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GRÖBNER-SHIRSHOV BASES FOR TEMPERLEY-LIEB ALGEBRAS

Sungsoon Kim† and Dong-il Lee*

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Abstract For Temperley-Lieb algebras of type B, we construct their Gröbner-Shirshov bases and the corresponding standard monomials, which give another combinatorial interpretation for the fully commutative elements.

1 Introduction

Originally, the Temperley-Lieb algebra appears in the context of statistical mechanics [19], and later its structure has been studied in connection with knot theory, where it is known to be a quotient of the Hecke algebra of type A [8].

Our approach to understanding the structure of Temperley-Lieb algebras is from the non-commutative Gröbner basis theory, called the *Gröbner-Shirshov basis theory*, which provides a powerful tool for understanding the structure of (non)associative algebras and their representations, especially in computational aspects. With the ever-growing power of computers, it is now viewed as a universal engine behind algebraic or symbolic computation.

The main interest of the notion of Gröbner-Shirshov bases stems from Shirshov's Composition Lemma and his algorithm [15] for Lie algebras and independently from Buchberger's algorithm [4] of computing Gröbner bases for commutative algebras. In [2], Bokut applied Shirshov's method to associative algebras, and Bergman mentioned the diamond lemma for ring theory [1].

The Gröbner-Shirshov bases for Coxeter groups of classical and exceptional types were completely determined in [3, 12, 13, 18]. The cases for Hecke algebras and Temperley-Lieb algebras of type A were calculated in [9].

In this paper, we deal with Temperley-Lieb algebras of type B, extending the result in [9, §6]. By completing the relations coming from a presentation of the Temperley-Lieb algebra, we compute its Gröbner-Shirshov basis to obtain the corresponding set of standard monomials. The explicit multiplication table between the monomials follows naturally. We remark that the set of standard monomials we constructed as a Gröbner-Shirshov basis corresponds to that of fully commutative elements which indexes a basis of the Temperley-Lieb algebra [6, 17].

2 Basic Definitions and Notations

In this section, we recall a basic theory of *Gröbner-Shirshov bases* for associative algebras so as to make the paper self-contained. There will be some properties listed without proofs which are well-known and necessary for this paper.

Let X be a set and let $\langle X\rangle$ be the free monoid of associative words on X. We denote the empty word by 1 and the *length* (or *degree*) of a word u by $l(u)$. We define a total-order $<$ on $\langle X\rangle$, called a *monomial order* as follows ;

$$\textit{if } x<y \textit{ implies } axb<ayb \textit{ for all } a,b\in\langle X\rangle.$$

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Fix a monomial order $<$ on $\langle X\rangle$ and let $\mathbb{F}\langle X\rangle$ be the free associative algebra generated by X over a field $\mathbb{F}$. Given a nonzero element $p \in \mathbb{F}\langle X\rangle$, we denote by $\overline{p}$ the monomial (called the *leading monomial*) appearing in p, which is maximal under the ordering $<$. Thus $p = \alpha\overline{p} + \sum \beta_i w_i$ with $\alpha, \beta_i \in \mathbb{F}$, $w_i \in \langle X\rangle$, $\alpha \neq 0$ and $w_i < \overline{p}$ for all i. If $\alpha = 1$, p is said to be *monic*.

Let S be a subset of monic elements in $\mathbb{F}\langle X\rangle$, and let I be the two-sided ideal of $\mathbb{F}\langle X\rangle$ generated by S. Then we say that the algebra $A = \mathbb{F}\langle X\rangle/I$ is *defined by* S.

Definition 2.1. Given a subset S of monic elements in $\mathbb{F}\langle X\rangle$, a monomial $u \in \langle X\rangle$ is said to be *S-standard* (or *S-reduced*) if $u \neq a\overline{s}b$ for any $s \in S$ and $a, b \in \langle X\rangle$. Otherwise, the monomial u is said to be *S-reducible*.

Lemma 2.2 ([1, 2]). Every $p \in \mathbb{F}\langle X\rangle$ can be expressed as

$$p = \sum \alpha_i a_i s_i b_i + \sum \beta_j u_j, \tag{2.1}$$

where $\alpha_i, \beta_j \in \mathbb{F}$, $a_i, b_i, u_j \in \langle X\rangle$, $s_i \in S$, $a_i\overline{s_i}b_i \leq \overline{p}$, $u_j \leq \overline{p}$ and u_j are S-standard.

Remark. The term $\sum \beta_j u_j$ in the expression (2.1) is called a *normal form* (or a *remainder*) of p with respect to the subset S (and with respect to the monomial order $<$). In general, a normal form is not unique.

As an immediate corollary of Lemma 2.2, we obtain:

Proposition 2.3. The set of S-standard monomials spans the algebra $A = \mathbb{F}\langle X\rangle/I$ defined by the subset S, as a vector space over $\mathbb{F}$.

Let p and q be monic elements in $\mathbb{F}\langle X\rangle$ with leading monomials $\overline{p}$ and $\overline{q}$. We define the *composition* of p and q as follows.

Definition 2.4. (a) If there exist a and b in $\langle X\rangle$ such that $\overline{p}a = b\overline{q} = w$ with $l(\overline{p}) > l(b)$, then the *composition of intersection* is defined to be $(p, q)_w = pa - bq$.

(b) If there exist a and b in $\langle X\rangle$ such that $a \neq 1$, $a\overline{p}b = \overline{q} = w$, then the *composition of inclusion* is defined to be $(p, q)_{a,b} = apb - q$.

Let $p, q \in \mathbb{F}\langle X\rangle$ and $w \in \langle X\rangle$. We define the *congruence relation* on $\mathbb{F}\langle X\rangle$ as follows: $p \equiv q \mod (S; w)$ if and only if $p - q = \sum \alpha_i a_i s_i b_i$, where $\alpha_i \in \mathbb{F}$, $a_i, b_i \in \langle X\rangle$, $s_i \in S$, $a_i\overline{s_i}b_i < w$.

Definition 2.5. A subset S of monic elements in $\mathbb{F}\langle X\rangle$ is said to be *closed under composition* if

$(p, q)_w \equiv 0 \mod (S; w)$ and $(p, q)_{a,b} \equiv 0 \mod (S; w)$ for all $p, q \in S$, $a, b \in \langle X\rangle$ whenever the compositions $(p, q)_w$ and $(p, q)_{a,b}$ are defined.

The following theorem is a main tool for our results in the subsequent sections.

Theorem 2.6 ([1, 2]). Let S be a subset of monic elements in $\mathbb{F}\langle X\rangle$. Then the following conditions are equivalent:

(a) S is closed under composition.

(b) For each $p \in \mathbb{F}\langle X\rangle$, a normal form of p with respect to S is unique.

(c) The set of S-standard monomials forms a linear basis of the algebra $A = \mathbb{F}\langle X\rangle/I$ defined by S.

Definition 2.7. A subset S of monic elements in $\mathbb{F}\langle X\rangle$ is a *Gröbner-Shirshov basis* if S satisfies one of the equivalent conditions in Theorem 2.6. In this case, we say that S is a *Gröbner-Shirshov basis* for the algebra A defined by S.

Let us now turn our attention to some combinatorial concepts for better understanding of the proof of our main theorem 4.2.

Definition 2.8. Let W be a Coxeter group. An element w is said to be *fully commutative* if any reduced word for w can be obtained from any other by interchange of adjacent commuting generators.

Stembridge [16] classified all of the Coxeter groups that have finitely many fully commutative elements. His results completed the work of Fan [5], who had done this for the simply-laced types. In the same paper [5], Fan showed that the fully commutative elements parameterized natural bases for corresponding quotients of Hecke algebras. In type A_n, these give rise to the Temperley–Lieb algebras (see [8]). Fan and Stembridge also enumerated the set of fully commutative elements. In particular, they showed the following.

Proposition 2.9 ([5, 17]). Let C_n be the n^{th} Catalan number, i.e. $C_n = \frac{1}{n+1}\binom{2n}{n}$. Then the numbers of fully commutative elements in the Coxeter group of types A_n, D_n and B_n are given as follows:

$$\begin{cases} C_{n+1} & \text{if the type is } A_n, \\ \frac{n+3}{2} \times C_n - 1 & \text{if the type is } D_n, \\ (n+2) \times C_n - 1 & \text{if the type is } B_n. \end{cases}$$

It is known by Kleshchev and Ram's work [11] that homogeneous representations of KLR algebras can be constructed from the fully commutative elements which are defined as reduced words having no subword of the form $s_i s_{i\pm 1} s_i$.

Motivated from their work, Feinberg and Lee computed in the article [7] the sets of reduced words of fully commutative elements of type D_n. In their work, we first decompose the set of fully commutative elements into disjoint subsets called *packets*, denoting the k-th packet by $\mathcal{P}_D(n,k)$. Then each packet is in turn represented as a partition of its subsets called *collections* depending on the shapes of suffixes of the words. Doing this process, Feinberg and Lee found that all collections of a packet $\mathcal{P}_D(n,k)$ have the same cardinality and each collection contains exactly $C(n,k)$ elements, thus finally obtained the following formula : ([7, Cor. 2.14])

$$\sum_{k=0}^{n} C(n,k)|\mathcal{P}_D(n,k)| = \frac{n+3}{2}C_n - 1 \tag{2.2}$$

where C_n is the n^{th} Catalan number, $C(n,k)$ is the (n,k)-entry of the Catalan triangle, and $|\mathcal{P}_D(n,k)|$ is the number of elements in the (n,k)-packet $\mathcal{P}_D(n,k)$.

We remark that the number $\frac{n+3}{2}C_n - 1$ on the right-hand side of the above formula is the dimension of the Temperley-Lieb algebra of type D_n.

We also note that using the exact values of $|\mathcal{P}_D(n,k)|$ in (2.2) ([7, Prop. 2.9]), we can have the following useful expansion :

$$2^{n-2} - 1 + \sum_{k=1}^{n-2} \frac{n-k+1}{n+1}\binom{n+k}{n}2^{n-k-2} + \frac{2}{n+1}\binom{2n}{n} = \frac{n+3}{2}C_n - 1.$$

Kim-Lee-Oh [10] also obtained an analogous formula for type B_n as well as the exact cardinality of each packet $\mathcal{P}_B(n,k)$:

$$\sum_{k=0}^{n} C(n,k)|\mathcal{P}_B(n,k)| = (n+2)C_n - 1, \tag{2.3}$$

which is the dimension of the Temperley-Lieb algebra of type B_n.

3 Review of results for the Temperley-Lieb algebra of type A_{n-1}

First, we review the results on Temperley-Lieb algebras $\mathcal{T}(A_{n-1})$ $(n \geq 2)$. Define $\mathcal{T}(A_{n-1})$ to be the associative algebra over the complex field $\mathbb{C}$, generated by $X = \{E_1, E_2, \ldots, E_{n-1}\}$ with defining relations:

$$R_{\mathcal{T}(A_{n-1})}: \quad \begin{aligned} E_i^2 &= \delta E_i && \text{for } 1 \leq i \leq n-1, \\ E_iE_j &= E_jE_i && \text{for } i > j+1 \quad \text{(commutative relations)}, \\ E_iE_jE_i &= E_i && \text{for } j = i \pm 1, \end{aligned}$$

where $\delta \in \mathbb{C}$ is a parameter. Our monomial order $<$ is taken to be the degree-lexicographic order with

$$E_1 < E_2 < \cdots < E_{n-1}.$$

We write $E_{i,j} = E_i E_{i-1} \cdots E_j$ for $i \geq j$ (hence $E_{i,i} = E_i$). By convention $E_{i,i+1} = 1$ for $i \geq 1$.

Proposition 3.1. ([9, Proposition 6.2]) The Temperley-Lieb algebra $\mathcal{T}(A_{n-1})$ has a Gröbner-Shirshov basis as follows:

$$\widehat{R}_{\mathcal{T}(A_{n-1})}: \quad \begin{array}{ll} E_i^2 - \delta E_i & \text{for } 1 \leq i \leq n-1, \\ E_i E_j - E_j E_i & \text{for } i > j+1, \\ E_{i,j} E_i - E_{i-2,j} E_i & \text{for } i > j, \\ E_j E_{i,j} - E_j E_{i,j+2} & \text{for } i > j. \end{array} \tag{3.1}$$

The corresponding $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard monomials are of the form

$$E_{i_1,j_1} E_{i_2,j_2} \cdots E_{i_p,j_p} \quad (0 \leq p \leq n-1) \tag{3.2}$$

where

$$1 \leq i_1 < i_2 < \cdots < i_p \leq n-1, \quad 1 \leq j_1 < j_2 < \cdots < j_p \leq n-1,$$
$$i_1 \geq j_1,\ i_2 \geq j_2,\ \ldots,\ i_p \geq j_p$$

(the case of $p = 0$ is the monomial 1). We denote the set of $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard monomials by $M_{\mathcal{T}(A_{n-1})}$ and the number $|M_{\mathcal{T}(A_{n-1})}|$ of $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard monomials is the n^{th} Catalan number,

$$C_n := \frac{1}{n+1}\binom{2n}{n}.$$

Example 3.2. Note that $|M_{\mathcal{T}(A_3)}| = C_4 = 14$. Explicitly, the $\widehat{R}_{\mathcal{T}(A_3)}$-standard monomials are as follows:

$$1, E_1, E_{2,1}, E_2, E_1E_2, E_{3,1}, E_{3,2}, E_3,$$
$$E_1E_{3,2}, E_1E_3, E_{2,1}E_{3,2}, E_{2,1}E_3, E_2E_3, E_1E_2E_3.$$

Remark. (1) One interesting point of considering standard monomials is that the product of two standard monomials becomes a standard monomial up to a scalar multiple. As an example, if we multiply E_1E_2 by $E_{2,1}E_{3,2}$ in the previous example then we obtain

$$(E_1E_2)(E_{2,1}E_{3,2}) = \delta E_1E_2E_1E_{3,2} = \delta E_1E_{3,2},$$

a multiple of another standard monomial $E_1E_{3,2}$. For another one, the multiplication of $E_{2,1}$ by $E_{3,1}$ leads us to have

$$E_{2,1}E_{3,1} = E_2(E_1E_{3,1}) = E_2(E_1E_3) = E_{2,1}E_3$$

by the Gröbner-Shirshov basis (3.1).

(2) One can also notice that the number of standard monomials equals the number of fully commutative elements, which is the dimension of the Temperley-Lieb algebra of type A.

4 Gröbner-Shirshov bases for the Temperley-Lieb algebras of type B_n

Let $\mathcal{T}(B_n)$ $(n \geq 2)$ be the Temperley-Lieb algebra of type B_n, that is, the associative algebra over the complex field $\mathbb{C}$, generated by $X = \{E_0, E_1, \ldots, E_{n-1}\}$ with defining relations:

$$R_{\mathcal{T}(B_n)}: \quad \begin{array}{ll} E_i^2 = \delta E_i & \text{for } 0 \leq i \leq n-1, \\ E_iE_j = E_jE_i & \text{for } i > j+1, \\ E_iE_jE_i = E_i & \text{for } j = i \pm 1,\ i, j > 0, \\ E_iE_jE_iE_j = 2E_iE_j & \text{for } \{i,j\} = \{0,1\}, \end{array} \tag{4.1}$$

where $\delta \in \mathbb{C}$ is a parameter. .

Fix our monomial order $<$ to be the degree-lexicographic order with

$$E_0 < E_1 < \cdots < E_{n-1}.$$

We write $E_{i,j} = E_i E_{i-1} \cdots E_j$ for $i \geq j \geq 0$, and $E^{i,j} = E_i E_{i+1} \cdots E_j$ for $i \leq j$. By convention, $E_{i,i+1} = 1$ and $E^{i+1,i} = 1$ for $i \geq 0$.

Lemma 4.1. The following relation holds in $\mathcal{T}(B_n)$:

$$E_{i,0} E^{1,j} E_i = E_{i-2,0} E^{1,j} E_i$$

for $i > j+1 \geq 1$.

Proof. Since $2 \leq i \leq n-1$ and $0 \leq j \leq i-2$, we calculate that

$$E_{i,0} E^{1,j} E_i = (E_i E_{i-1} E_i) E_{i-2,0} E^{1,j} = E_i E_{i-2,0} E^{1,j} = E_{i-2,0} E^{1,j} E_i$$

by the commutative relations and $E_i E_{i-1} E_i = E_i$. □

Let $\widehat{R}_{\mathcal{T}(B_n)}$ be the set of defining relations (4.1) combined with (3.1) and the relation in Lemma 4.1. From this, we define $M_{\mathcal{T}(B_n)}$ by the set of $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials. Among the monomials in $M_{\mathcal{T}(B_n)}$, we consider the monomials which are not $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard. That is, we take only $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials which are not of the form (3.2). This set is denoted by $M^0_{\mathcal{T}(B_n)}$. Note that each monomial in $M^0_{\mathcal{T}(B_n)}$ contains E_0. We decompose the set $M^0_{\mathcal{T}(B_n)}$ into two parts as follows :

$$M^0_{\mathcal{T}(B_n)} = M^{0+}_{\mathcal{T}(B_n)} \amalg M^{0-}_{\mathcal{T}(B_n)}$$

where the monomials in $M^{0+}_{\mathcal{T}(B_n)}$ are of the form

$$E_0 E_{i_1,j_1} E_{i_2,j_2} \cdots E_{i_p,j_p} \quad (0 \leq p \leq n-1) \tag{4.2}$$

with

$$\begin{aligned} &1 \leq i_1 < i_2 < \cdots < i_p \leq n-1, \quad 0 \leq j_1 \leq j_2 \leq \cdots \leq j_p \leq n-1, \\ &i_1 \geq j_1,\ i_2 \geq j_2,\ \ldots,\ i_p \geq j_p, \ \text{ and} \\ &j_k > 0 \ (1 \leq k < p) \ \text{ implies } j_k < j_{k+1} \end{aligned}$$

(the case of $p = 0$ is the monomial E_0), and the monomials in $M^{0-}_{\mathcal{T}(B_n)}$ are of the form

$$E'_{i_1,j_1} E_{i_2,j_2} \cdots E_{i_p,j_p} \quad (1 \leq p \leq n-1)$$

with

$$E'_{i,j} = E_{i,0} E^{1,j}$$

and the same restriction on i's and j's as above. It can be easily checked that $M^0_{\mathcal{T}(B_n)}$ is the set of $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials which are not $\widehat{R}_{\mathcal{T}(A_{n-1})}$-standard.

To each monomial $E_0 E_{i_1,0} E_{i_2,0} \cdots E_{i_k,0} E_{i_{k+1},j_{k+1}} \cdots E_{i_p,j_p}$ in $M^{0+}_{\mathcal{T}(B_n)}$ with $j_{k+1} > 0$, we can associate a unique path

$$(0,0) \to (i_1,0) \to (i_2,0) \to \cdots \to (i_k,0) \to (i_{k+1},j_{k+1}) \to \cdots \to (i_p,j_p) \to (n,n).$$

Here, a path consists of moves to the east or to the north, not above the diagonal in the lattice plane. The move from (i,j) to (i',j') ($i < i'$ and $j < j'$) is a concatenation of eastern moves followed by northern moves. As an example, the monomial $E_0 E_{1,0} E_{2,1} \in M^{0+}_{\mathcal{T}(B_3)}$ corresponds to

$$(0,0) \to (1,0) \to (2,1) \to (3,3).$$

Counting the number of elements in $M^0_{\mathcal{T}(B_n)}$, we obtain the following theorem.

Theorem 4.2. The algebra $\mathcal{T}(B_n)$ has a Gröbner-Shirshov basis $\widehat{R}_{\mathcal{T}(B_n)}$ with respect to our monomial order $<$:

$$\widehat{R}_{\mathcal{T}(B_n)}: \quad \begin{array}{cl} E_i^2 - \delta E_i & \text{for } 0 \leq i \leq n-1, \\ E_iE_j - E_jE_i & \text{for } i > j+1, \\ E_{i,j}E_i - E_{i-2,j}E_i & \text{for } i > j > 0, \\ E_jE_{i,j} - E_jE_{i,j+2} & \text{for } i > j > 0. \\ E_iE_jE_iE_j - 2E_iE_j & \text{for } \{i,j\} = \{0,1\}, \\ E_{i,0}E^{1,j}E_i - E_{i-2,0}E^{1,j}E_i & \text{for } i > j+1 \geq 1. \end{array}$$

The cardinality of the set $M_{\mathcal{T}(B_n)}$, i.e. the set of $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials, is

$$\dim \mathcal{T}(B_n) = (n+2)C_n - 1.$$

Proof. First, we consider a mapping

$$\phi : M^{0+}_{\mathcal{T}(B_n)} \setminus \{E_0\} \to M^{0-}_{\mathcal{T}(B_n)}$$

defined by $\phi(E_0E_{i_1,j_1}E_{i_2,j_2}\cdots E_{i_p,j_p}) = E'_{i_1,j_1}E_{i_2,j_2}\cdots E_{i_p,j_p}$. Then this map is a bijection. In order to compute $|M^0_{\mathcal{T}(B_n)}|$, it is enough to count the the number of elements in $M^{0+}_{\mathcal{T}(B_n)}$. For this, we consider the following procedure.

In the lattice plane, we plot the sequence of points $(i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p)$ corresponding to the monomial $E_0E_{i_1,j_1}E_{i_2,j_2}\cdots E_{i_p,j_p}$ in (4.2). Set $\ell > 0$ to be the largest i such that $(i, 0)$ belongs to the sequence of plotted points. Then the number of sequences of plotted points between $(\ell, 0)$ and (n, n) is the number of paths from $(\ell+1, 0)$ and (n, n) .

Counting the number of these paths, we have

$$\binom{2n-\ell-1}{n} - \binom{2n-\ell-1}{n+1} = \frac{\ell+2}{n+1}\binom{2n-\ell-1}{n}.$$

Thus the number of monomials of the form $E_0E_{i_1,0}\cdots E_{i_p,j_p}$ (4.2) is

$$\sum_{\ell=1}^{n-1} \frac{\ell+2}{n+1}\binom{2n-\ell-1}{n}2^{\ell-1},$$

which is the same quantity as $\frac{1}{2}\left(\sum_{k=0}^{n-2} C(n,k)|\mathcal{P}_B(n,k)| + 1\right) = \frac{n-1}{2}C_n$, as we have mentioned in (2.3) as well as in [7, Corollary 2.14].

Therefore we have

$$|M^{0+}_{\mathcal{T}(B_n)}| = C_n + \tfrac{n-1}{2}C_n = \tfrac{n+1}{2}C_n.$$

Then, the number of $\widehat{R}_{\mathcal{T}(B_n)}$-standard monomials becomes

$$|M_{\mathcal{T}(A_{n-1})}| + 1 + 2|M^{0+}_{\mathcal{T}(B_n)} \setminus \{E_0\}| = C_n + 1 + 2\left(\tfrac{n+1}{2}C_n - 1\right),$$

which gives exactly the number equal to

$$\dim \mathcal{T}(B_n) = (n+2)C_n - 1$$

as mentioned in [17, §5] and [6, §7]. Theorem 2.6 yields that $\widehat{R}_{\mathcal{T}(B_n)}$ is a Gröbner-Shirshov basis for $\mathcal{T}(B_n)$. □

Example 4.3. (1) We enumerate the $\widehat{R}_{\mathcal{T}(B_3)}$-standard monomials containing E_0:

$$E_0,\ E_0E_{1,0}, E_{1,0},\ E_0E_1, E'_1,\ E_0E_{2,0}, E_{2,0},\ E_0E_{2,1}, E'_{2,1},\ E_0E_2, E'_2,$$
$$E_0E_{1,0}E_{2,0}, E_{1,0}E_{2,0}, E_0E_{1,0}E_{2,1}, E_{1,0}E_{2,1}, E_0E_{1,0}E_2, E_{1,0}E_2, E_0E_1E_2, E'_1E_2.$$

(2) The product of two $\widehat{R}_{\mathcal{T}(B_3)}$-standard monomials is a scalar multiple of a standard monomial. For instance, we multiply $E_0E_{1,0}E_{2,0}$ by E_2 from the left:

$$E_2(E_0E_{1,0}E_{2,0}) = E_0E_{2,0}E_{2,0} = E_0E_0E_2E_{1,0} = \delta E_0E_{2,0}.$$

Remark. (1) Our monomials in $M_{\mathcal{T}(B_n)}$ are fully commutative, in the sense of [17, §5]. Note that the number of non-identity fully commutative top monomials is $\binom{2n}{n} - 1 = |M^0_{\mathcal{T}(B_n)}|$.

(2) We observe that the elements in (4.2) are in 1-1 correspondence with semistandard tableaux having at most two columns with entries in $\{1, 2, \ldots, n-1\}$. By the conjugate of Pieri's formula connecting Schur polynomials with elementary symmetric polynomials (See [14, I.(5.17)]), that is, $s_\mu e_r = \sum_\lambda s_\lambda$ (the sum is over all partitions λ such that $\lambda - \mu$ is a vertical r-strip), we get that the righthand side of the case of $\mu = 1^r$(or $\mu = 1^{r+1}$) is the sum of monomials associated to semistandard tableaux having at most two columns. So we count the number of monomials to obtain that

$$|M^{0+}_{\mathcal{T}(B_n)}| = \sum_{r=0}^{n-1} \binom{n-1}{r}^2 + \sum_{r=0}^{n-2} \binom{n-1}{r+1}\binom{n-1}{r}$$
$$= \sum_{r=0}^{n-1} \binom{n}{r+1}\binom{n-1}{r} = \binom{2n-1}{n-1} = \frac{1}{2}\binom{2n}{n} = \frac{n+1}{2} C_n.$$

The latter part of this formula is also computed in [17, §5].

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Author information

Sungsoon Kim†, LAMFA CNRS UMR 7352, Université de Picardie JV-Mathématiques, 80039 Amiens, (membre ext. IMJ-PRG Univ. Paris 7), France.
E-mail: `sungsoon.kim@u-picardie.fr`

Dong-il Lee*, Department of Mathematics, Seoul Women's University, Seoul 01797, Rep. of Korea.
E-mail: `dilee@swu.ac.kr`