_{n-\text{times}} \to \mathcal{N}$ is a permuting map, is called the trace of D.

Definition 4.2. Let n be any fixed positive integer. An n-additive (i.e.; additive in each argument) mapping $D : \mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N} \longrightarrow \mathcal{N}$ is called an *n-derivation* on $\mathcal{N}$ if the relations $D(x_1, x_2, \cdots, x_{i-1}, x_i x_i^{'}, x_{i+1}, \cdots, x_n) = D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)x_i^{'} + x_i D(x_1, x_2, \cdots, x_{i-1}, x_i^{'}, x_{i+1}, \cdots, x_n)$ hold for all $x_1, x_2, \cdots, x_{i-1}, x_i, x_i^{'}, x_{i+1}, \cdots, x_n \in \mathcal{N}$, $i = 1, 2, 3, \cdots, n$. If in addition, D is a permuting map then all the above conditions are equivalent and in this case D is called a *permuting n-derivation of $\mathcal{N}$* i.e.; a permuting n-derivation of $\mathcal{N}$ can also be defined as below.
An n-additive permuting mapping $D : \mathcal{N}\times\mathcal{N}\times\cdots\times\mathcal{N} \longrightarrow \mathcal{N}$ is called a *permuting n-derivation* of $\mathcal{N}$ if $D(x_1, x_2, \cdots, x_i x_i{}', \cdots, x_n) = D(x_1, x_2, \cdots, x_i, \cdots, x_n)x_i{}' + x_i D(x_1, x_2, \cdots, x_i{}', \cdots, x_n)$ holds for all $x_1, x_2, \cdots, x_i, x_i{}', \cdots, x_n \in \mathcal{N}$.

Example 4.3. Suppose that $\mathcal{N}$ is a commutative near-ring. Then

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b, 0 \in \mathcal{N} \right\}$$

is a non-commutative near-ring with regard to matrix addition and matrix multiplication. Define $D : \underbrace{R \times R \times \cdots \times R}_{n-\text{times}} \longrightarrow R$ such that

$$D\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} a_n & b_n \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & a_1 a_2 \cdots a_n \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that D is a permuting n-derivation of near-ring R.

Remark 4.4. By definition of n-derivation it is clear that a permuting n-derivation of $\mathcal{N}$ is also an n-derivation but the converse need not be true in general. For justification focus on the following example.

Example 4.5. Let R be a noncommutative ring and $\mathcal{N}_1$ a zero-symmetric left near-ring. Consider $S = R \times \mathcal{N}_1$. Then it is clear that S is a zero symmetric left near-ring with regard to matrix addition and multiplication. Now suppose that

$$\mathcal{N} = \left\{ \begin{pmatrix} (a,b) & (a^{'},b^{'}) \\ (0,0) & (0,0) \end{pmatrix} \mid (a,b),(a^{'},b^{'}),(0,0) \in S \right\}$$

.
It can be easily checked that $\mathcal{N}$ is a non-commutative zero-symmetric left near-ring with respect to matrix addition and matrix multiplication. Define $D : \mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N} \longrightarrow \mathcal{N}$ such that

$$D\left(\begin{pmatrix} (a_1,b_1) & (a_1^{'},b_1^{'}) \\ (0,0) & (0,0) \end{pmatrix}, \begin{pmatrix} (a_2,b_2) & (a_2^{'},b_2^{'}) \\ (0,0) & (0,0) \end{pmatrix}, \dots, \begin{pmatrix} (a_n,b_n) & (a_n^{'},b_n^{'}) \\ (0,0) & (0,0) \end{pmatrix} \right)$$
$$= \begin{pmatrix} (0,0) & (a_1a_2\cdots a_n,0) \\ (0,0) & (0,0) \end{pmatrix}.$$

It can be seen that D is an n-derivation of $\mathcal{N}$, however it is not a permuting n-derivation of $\mathcal{N}$.

Remark 4.6. In the above example, if we take R as a commutative ring, then D becomes a permuting n-derivation of $\mathcal{N}$ also. Since $\mathcal{N}$ is not additively commutative, it is always difficult to find the near-ring analogue of ring theoretic results. The following lemma felicitate our study for n-derivations in near-rings.

Lemma 4.7. *Let $\mathcal{N}$ be a near-ring. Then D is a permuting n-derivation of $\mathcal{N}$ if and only if $D(x_1x_1^{'}, x_2, \cdots, x_n) = x_1D(x_1^{'}, x_2, \cdots, x_n) + D(x_1, x_2, \cdots, x_n)x_1^{'}$ for all $x_1, x_1{'}, x_2, \cdots,$ $x_n \in \mathcal{N}$.*

In a left near-ring $\mathcal{N}$, right distributive law does not hold in general, however, the following partial distributive properties in $\mathcal{N}$ have been obtained in ([5], Lemma 2.4-2.6).

Theorem 4.8. *Let $\mathcal{N}$ be a near-ring. Let D be a permuting n-derivation of $\mathcal{N}$ and d be the trace of D. Then*

(i) $\{D(x_1, x_2, \cdots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \cdots, x_n)\}y =$
$D(x_1, x_2 \cdots, x_n)x_1^{'}y + x_1D(x_1^{'}, x_2, \cdots, x_n)y,$ *for every* $x_1, x_1^{'}, \cdots, x_n, y \in \mathcal{N}$.

(ii) $\{x_1D(x_1^{'}, x_2, \cdots, x_n) + D(x_1, x_2, \cdots, x_n)x_1{'}\}y =$
$x_1D(x_1^{'}, x_2, \cdots, x_n)y + D(x_1, x_2, \cdots, x_n)x_1{'}y,$ *for every* $x_1, x_1^{'}, \cdots, x_n, y \in \mathcal{N}$.

(iii) $\{d(x)x_1 + xD(x, x, \cdots, x, x_1)\}y =$
$d(x)x_1y + xD(x, x, \cdots, x, x_1)y,$ *for every* $x, x_1, x_1^{'}, \cdots, x_n, y \in \mathcal{N}$.

(iv) $\{xD(x, x, \cdots, x, x_1) + d(x)x_1\}y =$
$xD(x, x, \cdots, x, x_1)y + d(x)x_1y$ *for every* $x, x_1, x_1^{'}, \cdots, x_n, y \in \mathcal{N}$.

(v) *if $\mathcal{N}$ is 3-prime, $D \neq 0$, and $D(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N})x = \{0\}$ where $x \in \mathcal{N}$ then $x = 0$.*

(vi) *if $\mathcal{N}$ is 3-prime, $D \neq 0$, and $xD(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N}) = \{0\}$ where $x \in \mathcal{N}$, then $x = 0$.*

(vii) *if $\mathcal{N}$ is 3-prime, $D \neq 0$, then $D(C, C, \cdots, C) \neq \{0\}$, where $C \neq \{0\}$.*

Recently Öztürk and Jun ([35], Lemma 3.1) proved that in a 2-torsion free 3-prime near-ring which admits a symmetric bi-additive mapping D if the trace d of D is zero, then $D = 0$. Further, this result was generalized by K.H. Park and Y.S. Jun ([37], Lemma 2.2) for permuting tri-additive mapping in 3!-torsion free 3-prime near-ring. We have extended this result, as below, for permuting n-additive mapping in a $n!$-torsion free 3-prime near-ring under some constraints.

Theorem 4.9. *([5], Theorem 3.1). Let $\mathcal{N}$ be $n!$-torsion free 3-prime near-ring and D be a permuting n-additive mapping of $\mathcal{N}$ such that $D(\mathcal{N},\mathcal{N},\cdots,\mathcal{N}) \subseteq Z$. If $d(x) = 0$, for all $x \in \mathcal{N}$, then $D = 0$.*

In 1987 H.E.Bell([17], Theorem 2) proved that if a 2-torsion free zero-symmetric 3-prime near-ring $\mathcal{N}$ admits a non zero derivation D for which $D(\mathcal{N}) \subseteq Z$, then $\mathcal{N}$ is a commutative ring. Further, this result was generalized by K.H.Park and Y.S. Jun ([37], Theorem 3.1) in the year 2010 for permuting tri-derivation, who showed that if 3!-torsion free zero symmetric 3-prime near-ring $\mathcal{N}$ admits a nonzero permuting tri-derivation D for which $D(\mathcal{N},\mathcal{N},\mathcal{N}) \subseteq Z$, then $\mathcal{N}$ is a commutative ring. The following result shows that 2-torsion free and 3!-torsion free restrictions in the above results used by Bell and Park are superfluous. In fact, for permuting n-derivation in a 3-prime near-ring $\mathcal{N}$ we have obtained the following (see ([5], Theorem 3.2-3.4)):

Theorem 4.10. *Let D be a nonzero permuting n-derivation of 3-prime near-ring $\mathcal{N}$ such that $D(\mathcal{N},\mathcal{N},\cdots,\mathcal{N}) \subseteq Z$. Then $\mathcal{N}$ is a commutative ring.*

Theorem 4.11. *Let $\mathcal{N}$ be a 3-prime near-ring and D_1 and D_2 be any two nonzero permuting n-derivations of $\mathcal{N}$. If $[D_1(\mathcal{N},\mathcal{N},\cdots,\mathcal{N}), D_2(\mathcal{N},\mathcal{N},\cdots,\mathcal{N})] = \{0\}$, then $(\mathcal{N},+)$ is an abelian group.*

Theorem 4.12. *Let $\mathcal{N}$ be a 3-prime near-ring with nonzero permuting n-derivations D_1 and D_2 such that*

$$D_1(x_1,x_2,\cdots,x_n)D_2(y_1,y_2,\cdots,y_n) = -D_2(x_1,x_2,\cdots,x_n)D_1(y_1,y_2,\cdots,y_n)$$

for all $x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_n \in \mathcal{N}$. Then $(\mathcal{N},+)$ is an abelian group.

Corollary 4.13. *([16], Lemma 2.1). Let $\mathcal{N}$ be a 3-prime near-ring with nonzero derivations d_1 and d_2 such that $d_1(x)d_2(y) = -d_2(x)d_1(y)$ for all $x,y \in \mathcal{N}$. Then $(\mathcal{N},+)$ is an abelian group.*

Theorem 4.14. *([5], Theorem 3.7). Let $\mathcal{N}$ be a 3-prime near-ring and D be any nonzero permuting n-derivation of $\mathcal{N}$. If $K = \{a \in \mathcal{N} \mid [D(\mathcal{N},\mathcal{N},\cdots,\mathcal{N}), a] = \{0\}\}$,then*

(i) $a \in K$ implies either $a \in Z$ or $d(a) = 0$,

(ii) $d(K) \subseteq Z$,

(iii) K is a semigroup under multiplication,

(iv) If there exists an element $a \in K$ for which $d(a) \neq 0$ and $D(a^2,a,\cdots,a) \in Z$, then $(\mathcal{N},+)$ is an abelian group.

Theorem 4.15. *([5], Theorem 3.8). Let $\mathcal{N}$ be a 3-prime near-ring which admits a nonzero permuting n-derivation D such that $D(C,C,\cdots,C) \subseteq Z$. Then N is a commutative ring, where $C \neq \{0\}$.*

5 On Generalized n-Derivations in Near-rings

Motivated by the concept of generalized derivation in rings and near-rings the authors [10] generalized the concept of n-derivation of near-rings by introducing the notion of generalized n-derivations in near-rings.

Definition 5.1. Let n be a fixed positive integer. An n-additive mapping $F : \mathcal{N}\times\mathcal{N}\times\cdots\times\mathcal{N} \longrightarrow \mathcal{N}$ is called a *right generalized n-derivation* of $\mathcal{N}$ with associated n-derivation D if the relations
$F(x_1,x_2,\cdots,x_{i-1},x_ix_i^{'},x_{i+1},\cdots,x_n)$
$= F(x_1,x_2,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_n)x_i^{'} + x_iD(x_1,x_2,\cdots,x_{i-1},x_i^{'},x_{i+1},\cdots,x_n)$
hold for all $x_1,x_2,\cdots,x_{i-1},x_i,x_i^{'},x_{i+1},\cdots,x_n \in N$, $i = 1,2,3,\cdots,n$. If in addition, both F and D are permuting maps then all the above conditions are equivalent and in this case F is called a permuting right generalized n-derivation of N with associated permuting n-derivation D. An n-additive mapping $F : N\times\mathcal{N}\times\cdots\times\mathcal{N} \longrightarrow \mathcal{N}$ is called a *left generalized n-derivation* of N with associated n-derivation D if the relations

$F(x_1, x_2, \cdots, x_{i-1}, x_i x_i^{'}, x_{i+1}, \cdots, x_n)$
$\quad = D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)x_i^{'} + x_i F(x_1, x_2, \cdots, x_{i-1}, x_i^{'}, x_{i+1}, \cdots, x_n)$
hold for all $x_1, x_2, \cdots, x_{i-1}, x_i, x_i^{'}, x_{i+1}, \cdots, x_n \in \mathcal{N}$, $i = 1, 2, 3, \cdots, n$. If in addition, both F and D are permuting maps then all the above conditions are equivalent and in this case F is called a permuting left generalized n-derivation of $\mathcal{N}$ with associated permuting n-derivation D. An n-additive mapping $F : \mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N} \longrightarrow \mathcal{N}$ is called a *generalized n-derivation* of $\mathcal{N}$ with associated n-derivation D if it is both a right generalized n-derivation as well as a left generalized n-derivation of $\mathcal{N}$ with associated n-derivation D. If in addition, both F and D are permuting maps then F is called a permuting generalized n-derivation of N with associated permuting n-derivation D (see [10] for further reference). If $\mathcal{N}$ is a commutative ring, then it is trivial to see that the set of all left generalized n-derivations of $\mathcal{N}$ is equal to the set of all right generalized n-derivations of $\mathcal{N}$.

Example 5.2. (i) Let S be a commutative near-ring. For an example of left generalized n-derivation, consider
$\mathcal{N}_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b, 0 \in S \right\}$ is a noncommutative zero-symmetric left near-ring with regard to matrix addition and matrix multiplication.
Define $D_1 : \underbrace{\mathcal{N}_1 \times \mathcal{N}_1 \times \cdots \times \mathcal{N}_1}_{n-\text{times}} \longrightarrow \mathcal{N}_1$ such that

$$D_1\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} a_n & b_n \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & a_1 a_2 \cdots a_n \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that D_1 is an n-derivation of $\mathcal{N}_1$. Define $F_1 : \mathcal{N}_1 \times \mathcal{N}_1 \times \cdots \times \mathcal{N}_1 \longrightarrow \mathcal{N}_1$ such that

$$F_1\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} a_n & b_n \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & b_1 b_2 \cdots b_n \\ 0 & 0 \end{pmatrix}.$$

It can be easily verified that F_1 is a left generalized n-derivation of $\mathcal{N}_1$ with associated n-derivation D_1 but not a right generalized n-derivation of $\mathcal{N}_1$ with associated n-derivation D_1.
(ii) For an example of right generalized n-derivation, consider
$\mathcal{N}_2 = \left\{ \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} \mid c, d, 0 \in S \right\}$. It can be easily shown that $\mathcal{N}_2$ is a non-commutative zero symmetric left near-ring with regard to matrix addition and matrix multiplication. Define $D_2 : \underbrace{\mathcal{N}_2 \times \mathcal{N}_2 \times \cdots \times \mathcal{N}_2}_{n-\text{times}} \longrightarrow \mathcal{N}_2$ such that

$$D_2\left(\begin{pmatrix} 0 & c_1 \\ 0 & d_1 \end{pmatrix}, \begin{pmatrix} 0 & c_2 \\ 0 & d_2 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & c_n \\ 0 & d_n \end{pmatrix}\right) = \begin{pmatrix} 0 & c_1 c_2 \cdots c_n \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that D_2 is an n-derivation of $\mathcal{N}_2$. Define $F_2 : \mathcal{N}_2 \times \mathcal{N}_2 \times \cdots \times \mathcal{N}_2 \longrightarrow \mathcal{N}_2$ such that

$$F_2\left(\begin{pmatrix} 0 & c_1 \\ 0 & d_1 \end{pmatrix}, \begin{pmatrix} 0 & c_2 \\ 0 & d_2 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & c_n \\ 0 & d_n \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & d_1 d_2 \cdots d_n \end{pmatrix}.$$

It can be easily verified that F_2 is a right generalized n-derivation of $\mathcal{N}_2$ with associated n-derivation D_2 but not a left generalized n-derivation of $\mathcal{N}_2$ with associated n-derivation D_2.
(iii) For an example of generalized n-derivation, consider
$\mathcal{N}_3 = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z, 0 \in S \right\}$. It can be easily seen that $\mathcal{N}_3$ is a non-commutative zero symmetric left near-ring with regard to matrix addition and matrix multiplication. Define $D_3 : \underbrace{\mathcal{N}_3 \times \mathcal{N}_3 \times \cdots \times \mathcal{N}_3}_{n-\text{times}} \longrightarrow \mathcal{N}_3$ such that

$$D_3\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & z_n \end{pmatrix}\right) = \begin{pmatrix} 0 & x_1 x_2 \cdots x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that D_3 is an n-derivation of $\mathcal{N}_3$. Define $F_3 : \mathcal{N}_3 \times \mathcal{N}_3 \times \cdots \times \mathcal{N}_3 \longrightarrow \mathcal{N}_3$ such that

$$F_3\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & z_n \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be easily verified that F_3 is a generalized n-derivation (i.e.; both left generalized n-derivation and right generalized n-derivation) of $\mathcal{N}_3$ with associated n-derivation D_3.

In a left near-ring additive group $(\mathcal{N}, +)$ need not be abelian and right distributive property does not hold in general. The authors ([10], Lemma 2.6-2.11)proved the following results for generalized n-derivations on near-rings which allows limited additive abelian property as well as limited distributive properties:

- F is a right generalized n-derivation of $\mathcal{N}$ with associated n-derivation D if and only if $F(x_1, x_2, \cdots, x_i x_i^{'}, \cdots, x_n) = x_i D(x_1, x_2, \cdots, x_i^{'}, \cdots, x_n) + F(x_1, x_2, \cdots, x_i, \cdots, x_n) x_i^{'}$ hold for all $x_1, , x_2, \cdots, x_i, x_i^{'}, \cdots x_n \in \mathcal{N}$, $i = 1, 2, \cdots, n$.
- If $\mathcal{N}$ admits a right generalized n-derivation F with associated n-derivation D of $\mathcal{N}$, then $\{F(x_1, x_2, \cdots, x_i, \cdots, x_n) x_i^{'} + x_i D(x_1, x_2, \cdots, x_i^{'}, \cdots, x_n)\} y = F(x_1, x_2, \cdots, x_i, \cdots, x_n) x_i^{'} y + x_i D(x_1, x_2, \cdots, x_i^{'}, \cdots, x_n) y$, hold for all $x_1, , x_2, \cdots, x_i, x_i^{'}, \cdots, x_n, y \in \mathcal{N}, i = 1, 2, \cdots, n$.
- If $\mathcal{N}$ admits a right generalized n-derivation F with associated n-derivation D of $\mathcal{N}$, then $\{x_i D(x_1, x_2, \cdots, x_i^{'}, \cdots, x_n) + F(x_1, x_2, \cdots, x_i, \cdots, x_n) x_i^{'}\} y = x_i D(x_1, x_2, \cdots, x_i^{'}, \cdots, x_n) y + F(x_1, x_2, \cdots, x_i, \cdots, x_n) x_i^{'} y$ hold for all $x_1, , x_2, \cdots, x_i, x_i^{'}, \cdots, x_n, y \in \mathcal{N}, i = 1, 2, \cdots, n$.
- F is a left generalized n-derivation of $\mathcal{N}$ with associated n-derivation D if and only if $F(x_1, x_2, \cdots, x_i x_i^{'}, \cdots, x_n) = x_i F(x_1, x_2, \cdots, x_i^{'}, \cdots, x_n) + D(x_1, x_2, \cdots, x_i, \cdots, x_n) x_i^{'}$, hold for all $x_1, , x_2, \cdots, x_i, x_i^{'}, \cdots x_n \in \mathcal{N}$, $i = 1, 2, \cdots, n$.
- If $\mathcal{N}$ admits a generalized n-derivation F with associated n-derivation D of $\mathcal{N}$, then $\{D(x_1, x_2, \cdots, x_i, \cdots, x_n) x_i^{'} + x_i F(x_1, x_2, \cdots, x_i^{'}, \cdots, x_n)\} y = D(x_1, x_2, \cdots, x_i, \cdots, x_n) x_i^{'} y + x_i F(x_1, x_2, \cdots, x_i^{'}, \cdots, x_n) y$, hold for all $x_1, , x_2, \cdots, x_i, x_i^{'}, \cdots, x_n, y \in \mathcal{N}, i = 1, 2, \cdots, n$.
- If $\mathcal{N}$ admits a generalized n-derivation F with associated n-derivation D of $\mathcal{N}$, then $\{x_i F(x_1, x_2, \cdots, x_i^{'}, \cdots, x_n) + D(x_1, x_2, \cdots, x_i, \cdots, x_n) x_i^{'}\} y = x_i F(x_1, x_2, \cdots, x_i^{'}, \cdots, x_n) y + D(x_1, x_2, \cdots, x_i, \cdots, x_n) x_i^{'} y$ hold for all $x_1, , x_2, \cdots, x_i, x_i^{'}, \cdots, x_n, y \in \mathcal{N}, i = 1, 2, \cdots, n$.

Recently Öznur Gölbasi([28], Theorem 2.6) proved that if $\mathcal{N}$ is a 3-prime near-ring with a nonzero generalized derivation f such that $f(\mathcal{N}) \subseteq Z$, then $(\mathcal{N}, +)$ is an abelian group. Moreover if $\mathcal{N}$ is 2-torsion free, then $\mathcal{N}$ is a commutative ring. The following result shows that "2-torsion free restriction" in the above result used by Öznur Gölbasi is superfluous. In fact, for generalized n-derivation in a prime near-ring $\mathcal{N}$, we have obtained the following.

Theorem 5.3. *([10], Theorem 3.1). Let $\mathcal{N}$ be a 3-prime near-ring admitting a nonzero generalized n-derivation F with associated n-derivation D of $\mathcal{N}$. If $F(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N}) \subseteq Z$, then $\mathcal{N}$ is a commutative ring.*

Corollary 5.4. *([5], Theorem 3.2). Let $\mathcal{N}$ be a 3-prime near-ring admitting a nonzero permuting n-derivation D such that $D(\mathcal{N}, \mathcal{N}, ..., \mathcal{N}) \subseteq Z$, then $\mathcal{N}$ is a commutative ring.*

Very recently Öznur Gölbasi ([30], Theorem 3.1) proved that if $\mathcal{N}$ is a semiprime near-ring and f is a nonzero generalized derivation on $\mathcal{N}$ with an associated derivation d such that $f(x)y = xf(y)$ for all $x, y \in \mathcal{N}$, then $d = 0$. While proving the theorem it has been assumed that f is a right generalized derivation of $\mathcal{N}$ with associated derivation d. We have extended this result in the setting of generalized n-derivation. In fact we proved the following :

Theorem 5.5. *([10], Theorem 3.10). Let $\mathcal{N}$ be a semiprime near-ring admitting a generalized n-derivation F with associated n-derivation D of $\mathcal{N}$. If $F(x_1, x_2, \cdots, x_n)y_1 = x_1F(y_1, y_2, \cdots, y_n)$ for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathcal{N}$, then $D = 0$.*

Corollary 5.6. *([5], Theorem 3.6). Let $\mathcal{N}$ be a semiprime near-ring and D an n-derivation of $\mathcal{N}$. If $D(x_1, x_2, \cdots, x_n)y_1 = x_1D(y_1, y_2, \cdots, y_n)$, for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathcal{N}$, then $D = 0$.*

Theorem 5.7. *([10], Theorem 3.13). Let $\mathcal{N}$ be a 3-prime near-ring admitting a generalized n-derivation F with associated n-derivation D of $\mathcal{N}$ such that $D(Z, \mathcal{N}, \cdots, \mathcal{N}) \neq \{0\}$. If $G : \mathcal{N} \times \mathcal{N} \times \cdots \mathcal{N} \longrightarrow \mathcal{N}$ is a map such that $[F(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N}), G(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N})] = \{0\}$, then $G(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N}) \subseteq Z$.*

Theorem 5.8. *([10], Theorem 3.14). Let $\mathcal{N}$ be a 3-prime near-ring admitting a generalized n-derivation F with associated n-derivation D of $\mathcal{N}$ such that $D(Z, \mathcal{N}, \cdots, \mathcal{N}) \neq \{0\}$. If G is a nonzero generalized n-derivation of $\mathcal{N}$ such that $[F(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N}), G(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N})] = \{0\}$, then $\mathcal{N}$ is a commutative ring.*

Theorem 5.9. *([10], Theorem 3.16). Let F_1 and F_2 be generalized n-derivations of 3-prime near-ring $\mathcal{N}$ with associated nonzero n-derivations D_1 and D_2 of $\mathcal{N}$ respectively such that $[F_1(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N}), F_2(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N})] = \{0\}$. Then $(\mathcal{N}, +)$ is an abelian group.*

Corollary 5.10. *([5],Theorem 3.3). Let $\mathcal{N}$ be a 3-prime near-ring and D_1, D_2 be any two nonzero permuting n-derivations of $\mathcal{N}$. If $[D_1(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N}), D_2(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N})] = \{0\}$, then $(\mathcal{N}, +)$ is an abelian group.*

6 Semigroup ideals and generalized n-derivations in near-rings

A nonempty subset $\mathfrak{A}$ of $\mathcal{N}$ is called semigroup left ideal (resp. semigroup right ideal) if $\mathcal{N}\mathfrak{A} \subseteq \mathfrak{A}$ (resp. $\mathfrak{A}\mathcal{N} \subseteq \mathfrak{A}$) and if $\mathfrak{A}$ is both a semigroup left ideal and a semigroup right ideal, it will be called a semigroup ideal. Recently many authors have studied commutativity of addition and ring behavior of 3-prime near-rings satisfying certain properties and identities involving derivations and generalized derivations on semigroup ideals (see [2],[18],[32][33], where further references can be found). In the present section we study the commutativity of addition and ring behavior of 3-prime near-rings satisfying certain properties and identities involving generalized n-derivations on semigroup ideals. In fact, the results presented in this section generalize, extend, compliment and improve several results obtained earlier on derivations, generalized derivations, permuting n-derivations and generalized n-derivations for 3-prime near-rings; for example Theorem 1.2 of [2], Theorems 3.2−3.4&3.7 of [5], Theorems 3.1, 3.11, 3.15, 3.16 of [10], Theorems 3.2 − 3.3 of [18] etc.- to mention a few only. We begin with the following theorem obtained in ([12], Theorem 3.1).

Theorem 6.1. *Let $\mathcal{N}$ be a 3-prime near-ring and $\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n$ be nonzero semigroup ideals of $\mathcal{N}$. If it admits a nonzero generalized n-derivation F with associated n-derivation D of $\mathcal{N}$ such that $F(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) \subseteq Z$, then $\mathcal{N}$ is a commutative ring.*

Corollary 6.2. *([10], Theorem 3.1). Let $\mathcal{N}$ be a 3-prime near-ring admitting a nonzero generalized n-derivation F with associated n-derivation D of $\mathcal{N}$. If $F(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N}) \subseteq Z$, then $\mathcal{N}$ is a commutative ring.*

The following example demonstrates that $\mathcal{N}$ to be 3-prime is essential in the hypothesis of the above theorem.

Example 6.3. ([12], Example 3.1). Let $\mathbb{Z}$ be the usual ring of integers and $(\mathbb{C}, +, *)$ be the left near-ring of complex numbers. Here '$*$' is defined by $z_1 * z_2 = |z_1|.z_2$ for all $z_1, z_2 \in \mathbb{C}$, where '$+$' and '$.$' denote the usual addition and multiplication of complex numbers. Assume $\mathcal{N} = \mathbb{Z}\times\mathbb{C}$ and $\mathfrak{A}_1 = m_1\mathbb{Z}\times\{0\}, \mathfrak{A}_2 = m_2\mathbb{Z}\times\{0\}, \cdots, \mathfrak{A}_n = m_n\mathbb{Z}\times\{0\}$, where $m_1, m_2, \cdots, m_n$ are different positive integers. Then it can be easily verified that $\mathcal{N}$ is a zero-symmetric left near-ring with regard to componentwise addition and multiplication, having $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n$ its nonzero semigroup ideals. Define $F : \mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N} \longrightarrow \mathcal{N}$ such that $F((a_1, z_1), (a_2, z_2), \cdots, (a_n, z_n)) = (\lambda a_1 a_2 \cdots a_n, 0)$, where λ is any integer. It is easy to show that $\mathcal{N}$ is a semiprime near-ring but not a 3-prime near-ring and F is a nonzero generalized n-derivation of $\mathcal{N}$ with associated n-derivation $D = 0$, the zero map from $\mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N}$ to $\mathcal{N}$ such that $F(\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n) \subseteq Z$. However, $\mathcal{N}$ is not a commutative ring.

Theorem 6.4. *([12], Theorem 3.2). Let $\mathcal{N}$ be a 3-prime near-ring and $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n$ nonzero semigroup ideals of $\mathcal{N}$. If it admits generalized n-derivations F and G with associated nonzero n-derivations D and H of $\mathcal{N}$ respectively such that*

$$F(x_1, x_2, \cdots, x_n)H(y_1, y_2, \cdots, y_n) = -G(x_1, x_2, \cdots, x_n)D(y_1, y_2, \cdots, y_n)$$

for all $x_1, y_1 \in \mathfrak{A}_1; x_2, y_2 \in \mathfrak{A}_2; \cdots; x_n, y_n \in \mathfrak{A}_n$, then $(\mathcal{N}, +)$ is abelian.

Corollary 6.5. *([10], Theorem 3.15). Let F and G be generalized n-derivations of 3-prime near-ring $\mathcal{N}$ with associated nonzero n-derivations D and H of $\mathcal{N}$ respectively such that $F(x_1, x_2, \cdots, x_n)H(y_1, y_2, \cdots, y_n) = -G(x_1, x_2, \cdots, x_n)D(y_1, y_2, \cdots, y_n)$ for all $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \in \mathcal{N}$. Then $(\mathcal{N}, +)$ is an abelian group.*

Let X and Y be nonempty subsets of $\mathcal{N}$ and $a \in \mathcal{N}$. By the notations $[X, Y]$ and $[X, a]$ we mean the subsets of $\mathcal{N}$ defined by $[X, Y] = \{[x, y] \mid x \in X, y \in Y\}$ and $[X, a] = \{[x, a] \mid x \in X\}$ respectively.

Very recently A. Ali *et al.* ([2], Theorem 12) proved that if $\mathcal{N}$ is a 3-prime near-ring, admitting a nonzero generalized derivation f with associated nonzero derivation d such that $[f(\mathfrak{A}), f(\mathfrak{A})] = \{0\}$, where $\mathfrak{A}$ is a nonzero semigroup ideal of $\mathcal{N}$, then $(\mathcal{N}, +)$ is abelian. We have improved and extended this result for generalized n-derivation in the setting of 3-prime near-rings. In fact we obtained the following.

Theorem 6.6. *([12], Theorem 3.3). Let $\mathcal{N}$ be a 3-prime near-ring and $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n$ nonzero semigroup ideals of $\mathcal{N}$. If it admits generalized n-derivations F_1 and F_2 with associated nonzero n-derivations D_1 and D_2 of N respectively such that*

$$[F_1(\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n), F_2(\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n)] = \{0\},$$

then $(\mathcal{N}, +)$ is abelian.

Corollary 6.7. *([10], Theorem 3.16). Let F_1 and F_2 be generalized n-derivations of 3-prime near-ring $\mathcal{N}$ with associated nonzero n-derivations D_1 and D_2 of $\mathcal{N}$ respectively such that $[F_1(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N}), F_2(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N})] = \{0\}$. Then $(\mathcal{N}, +)$ is an abelian group.*

The following example shows that the restriction of 3-primeness imposed on the hypotheses of Theorems 6.2 & 6.3 is not superfluous.

Example 6.8. ([12], Example 3.2). Let Q be the usual ring of real quaternions and $(S_3, +)$ be the symmetric group of degree 3. Let $S = Q \times S_3$. Define multiplication '$*$' in S by $(q_1, p_1) * (q_2, p_2) = (q_1.q_2, 0)$ for all $(q_1, p_1), (q_2, p_2) \in S$, where '.' is the usual multiplication of the ring Q and 0 stands for the identity of the group $(S_3, +)$. Then it can be easily seen that $(S, +, *)$ is a distributive near-ring, where '$+$' stands for componentwise addition. Consider $\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z, 0 \in S \right\}$. It can be easily seen that $\mathcal{N}$ is a zero-symmetric left near-ring with regard to matrix addition and matrix multiplication but not a 3-prime near-ring. Define $D_1, D_2 : \underbrace{\mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N}}_{n-\text{times}} \longrightarrow \mathcal{N}$ respectively as

$$D_1\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & x_1 x_2 \cdots x_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$D_2\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & z_1z_2\cdots z_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that D_1 & D_2 are nonzero n-derivations of $\mathcal{N}$. If we take $F_1 = D_2$ & $F_2 = D_1$, then it can be easily verified that F_1 & F_2 are nonzero generalized n-derivations of $\mathcal{N}$ with associated nonzero n-derivations D_1 & D_2 of $\mathcal{N}$ respectively. Let $\mathfrak{A} = \left\{ \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid y, 0 \in S \right\}$. It is obvious to observe that $\mathfrak{A}$ is a nonzero semigroup ideal of $\mathcal{N}$. If we choose $\mathfrak{A}_1 = \mathfrak{A}_2 = \cdots = \mathfrak{A}_n = \mathfrak{A}$, then we have the following:
(i) $F_1(x_1, x_2, \cdots, x_n)D_2(y_1, y_2, \cdots, y_n) = -F_2(x_1, x_2, \cdots, x_n)D_1(y_1, y_2, \cdots, y_n)$
for all $x_1, y_1 \in \mathfrak{A}_1; x_2, y_2 \in \mathfrak{A}_2; \cdots; x_n, y_n \in \mathfrak{A}_n$ and
(ii) $[F_1(\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n), F_2(\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n)] = \{0\}$. However, $(\mathcal{N}, +)$ is not abelian.

Theorem 6.9. *([12], Theorem 3.4). Let $\mathcal{N}$ be a 3-prime near-ring admitting a generalized n-derivation F with associated n-derivation D of $\mathcal{N}$. If $K = \{a \in \mathcal{N} \mid [F(u_1, u_2, \cdots, u_n), a] = 0\}$ for all $u_1 \in \mathfrak{A}_1, u_2 \in \mathfrak{A}_2, \cdots, u_n \in \mathfrak{A}_n$, where $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n$ are nonzero semigroup ideals of $\mathcal{N}$ and d stands for the trace of D, then $a \in K$ implies either $a \in Z$ or $d(a) = 0$.*

Corollary 6.10. *([10], Theorem 3.11). Let $\mathcal{N}$ be a 3-prime near-ring admitting a generalized n-derivation F with associated n-derivation D of $\mathcal{N}$. If $K = \{a \in \mathcal{N} \mid [F(\mathcal{N}, \mathcal{N}, \cdots, \mathcal{N}), a] = \{0\}\}$ and d stands for the trace of D, then $a \in K$ implies either $a \in Z$ or $d(a) = 0$.*

We close our discussion with the following example which justifies the existence of 3-primeness in the hypothesis of the Theorem 6.4.

Example 6.11. ([12], Example 3.3). Consider $\mathcal{N}$, F_1, D_1 and $\mathfrak{A}_1 = \mathfrak{A}_2 = \cdots, = \mathfrak{A}_n = \mathfrak{A}$ as discussed in the Example 6.2. Let us choose $a = \begin{pmatrix} 0 & l & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $l = (q, p)$ and $0 \neq q \in Q, p \in S_3$. Then clearly $a \in \mathcal{N}$ and it can be easily shown that $[F_1(u_1, u_2, \cdots, u_n), a] = 0$ for all $u_1 \in \mathfrak{A}_1, u_2 \in \mathfrak{A}_2, \cdots, u_n \in \mathfrak{A}_n$. However $a \notin Z$ and $d_1(a) \neq 0$, where d_1 stands for the trace of D_1.

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