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COPRIMELY STRUCTURED MODULES

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Abstract Let R be a commutative ring with identity. A prime submodule P of an R-module M is called coprimely structured if, whenever P is coprime to each element of an arbitrary family of submodules of M, the intersection of the family is not contained in P. An R-module M is called coprimely structured provided each prime submodule of M is coprimely structured. In this paper, properties of coprimely structured modules are examined. Severals results for coprimely structured finitely generated modules and coprimely structured multiplication modules are obtained.

1 Introduction

Throughout this paper, all rings are commutative and with identity. Coprimely structured rings are introduced in [11]. In this paper, we generalize this concept to the class of modules. Let R be a ring. A prime submodule Q of an R-module M is called coprimely structured if for each family $\{N_i\}_{i\in I}$ of submodules of R whenever $N_i + Q = M$ for each $i \in I$, we have $\bigcap_{i\in I} N_i \nsubseteq Q$. M is called a copimely structured module if each of its prime submodules is coprimely structured. In Section 2, definitions and general results are given. Direct sums of modules are examined and a sufficient condition for a direct sum of coprimely structured modules being coprimely structured is stated [Theorem 2.7].

In Section 3, finitely generated modules are discussed. For a finitely generated module, it can be decided whether the module is coprimely structured or not by examining only maximal ideals instead of all prime ideals [Theorem 3.1]. The property of an R-module's being coprimely structured is transported to its localization in case of R is local and the module is a finitely generated distributive R-module [Theorem 3.4].

An R-module M is called a multiplication module if each submodule N of M is of the form IM for some ideal I of R. Multiplication modules are studied widely in the literature, see [1, 5]. In the category of multiplication modules, many properties of coprimely structured modules can be characterized. The radical of an ideal I of R is defined as the intersection of all prime ideals that contain I. Similarly, the radical of a submodule N, rad(N), of a module M is the intersection of all prime submodules of M that contains N. When R is viewed as a module over itself, the definitions of the radical of an ideal and the radical of a submodule coincide. In [2], the product of two submodules $N = IM$ and $K = JM$ of a multiplication R-module M is defined as $(IJ)M$. Accordingly, the product of two elements $m, m' \in M$ is defined as the product of the submodules Rm and Rm'. Using this definition, it is proved in [2] that

$$\text{rad}(N) = \{m \in M : m^k \subseteq N \text{ for some } k \geq 0\}$$

for a submodule N of a multiplication R-module M. Section 4 is reserved for multiplication modules. A family $\{N_i\}_{i\in I}$ of submodules of a multiplication R-module M is said to satisfy property (*) if for each $x \in M$, there is an $n \in \mathbb{N}$ such that $x \in \text{rad}(N_i)$ implies $x^n \subseteq N_i$. With the aid of the property (*), it is possible to characterize coprimely structured multiplication modules in terms of prime submodules and maximal submodules [Theorem 4.5]. Besides, the property (*) is proved to be useful to give a sufficient condition for a module to be coprimely structured provided a particular quotient of the module is coprimely structured [Theorem 4.6]. In case of M is a finitely generated faithful multiplication R-module, M is coprimely structured if and only if R is coprimely structured [Theorem 4.7].

In Section 5, the relation between the property (*) and finitely generated multiplication modules is examined. In particular, if R is a principal ideal ring and M is a finitely generated faithful multiplication module, this property can be used to obtain information about M's being zero-dimensional [Theorem 5.3].

2 Coprimely Structured Modules

In this section, we define coprimely structured modules and investigate some basic properties of them. Also we consider the relation between coprimely structured modules and strongly 0-dimensional modules.

Definition 2.1. Let M be an R-module and $\{N_i\}_{i\in I}$ a family of submodules of M. A prime submodule P of M is said to be a coprimely structured submodule of M if $N_i + P = M$, for all $i \in I$, implies $\bigcap_{i\in I} N_i \nsubseteq P$. An R-module M is called a coprimely structured module if every prime submodule of M is coprimely structured.

Note that not every module is a coprimely structured module. Here is an example:

Example 2.2. Let $R = \mathbb{Z}$, $M = \mathbb{Z} \times \mathbb{Z}$. Conside the family of submodules $\{N_n = n\mathbb{Z} \times n\mathbb{Z} : n \in \mathbb{N}, n \text{ odd} \}$ of M. We have $\bigcap_{\substack{n\in\mathbb{N}\\ n \text{odd}}} N_n = (0)$. Observe that $2\mathbb{Z} \times 2\mathbb{Z}$ is a prime submodule of M and $\bigcap_{\substack{n\in\mathbb{N}\\ n \text{odd}}} N_n \subset 2\mathbb{Z} \times 2\mathbb{Z}$. However, for each $n \in \mathbb{N}$, n odd, we have $N_n + 2\mathbb{Z} \times 2\mathbb{Z} = M$. Thus M is not coprimely structured.

Theorem 2.3. *Every homomorphic image of a coprimely structured module is coprimely structured.*

Proof. Let M be a coprimely structured R-module, and M' an R-module. Let $f : M \to M'$ be an R-module homomorphism. Assume that for a family of submodules $\{N_i'\}_{i\in I}$ and a prime submodule P' of $f(M)$, the inclusion $\bigcap_{i\in I} N_i' \subseteq P'$ holds. Then there exist a family of submodules $\{N_i\}_{i\in I}$ of M and a prime submodule P of M such that each N_i and P contain $\ker f$, and for all $i \in I$, the equalities $f(N_i) = N_i'$ and $f(P) = P'$ hold. Then

$$f(\bigcap_{i\in I} N_i) \subseteq \bigcap_{i\in I} f(N_i) = \bigcap_{i\in I} N_i' \subseteq P' = f(P)$$

and hence $\bigcap_{i\in I} N_i \subseteq P$. As M is coprimely structured, there exists $j \in I$ such that $N_j + P \neq M$. Since both N_j and P contain $\ker f$, we conclude that $N_j' + P' = f(N_j + P) \neq f(M)$. Thus $f(M)$ is coprimely structured. □

Corollary 2.4. *Let M be an R-module and N a submodule of M. If M is coprimely structured, so is M/N.*

A prime submodule P of an R-module M is called a strongly prime submodule provided for any family $\{N_i\}_{i\in I}$ of submodules of M, the inclusion $\bigcap_{i\in I} N_i \subseteq P$ implies $N_j \subseteq P$ for some $j \in I$. An R-module M is a strongly 0-dimensional module if each of its prime submodules is strongly prime. Strongly 0-dimensional multiplication modules are introduced and studied in [10]. The following theorem states the relation between coprimely structured modules and strongly 0-dimensional modules.

Theorem 2.5. *Every strongly 0-dimensional R-module is coprimely structured.*

Proof. Let M be a strongly 0-dimensional R-module, $\{N_i\}_{i\in I}$ a family of submodules of M and P a prime submodule of M. Suppose that the equation $N_i + P = M$ is satisfied for all $i \in I$. Assume that $\bigcap_{i\in I} N_i \subseteq P$. Since M is strongly 0-dimensional, there exists $j \in I$ such that $N_j \subseteq P$. Then we have $M = N_j + P = P$ which is a contradiction. Therefore $\bigcap_{i\in I} N_i \nsubseteq P$. □

By Theorem 2.3, if a direct sum is coprimely structured so is each of its direct summands. We are to investigate when the converse is true. In [7, 2.1], Erdogdu characterizes the submodule structure of direct sum of two modules. For two R-modules M and N, if $\text{Ann}(x) + \text{Ann}(y) = R$ for each $x \in M$ and $y \in N$, then every submodule of $M \oplus N$ is of the form $A \oplus B$ for some submodule A of M and some submodule B of N. We generalize this result to an arbitrary direct sum.

Lemma 2.6. *Let M_i, $i \in I$, be R-modules. The following are equivalent:*

(i) $Ann(m_i) + Ann(m_j) = R$ for each $m_i \in M_i$ and $i \neq j$.

(ii) Each submodule of $\bigoplus_{i\in I} M_i$ is of the form $\bigoplus_{i\in I} N_i$, where N_i is a submodule of M_i for each $i \in I$.

Proof. (i)⇒(ii) Suppose that $\text{Ann}(m_i) + \text{Ann}(m_j) = R$ for each $m_i \in M_i$ and $i \neq j$. We first prove that $\text{Ann}((m_{i_1}, ..., m_{i_{n-1}})) + \text{Ann}(m_{i_n}) = R$ for each $n \in \mathbb{N}$, $m_{i_j} \in M_{i_j}$ and $1 \leq j \leq n$. Let $n \in \mathbb{N}$. We prove the statement by induction on n. Let $m_i \in M_i$. For $n = 2$ the result follows from the assumption. Let $n = 3$. By assumption, we have $1 = a + b = c + d$ where $a \in \text{Ann}(m_1)$, $c \in \text{Ann}(m_2)$ and $b, d \in \text{Ann}(m_3)$. Then

$$1 = (a + b)(c + d) = ac + ad + bc + bd.$$

Since $ac \in \text{Ann}((m_1, m_2))$ and $ad + bc + bd \in \text{Ann}(m_3)$ we have

$$\text{Ann}((m_1, m_2)) + \text{Ann}(m_3) = R.$$

Let $n = k$. Assume that $\text{Ann}((m_{i_1}, ..., m_{i_{l-1}}))+\text{Ann}(m_{i_l}) = R$ for $2 \leq l \leq k-1$, $i_j \in \{1, ..., k\}$. Then,

$$\begin{aligned}\text{Ann}((m_{i_1}, ..., m_{i_{k-2}}, m_{i_{k-1}})) &+ \text{Ann}(m_{i_k}) \\ &= \text{Ann}(((m_{i_1}, ..., m_{i_{k-2}}), m_{i_{k-1}})) + \text{Ann}(m_{i_k}) \\ &= R\end{aligned}$$

since, by assumption, we have

$$\text{Ann}((m_{i_1}, ..., m_{i_{k-2}})) + \text{Ann}(m_{i_k}) = R$$

and $\text{Ann}(m_{i_{k-1}}) + \text{Ann}(m_{i_k}) = R$.

Now, let N be a submodule of $M = \bigoplus_{i\in I} M_i$. Let $n \in N$. Then $n = \sum_{i\in I} m_i$, where $m_{i_1}, ..., m_{i_k}$ are nonzero and $m_i = 0$ for $i \neq i_1, ..., i_k$. For each $l \in \{1, ..., k\}$,

$$1 = a_l + b_l$$

where $a_l \in \text{Ann}(((m_{i_1}, ..., m_{i_{l-1}}, m_{i_{l+1}}, ..., m_{i_k})))$ and $b_l \in \text{Ann}(m_{i_l})$. Then

$$\begin{aligned}1 = \prod_{l=1}^{k}(a_l + b_l) &= (a_1b_2...b_k + a_2b_1b_3...b_k + ... + a_kb_1...b_{k-1}) \\ &\quad + b_1...b_k + \left(\sum_{2\leq r} a_{t_1}...a_{t_r}b_{s_1}...b_{s_p}\right).\end{aligned}$$

Observe that the terms in the second line of the right hand side are contained in $\text{Ann}(m_{i_l})$ for each $l \in \{1, ..., k\}$. On the other hand, for each $j \in \{1, ..., k\}$, we have

$$(a_jb_1...b_{j-1}b_{j+1}...b_k)n = (a_jb_1...b_{j-1}b_{j+1}...b_k)\iota_{i_j}(m_{i_j}) \in N \cap \iota_{i_j}(M_{i_j})$$

where $\iota_i : M_i \to \bigoplus_{i\in I} M_i$ is the ith natural injection. Hence

$$n = 1.n = \left[\prod_{l=1}^{k}(a_l + b_l)\right] n = \sum_{i\in I} c_i$$

where $c_{i_1} = (a_1b_2...b_k)m_{i_1}$, $c_{i_2} = (a_2b_1b_3...b_k)m_{i_2}$,..., $c_{i_k} = (a_kb_1...b_{k-1})m_{i_k}$ and $c_i = 0$ for $i \in I - \{i_1, ..., i_k\}$. Then $n \in \bigoplus_{i\in I}(N \cap \iota_i(M_i)) \subseteq N$. Therefore $N = \bigoplus_{i\in I}(N \cap \iota_i(M_i))$.

(ii)⇒(i)Assume that N is a submodule of $M = \bigoplus_{i\in I} M_i$. Then, by assumption, $N = \bigoplus_{i\in I} N_i$ for some submodule N_i of M_i for each $i \in I$. Observe that

$$N \cap \iota_i(M_i) = \left(\bigoplus_{i\in I} N_i\right) \cap \iota_i(M_i) = \iota_i(N_i).$$

Hence $N = \bigoplus_{i\in I} N_i = \bigoplus_{i\in I}(N \cap \iota_i(M_i))$. Let $i, j \in I$ and $m_i \in M_i$, $m_j \in M_j$. Set $a = \sum_{k\in I} a_k \in M$ where $a_i = m_i$, $a_j = m_j$ and $a_k = 0$ for $k \neq i, j$. Since Ra is a submodule of M, by the above argument $Ra = \bigoplus_{i\in I}(Ra \cap \iota_i(M_i))$. Then $a = \sum_{k\in I} n_k$ where $n_k \in Ra \cap \iota_k(M_k)$. Set $b = \sum_{k\in I} b_k \in M$ where $b_i = m_i$, $b_j = n_j$ and $b_k = 0$ for $k \neq i, j$. Then $a - b = \sum_{k\in I} a_k - \sum_{k\in I} b_k = \sum_{k\in I} n_k - \sum_{k\in I} b_k$. Comparing the corresponding indices, we obtain $n_i - m_i = 0$ and $m_j - n_j = 0$. Hence $m_i = n_i$ and $m_j = n_j$. For each $k \in I$, since $\iota_k(n_k) \in Ra \cap \iota_k(M_k)$, there exists $r_k \in R$ such that $r_k a = \iota_k(n_k) \in \iota_k(M_k)$. In particular, $\iota_j(n_j) = r_j a$. Using the equailities $n_i = m_i$ and $n_j = m_j$, we get $r_j m_i = 0$ and $m_j = r_j m_j$. Then $r_j \in \text{Ann}(m_i)$ and $1 - r_j \in \text{Ann}(m_j)$. Therefore

$$1 = r_j + (1 - r_j) \in \text{Ann}(m_i) + \text{Ann}(m_j)$$

and hence $\text{Ann}(m_i) + \text{Ann}(m_j) = R$. □

Theorem 2.7. *Let M_i, $i \in I$, be coprimely structured R-modules and assume that Ann(m_i) + Ann$(m_j) = R$ for each, $m_i \in M_i$, $i, j \in I$, $i \neq j$. Then $M = \bigoplus_{i\in I} M_i$ is coprimely structured.*

Proof. Let N_λ, $\lambda \in \Lambda$, be a family of submodules and P a prime submodule of M such that $\bigcap_{\lambda\in\Lambda} N_\lambda \subseteq P$. By Lemma 2.6, each submodule of M is of the form $\bigoplus_{i\in I} N_i$ where N_i is a submodule of M_i for each $i \in I$. Then for each $\lambda \in \Lambda$, for each $i \in I$, there exists $N_{i,\lambda}$, submodule of M_i such that $N_\lambda = \bigoplus_{i\in I} N_{i,\lambda}$ and there exists P_i, submodule of M_i, such that $P = \bigoplus_{i\in I} P_i$. Since P is prime, there exists a unique $k \in I$ such that $P_k \neq M_k$. Let $r \in R$, $m \in M_k$. Assume that $rm \in P_k$ and $m \notin P_k$. Set $a = \sum_{i\in I} a_i$ where $a_k = m$ and $a_i = 0$ for $i \neq k$. Then $ra \in P$ and $a \notin P$. Since P is prime, $r \in (P : M)$. Then $rM_k \subseteq P_k$ and hence we conclude that $r \in (P_k : M_k)$. Therefore P_k is a prime submodule of M_k. We have

$$\bigoplus_{i\in I}\left(\bigcap_{\lambda\in\Lambda} N_{i,\lambda}\right) \subseteq \bigcap_{\lambda\in\Lambda}\left(\bigoplus_{i\in I} N_{i,\lambda}\right) = \bigcap_{\lambda\in\Lambda} N_\lambda \subseteq P = \bigoplus_{i\in I} P_i \ .$$

Then, we have $\bigcap_{\lambda\in\Lambda} N_{k,\lambda} \subseteq P_k$. Since M_k is coprimely structured, there exists $\gamma \in \Lambda$ such that $N_{k,\gamma} + P_k \neq M_k$. Therefore

$$N_\gamma + P = (N_{k,\gamma} + P_k) \oplus \left(\bigoplus_{\substack{i\in I\\ i\neq k}} N_{i,\gamma}\right) + \left(\bigoplus_{\substack{i\in I\\ i\neq k}} P_i\right) \neq M_k + \left(\bigoplus_{\substack{i\in I\\ i\neq k}} M_i\right) = M \ .$$

Thus, M is coprimely structured. □

Example 2.8. Let $R = \mathbb{Z}$ and $M = \bigoplus_{p \text{ prime}} \mathbb{Z}_p$. Then M is an R-module. For each prime number p, $\mathbb{Z}_p$, being finite, is coprimely structured. Let p and q be two different prime numbers. Since p and q are coprime, there exists x and y in $\mathbb{Z}$ such that $px + qy = 1$. For each $m_p \in \mathbb{Z}_p$ and $m_q \in \mathbb{Z}_q$, since $px \in \text{Ann}(m_p)$ and $qy \in \text{Ann}(m_q)$ we have $1 \in \text{Ann}(m_p) + \text{Ann}(m_q)$. Therefore, by Theorem 2.7, M is coprimely structured.

3 Coprimely Structured Property on Finitely Generated Modules

It is known that every proper submodule of a finitely generated R-module is contained in a maximal submodule, [3, 2.8]. Provided we work on the class of finitely generated modules, it is enough to consider maximal submodules to decide whether a module is coprimely structured, or not. The following theorem states this result.

Theorem 3.1. *Let M be a finitely generated R-module. If every maximal submodule of M is coprimely structured, then M is coprimely structured.*

Proof. Assume that every maximal submodule of M is coprimely structured. Let $\{N_i\}_{i\in I}$ be a family of submodules of M and P a prime submodule of M satisfying $\bigcap_{i\in I} N_i \subseteq P$. Since M is finitely generated, the submodule P is contained in a maximal submodule K of M. Then $\bigcap_{i\in I} N_i \subseteq K$, and since K is coprimely structured, there exists $j \in I$ such that $N_j + K \neq M$. Then $N_j + P \neq M$. Thus, we conclude that M is coprimely structured. □

Lemma 3.2. *Let M be a finitely generated R-module. The following are equivalent:*

(i) *M is coprimely structured.*

(ii) *Every maximal submodule K of M is strongly prime.*

(iii) *For any maximal submodule K and any family $\{N_i\}_{i\in I}$ of submodules of M, $K+N_i = M$, for all $i \in I$, implies $K + \bigcap_{i\in I} N_i = M$.*

Proof. (i)⇒(ii) Let $\{N_i\}_{i\in I}$ be a family of submodules of M and K a maximal submodule of M satisfying $\bigcap_{i\in I} N_i \subseteq K$. Since M is coprimely structured, $N_j + K \neq M$ for some $j \in I$. Then $N_j \subseteq K$ and hence K is strongly prime.
(ii)⇒(iii) Let $\{N_i\}_{i\in I}$ be a family of submodules of M and K a maximal submodule of M such that $K + N_i = M$ holds for each $i \in I$. We have $N_i \nsubseteq K$ for each $i \in I$. Since K is strongly prime, we obtain $\bigcap_{i\in I} N_i \nsubseteq K$. This implies $K + \bigcap_{i\in I} N_i = M$.
(iii)⇒(i) Let $\{N_i\}_{i\in I}$ be a family of submodules of M and P a prime submodule of M satisfying $\bigcap_{i\in I} N_i \subseteq P$. Since M is finitely generated, the submodule P is contained in a maximal submodule K of M. Then we have $\bigcap_{i\in I} N_i \subseteq K$, and hence $K + \bigcap_{i\in I} N_i \neq M$. This implies $P + N_j \subseteq K + N_j \neq M$ for some $j \in I$. Therefore M is coprimely structured. □

It is proved in [10, 2.4] that a strongly 0-dimensional multiplication module is zero-dimensional. Actually, the proof is valid if we drop the assumption that the module is a multiplication module.

Theorem 3.3. *Let M be a finitely generated R-module. Then M is a zero-dimensional coprimely structured module if and only if M is a strongly 0-dimensional module.*

Proof. Follows from Theorem 2.5 and Lemma 3.2. □

An R-module M is said to be a distributive module if the lattice of submodules of M is distributive, that is, for any submodules A, B, C of M, the equality $A \cap (B + C) = (A \cap B) + (A \cap C)$ holds. In [12, 2.4], Stephenson proved that for a local ring R and a distributive R-module M, submodules of M are comparable. For a comprehensive study on distributive modules the reader may refer to [7, 12].

Theorem 3.4. *Let R be a local ring and M a finitely generated distributive module. Let S be a multiplicatively closed subset of R. If M is coprimely structured then $S^{-1}M$ is coprimely structured.*

Proof. Let $\{N_i\}_{i\in I}$ be a family of submodules and P a prime submodule of $S^{-1}M$. Then for some family $\{K_i\}_{i\in I}$ of submodules and some prime submodule Q of M we have $N_i = S^{-1}K_i$ and $P = S^{-1}Q$. Assume that $\bigcap_{i\in I} N_i \subseteq P$. Then

$$S^{-1}\left(\bigcap_{i\in I} K_i\right) \subseteq \bigcap_{i\in I} S^{-1}K_i = \bigcap_{i\in I} N_i \subseteq P = S^{-1}Q.$$

Hence $\bigcap_{i\in I} K_i \subseteq Q$. Since M is coprimely structured, we have $K_j + Q \neq M$ for some $j \in I$. Since M is finitely generated, there exists a maximal submodule K of M such that $K_j + Q \subseteq K$. As $K_j \subseteq K$, there exists a minimal prime submodule Q_j of M such that $K_j \subseteq Q_j \subseteq K$. Then $Q_j + Q \subseteq K$. Since R is local and M is distributive, either $Q_j \subseteq Q$ or $Q \subseteq Q_j$. Then $S^{-1}Q_j \subseteq S^{-1}Q$ or $S^{-1}Q \subseteq S^{-1}Q_j$. Hence we obtain $N_j + P \neq S^{-1}M$. Therefore $S^{-1}M$ is coprimely structured. □

4 Coprimely Structured Multiplication Modules

In this section we study some properties of coprimely structured multiplication modules. An R-module M is called a multiplication module if each submodule N of M is of the form IM for some ideal I of R As finitely generated modules, nonzero multiplication modules admits the property that every proper submodule is contained in a maximal submodule, by [1, 2.5], we have the following theorems on multiplication modules similar to results on finitely generated modules mentioned above. The proofs are exactly the same, and hence omitted.

Theorem 4.1. *Let M be a multiplication R-module. If every maximal submodule of M is coprimely structured, then M is coprimely structured.*

Lemma 4.2. *Let M be a multiplication R-module. The following are equivalent:*

(i) M is coprimely structured.

(ii) Every maximal submodule K of M is strongly prime.

(iii) For any maximal submodule K and any family $\{N_i\}_{i\in I}$ of submodules of M, $K+N_i = M$, for all $i \in I$, implies $K + \bigcap_{i\in I} N_i = M$.

Theorem 4.3. *Let M be a zero-dimensional multiplication R-module. Then M is coprimely structured if and only if M is strongly 0-dimensional.*

Next, we prove a theorem that gives a characterization of coprimely structured multiplication modules in terms of families of prime submodules and maximal submodules. To this aim, we state some definitions and notations. For a submodule N of M, the radical of N, denoted by rad(N), is defined as the intersection of all prime submodules of M that contain N. In [2, 3.3], Ameri defines the product of two submodules $N = IM$ and $K = JM$ of a multiplication R-module M as $(IJ)M$. Accordingly, the product of two elements $m, m' \in M$ is defined as the product of the submodules Rm and Rm'. It is shown in [2, 3.13] that $\text{rad}(N) = \{m \in M : m^k \subseteq N \text{ for some } k \geq 0\}$ for a submodule N of a multiplication R-module M.

A family $\{N_i\}_{i\in I}$ of submodules of a multiplication R-module M is said to satisfy property (*) if for each $x \in M$, there is an $n \in \mathbb{N}$ such that $x \in \text{rad}(N_i)$ implies $x^n \subseteq N_i$. We note that if we consider R as a module over itself, this property is the same as the condition A2 in [4, 7]. Accordingly, the following lemma is a generalization of [6, 2].

Lemma 4.4. *A family $\{N_i\}_{i\in I}$ of submodules of a multiplication R-module M satisfies the property (*) if and only if for each subset $J \subseteq I$,*

$$\text{rad}(\bigcap_{i\in J} N_i) = \bigcap_{i\in J} \text{rad}(N_i).$$

Proof. Let $\{N_i\}_{i\in I}$ be a family of submodules of a multiplication R-module M. Assume that the family $\{N_i\}_{i\in I}$ satisfies the property (*). Let J be a subset of I. The inclusion $\text{rad}(\bigcap_{i\in J} N_i) \subseteq \bigcap_{i\in J} \text{rad}(N_i)$ always holds. For the reverse inclusion let $x \in \bigcap_{i\in J} \text{rad}(N_i)$. Then for all $i \in J$ we have $x \in \text{rad}(N_i)$. Since $\{N_i\}_{i\in I}$ satisfies the property (*), there exists an $n \in \mathbb{N}$ such that $x^n \subseteq N_i$ for each $i \in J$. This implies $x^n \subseteq \bigcap_{i\in J} N_i$. Hence we obtain $x \in \text{rad}(\bigcap_{i\in J} N_i)$. Conversely, assume for each subset J of I, that the equation $\text{rad}(\bigcap_{i\in J} N_i) = \bigcap_{i\in J} \text{rad}(N_i)$ holds. Let $x \in M$ and set $J = \{i \in I : x \in \text{rad}(N_i)\}$. Then $x \in \bigcap_{i\in J} \text{rad}(N_i) = \text{rad}(\bigcap_{i\in J} N_i)$. Therefore there is an $n \in \mathbb{N}$ such that $x^n \in \bigcap_{i\in J} N_i$. Since $\bigcap_{i\in J} N_i \subseteq N_i$ for each $i \in J$, we conclude that $x^n \subseteq N_i$ for each $i \in J$. Therefore $\{N_i\}_{i\in I}$ satisfies the property (*).

□

Theorem 4.5. *Let M be a multiplication R-module. If M is coprimely structured, then for any family $\{P_i\}_{i\in I}$ of prime submodules and any maximal submodule K of M, the inclusion $\bigcap_{i\in I} P_i \subseteq K$ implies $P_j \subseteq K$ for some $j \in I$. The converse is true if the property (*) is satisfied by any family of submodules of M.*

Proof. Assume that M is coprimely structured. Let $\{P_i\}_{i\in I}$ be a family of submodules of M and K a maximal submodule of M such that $\bigcap_{i\in I} P_i \subseteq K$. We have $P_j + K \neq M$ for some $j \in I$. Thus $P_j \subseteq K$. Conversely, assume that the property (*) is satisfied by any family of submodules of M. Further, assume, for any family $\{P_i\}_{i\in I}$ of prime submodules and any maximal submodule K of M, that the inclusion $\bigcap_{i\in I} P_i \subseteq K$ implies $P_j \subseteq K$ for some $j \in I$. Let $\{N_\alpha\}_{\alpha\in A}$ be a family of submodules of M and P a prime submodule of M such that $\bigcap_{\alpha\in A} N_\alpha \subseteq P$. Then $\text{rad}(\bigcap_{\alpha\in A} N_\alpha) \subseteq \text{rad}(P) = P$. As M is a multiplication module, P is contained in a maximal submodule L of M. Since, by assumption, the property (*) is satisfied by $\{N_\alpha\}_{\alpha\in A}$, using Lemma 4.4, we obtain $\bigcap_{\alpha\in A} \text{rad}(N_\alpha) = \text{rad}(\bigcap_{\alpha\in A} N_\alpha) \subseteq P \subseteq L$. Besides, for each $\alpha \in A$,

$$\text{rad}(N_\alpha) = \bigcap_{\substack{\beta\in B \\ N_\alpha \subseteq P_{\beta,\alpha}}} P_{\beta,\alpha}$$

for some family $\{P_{\beta,\alpha}\}_{\beta\in B}$ of prime submodules of M. Therefore,

$$\bigcap_{\substack{(\alpha,\beta)\in A\times B\\ N_\alpha\subseteq P_{\beta,\alpha}}} P_{\beta,\alpha} = \bigcap_{\alpha\in A}\ \bigcap_{\substack{\beta\in B\\ N_\alpha\subseteq P_{\beta,\alpha}}} P_{\beta,\alpha} = \bigcap_{\alpha\in A}\mathrm{rad}(N_\alpha)\subseteq L.$$

Then, by assumption, we have $P_{\lambda,\kappa}\subseteq L$ for some $(\lambda,\kappa)\in A\times B$. This implies $P_{\lambda,\kappa}+L\neq M$ for some $(\lambda,\kappa)\in A\times B$. Hence, for some $\lambda\in A$, we have $N_\lambda+P\subseteq P_{\lambda,\kappa}+P\subseteq P_{\lambda,\kappa}+L\neq M$. Thus M is coprimely structured. □

Theorem 4.6. *Let M be a multiplication R-module. Assume that the property (*) is satisfied for any family of submodules of M. Let N be a submodule of M which is contained in* rad(0). *Then M/N is coprimely structured if and only if M is coprimely structured.*

Proof. Assume that M/N is coprimely structured. Let $\{P_i\}_{i\in I}$ be a family of prime submodules of M and K a maximal submodule of M satisfying $\bigcap_{i\in I}P_i\subseteq K$. Then, as $N\subseteq \mathrm{rad}(0)$, we obtain

$$\bigcap_{i\in I}P_i/N=(\bigcap_{i\in I}P_i)/N\subseteq K/N.$$

Since K is maximal, K/N is maximal in M/N. Then, for some $j\in I$, we have $P_j/N+K/N\neq M/N$. This implies $P_j/N\subseteq K/N$. Therefore, for some $j\in I$ the inclusion $P_j\subseteq K$ holds. Using Theorem 4.5 we conclude that M is coprimely structured. The converse follows from Corollary 2.4. □

Theorem 4.7. *Let M be a finitely generated faithful multiplication R-module. M is coprimely structured if and only if R is coprimely structured.*

Proof. Assume that M is a coprimely structured module. Let $\{I_\alpha\}_{\alpha\in A}$ be a family of ideals of R and P a prime ideal of R satisfying $\bigcap_{\alpha\in A}I_\alpha\subseteq P$. Then, we have $\bigcap_{\alpha\in A}(I_\alpha M)=\left(\bigcap_{\alpha\in A}I_\alpha\right)M\subseteq PM$. Since M is coprimely structured, there exists $\beta\in A$ such that $(I_\beta+P)M=I_\beta M+PM\neq M$. Therefore, by [1, 3.1], we obtain $I_\beta+P\neq R$, and hence R is coprimely structured. Conversely, assume that R is coprimely structured. Let $\{N_\lambda\}_{\lambda\in L}$ be a family of submodules of M and Q' a prime submodule of M. Suppose $\bigcap_{\lambda\in L}N_\lambda\subseteq Q'$. Since M is a multiplication module there exist a family $\{I_\lambda\}_{\lambda\in L}$ of ideals of R and a prime ideal Q of R such that $N_\lambda=I_\lambda M$, for all $\lambda\in L$, and $Q'=QM$, by [1, 2.11]. Then, using [1, 1.6], we have $(\bigcap_{\lambda\in L}I_\lambda)M=\bigcap_{\lambda\in L}(I_\lambda M)=\bigcap_{\lambda\in L}N_\lambda\subseteq Q'=QM$, and by [1, 3.1], we obtain $\bigcap_{\lambda\in L}I_\lambda\subseteq Q$. Since R is coprimely structured, there exists $\kappa\in L$ such that $I_\kappa+Q\neq R$. Then, using [1, 3.1], we conclude that $N_\kappa+Q'=(I_\kappa M+P)M\neq M$. Thus M is coprimely structured. □

Theorem 4.8. *Every Artinian multiplication module is coprimely structured.*

Proof. Every Artinian multiplication module is strongly 0-dimensional by [10, 2.6]. The result follows from Theorem 2.5. □

5 The Property (*)

In [6], Brewer and Richman give some characterizations of zero-dimensional rings. Here we generalize some of these results under certain conditions. In particular, if R is a principal ideal ring and M is a finitely generated faithful multiplication R-module, the property (*) we introduced in Section 4 can be used to determine whether M is zero-dimensional or not. Before giving that result we need some lemma. R is assumed to be a principal ideal ring in the following.

Lemma 5.1. *Let M be a finitely generated multiplication R-module and m an element of M such that $Rm=IM$. The following conditions are equivalent:*

(i) There exists an $n\in\mathbb{N}$ such that $I^nM=I^{n+1}M$.

(ii) $IM+\bigcup_{n=1}^{\infty}(0:_M I^n)=M$.

Proof. (i)⇒(ii) Suppose that $I^nM = I^{n+1}M$ for some $n \in \mathbb{N}$. Since R is a principal ideal ring there exists an $r \in R$ such that $I = (r)$. Then $r^nM = r^{n+1}M$. That means for each $m \in M$ there exists a $m' \in M$ such that $r^nm = r^{n+1}m'$. Then $m - rm' \in (0 :_M r^n) = (0 :_M I^n)$. Hence we have $M \subseteq IM + \bigcup_{n=1}^{\infty}(0 :_M I^n)$. The result follows.

(ii)⇒(i) Assume that $IM + \bigcup_{n=1}^{\infty}(0 :_M I^n) = M$ holds. Since I is a principal ideal, $I = (x)$ for some $x \in R$. Then we have $xM + \bigcup_{n=1}^{\infty}(0 :_M x^n) = M$. Let $\{a_1, a_2, ..., a_n\}$ be a generator set for M. Then, by assumption, we have

$$a_i \in xM + \bigcup_{n=1}^{\infty}(0 :_M x^n)$$

for each $i \in \{1, 2, ..., n\}$. Then, for each $i \in \{1, 2, ..., n\}$, there exists $m_i \in M$ and $n_i \in (0 : x^{k_i})$, $k_i \in \mathbb{N}$ such that $a_i = xm_i + n_i$. Set $k = max\{k_1, ..., k_n\}$. Then

$$x^ka_i = x^{k+1}m_i + x^kn_i = x^{k+1}m_i \in x^{k+1}M.$$

Hence we have $x^kM \subseteq x^{k+1}M$. The other inclusion is always true. Therefore we obtain $x^kM = x^{k+1}M$, that is $I^kM = I^{k+1}M$. □

Theorem 5.2. *A finitely generated faithful multiplication R-module M is zero-dimensional if and only if one of the conditions of Lemma 5.1 are satisfied for every $m \in M$.*

Proof. Suppose that the condition (ii) of Lemma 5.1 is not satisfied for some $m \in M$. Then $IM + \bigcup_{n=1}^{\infty}(0 :_M I^n)$ is a proper submodule of M, and hence it is contained in a prime submodule Q' of M. Then the ideal $Q = (Q' : M)$ is a prime ideal of R. Since R is a principal ideal ring, we have $I = (a)$ for some $a \in R$. Set $S := \{a^nr : n \in \mathbb{N}, r \in R\backslash Q\}$. Assume $0 \in S$. Then there exists an $r \in R\backslash Q$ such that $a^nr = 0$. Let $x \in M$. Then we have $rx \in (0 :_M a^n) = (0 :_M I^n) \subseteq Q'$. As $r \notin Q = (Q' : M)$ and Q' is prime, we conclude that $x \in Q'$, and hence $M \subseteq Q'$, a contradiction. Hence $0 \notin S$. Then there exists a prime ideal P of R such that $P \cap S = \emptyset$. Since $R\backslash Q \subseteq S$ we have $P \subseteq Q$. Besides $aM \subseteq Q'$, and hence $a \in Q$. However, $a \notin P$ since $a \in S$. Therefore P is a proper ideal of Q. Then, by [1, 3.1], PM is a proper submodule of QM. Thus M is not zero-dimensional.

Conversely assume that M is not zero-dimensional. Then there exist prime submodules P and Q of M such that $P \subset Q$. Let $m \in Q\backslash P$. Then $Rm = IM$ for some ideal I of R. Suppose $\bigcup_{n=1}^{\infty}(0 :_M I^n) \nsubseteq P$. Then there exists an $x \in M$ such that $I^nx = 0$ for some $n \in \mathbb{N}$ and $x \notin P$. Since $I = (b)$ for some $b \in R$, we have $b^nx = 0 \in P$. As P is prime, we have $b^n \in (P : M)$. Then $b \in (P : M)$. Hence we obtain $m \in Rm = bM \subseteq P$, a contradiction. Therefore $\bigcup_{n=1}^{\infty}(0 :_M I^n) \subseteq P \subset Q$. Besides $IM = Rm \subseteq Q$. Hence $IM + \bigcup_{n=1}^{\infty}(0 :_M I^n) \subseteq Q \neq M$. This is the contrapositive of the condition (ii) of Lemma 5.1. □

Theorem 5.3. *Let M be a finitely generated faithful multiplication R-module. The following conditions are equivalent:*

(i) M is zero-dimensional.

(ii) Property (*) holds for the family of all submodules of M.

(iii) Property (*) holds for the family of all primary submodules of M.

Proof. (i)⇒(ii) Suppose that M is zero-dimensional. Then by 5.2, for each $m \in M$ there exists an $n \in \mathbb{N}$ such that $I^nM = I^{n+1}M$, where $Rm = IM$. Observe that the equality $I^{n+t}M = I^nM$ holds for all $t \in \mathbb{N}$. Let N be a submodule of M. If $x \in \text{rad}N$, then there exists a $k \in \mathbb{N}$ such that $J^kM \subseteq N$ where $Rx = JM$. If $n > k$ then $J^nM = J^{n-k}(J^kM) \subseteq J^{n-k}N \subseteq N$. If $n \leq k$ then $J^nM = J^kM \subseteq N$. In both cases we have $x^n = J^nM \subseteq N$. Therefore the property (*) holds for the family of all submodules of M.

(ii)⇒(iii) Trivial.

(iii)⇒(i) Suppose that the property (*) holds for the family of all primary submodules of M and M is not zer-dimensional. Assume that P is a prime submodule of M that is not maximal. Let $x \in M$. Since M is a multiplication module there is an ideal I of R such that $Rx = IM$. Let P be a minimal prime submodule of IM. For each $n \in \mathbb{N}$, define

$$Q_n = \{m \in M : \ sm \in I^nM \text{ for some } s \in R\backslash(P : M)\}.$$

Set $Q = (P : M)$. Then, by [9, 6], M_Q is a local module. Hence P_Q is the unique maximal submodule of M_Q. Observe that P is also a minimal prime submodule of $I^n M$. Then we have $\text{rad}(I^n M)_Q = P_Q$. Thus,

$$\text{rad}Q_n = \text{rad}((I^n M)_Q \cap R) = (\text{rad}(I^n M)_Q) \cap R = P_Q \cap R = P,$$

and hence we obtain

$$(P : M)M = P = \text{rad}Q_n = \text{rad}((Q_n : M)M) = \sqrt{(Q_n : M)}M.$$

Since M is a finitely generated faithful multiplication module, by [1, 3.1], we conclude that $\sqrt{(Q_n : M)} = (P : M)$. Now, let $r \in R$, $m \in M$ such that $rm \in Q_n$. Then there exists $s \in R\backslash(P : M)$ such that $srm \in I^n M$. If $r \notin \sqrt{(Q_n : M)}$, then $sr \in R\backslash(P : M)$. Then since $srm \in I^n M$ we obtain $m \in Q_n$. Therefore Q_n is $(P : M)$-primary. Then $\{Q_n\}_n \in \mathbb{N}$, is a family of primary submodules of M. Observe that $x \in P = \bigcap_{n\in\mathbb{N}} \text{rad}(Q_n)$. We are to show that $x \notin \text{rad}(\bigcap_{n\in\mathbb{N}} Q_n)$. Assume, on the contrary, that $x \in \text{rad}(\bigcap_{n\in\mathbb{N}} Q_n)$. Then for some $k \in \mathbb{N}$ we have $x^k \subseteq \bigcap_{n\in\mathbb{N}} Q_n$. In particular, $x^k \subseteq Q_{k+1}$. Note that I is a principal ideal, hence there exists $a \in R$ such that $I = (a)$. Since $P \neq M$ there exists $m \in M\backslash P$ and $s \in R\backslash(P : M)$ such that $sa^k m = a^{k+1} m'$ for some $m' \in M$. Since M is torsion-free, we conclude that $sm = am' \in IM \subseteq P$ and this contradicting our choice $m \in M\backslash P$. Hence we must have $x \notin \text{rad}(\bigcap_{n\in\mathbb{N}} Q_n)$. Therefore we obtain a family of primary submodules Q_n, $n \in M$, of M for which the property (*) does not hold. □

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