Palestine Journal of Mathematics

Vol. 7(Special Issue: I, 2018) , 18–22

ON COFINITELY WEAK* RAD$-\oplus-$SUPPLEMENTED MODULES

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Communicated by Mohammad Ashraf

MSC 2010 Classifications: Primary 16D10; Secondary 16D80.

Keywords and phrases: Rad$-\oplus-$supplemented module, weak* Rad$-\oplus-$supplemented module, cofinitely weak* Rad$-\oplus-$supplemented module.

Abstract In this paper we introduce the idea of cofinitely weak* Rad$-\oplus-$supplemented module as a generalization of weak* Rad$-\oplus-$supplemented module. Some relevant counter examples are given to distinguish these structure of modules. We establish several properties of cofinitely weak* Rad$-\oplus-$supplemented module related with $w-$local modules. Finally, we prove that the class of cofinitely weak* Rad$-\oplus-$supplemented modules is closed under arbitrary direct sums.

1 Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary left $R-$modules, unless otherwise specified. Let M be an $R-$module. A submodule N of M denoted by $N \subseteq M$ and $Rad(M)$ will indicate the Jacobson radical of M. A submodule N of a module M is called small in M (denoted by $N \ll M$), if $M \neq N + K$ for every proper submodule K of M. A non zero module M is said to be hollow if every proper submodule of M is small in M, and it is said to be local if the sum of all the proper submodules of M is also a proper submodule of M, equivalently M is hollow and finitely generated. A non zero module M is said to be $w-$local, if it has a unique maximal submodule (cf. [4]).

A module M is said to have property (p^*), if for every submodule N of M, there exists a direct summand K of M such that $K \subseteq N$ and $N/K \subseteq Rad(M/K)$ (cf. [3]). Recall that a module M is called radical if M has no maximal submodule i.e., $RadM = M$ (cf. [6]). For a module $M, P(M)$ will indicate the sum of all radical submodules of M. Note that $P(M)$ is the largest radical submodule of M.

If N and L are submodules of M, then N is called a supplement of L , if $N + L = M$ and $N \cap L \ll N$. A module M is called supplemented if each of its submodules has a supplement in M. A module M is called $\oplus-$supplemented (completely $\oplus-$supplemented) if every submodule (direct summand) of M has a supplement that is a direct summand of M (cf. [5, 7, 9]). A submodule N of a module M is called cofinite if M/N is finitely generated and a module M is called cofinitely supplemented if every cofinite submodule of M has a supplement in M (cf. [2, 7]). A submodule N of a module M has a Rad-supplement K in M if $N + K = M$ and $N \cap K \subseteq RadK$. A module M is called Rad-supplemented if every submodule of M has a Rad-supplement (cf. [5, 7]). M is called Rad$-\oplus-$supplemented if every submodule of M has a Rad-supplement that is a direct summand of M. The $\mathbb{Z}-$module $\mathbb{Q}$ is Rad$-\oplus-$supplemented but not $\oplus-$supplemented. Every module with (p^*) is Rad$-\oplus-$supplemented. A module M is called completely Rad$-\oplus-$supplemented if every direct summand of M is Rad$-\oplus-$supplemented (cf. [5]). Recall that a module M is called weak* Rad$-\oplus-$supplemented if every semi-simple submodule of M has a Rad-supplement that is a direct summand of M (cf. [5]).

Motivated by the above notions, we introduce a new concept known as cofinitely weak* Rad$-\oplus-$supplemented module as a generalization of the class of Rad$-\oplus-$supple-mented modules and weak* Rad$-\oplus-$supplemented modules.

Remark 1.1. Let K and N be submodules of a module M with $K \subseteq N \subseteq M$. Then

(i) if K is a cofinite submodule of N and N is a cofinite submodule of M, then K is also a cofinite submodule of M (transitive property).

(ii) N is a cofinite submodule of M if and only if N/K is a cofinite submodule of M/K.

2 Cofinitely weak* Rad$-\oplus-$supplemented Modules

Definition 2.1. An $R-$module M is called a cofinitely weak* Rad$-\oplus-$supplemented module if every cofinite semi simple submodule of M has a Rad-supplement that is a direct summand of M.

For example, hollow modules and modules with (p^*) are cofinitely weak* Rad$-\oplus-$supplemented modules. Clearly, every Rad$-\oplus-$supplemented module is a weak* Rad$-\oplus-$supplemented module and weak* Rad$-\oplus-$supplemented module is a cofinitely weak* Rad$-\oplus-$supplemented module but the converses are not true in general. Thus we have the following implications:

$\oplus-$supplemented $\Rightarrow$ Rad$-\oplus-$supplemented $\Rightarrow$ weak* Rad$-\oplus-$supplemented $\Rightarrow$ cofinitely weak* Rad$-\oplus-$supplemented.

Example 2.2. (1). Every $\oplus-$supplemented module is Rad$-\oplus-$supplemented module, but the converse is not always true. For example, let M be a non torsion module over $\mathbb{Z}$ with $RadM = M$. Then M is Rad$-\oplus-$supplemented but not supplemented. Consider the $\mathbb{Z}-$module $M = \mathbb{Q} \oplus \mathbb{Z}/p\mathbb{Z}$ for any prime p. Then M has a unique maximal submodule, $RadM \neq M$, i.e., M is $w-$local so M is Rad$-\oplus-$supplemented but not $\oplus-$supplemented. On the other hand if M is $\oplus-$supplemented, then $\mathbb{Q}$ is supplemented which is not true because $\mathbb{Q}$ is not torsion.
(2). Let R be a non local Dedekind domain with quotient field K. Then the module K is Rad$-\oplus-$supplemented but not $\oplus-$supplemented. On the other hand if K is $\oplus-$supplemented, then R is a local ring which contradicts our assumption.
(3). (cf. [7, Example 2.15(1)]), Let R be a local commutative ring having a radical module K (e.g., R is a discrete valuation ring with quotient field K). Then there exists a free module F and a semi simple submodule X of F such that $F/X \cong K$. Suppose that there is a direct summand Y of F such that Y is a weak* Rad$-\oplus-$supplement of X in F. By (cf. [8, Theorem 1]), Y is a direct sum of a local module. It follows that $Rad(Y) \neq Y$. On the other hand we have $F/X \cong Y/(X \cap Y) \cong K$, so $Y/(X \cap Y)$ has no maximal submodule. But by definition of weak* Rad$-\oplus-$supplement $(X \cap Y) \subseteq Rad(Y)$. Then $Y/Rad(Y)$ has no maximal submodules. Since $Y/Rad(Y)$ is semi simple, we get $Y = Rad(Y)$, a contradiction. Therefore, F is not weak* Rad$-\oplus-$supplemented. However, it is easily seen that if N is a proper cofinite semi simple submodule of F, then there exist local direct summands $K_1, K_2, ...K_r$ of F such that $K_1 + K_2 + ... + K_r$ is direct and a direct summand of F, so $F = N + K_1 + K_2 + ... + K_r$ and this sum is irredundant. Hence $K_1 + K_2 + ... + K_r$ is a weak* Rad$-\oplus-$supplement of N in F by Proposition 2.13. Consequently, F is cofinitely weak* Rad$-\oplus-$supplemented but not weak* Rad$-\oplus-$supplemented.

Proposition 2.3. *Let M be a weak* Rad$-\oplus-$supplemented module. A cofinite fully invariant submodule N of M is weak* Rad$-\oplus-$supplemented if it is a direct summand of M.*

Proof. Let K be any submodule of M contained inside N with $Rad(N) \subseteq K$. By assumption $M = N \oplus H$ for some finitely generated sumodule H of M. Thus $Rad(H) \ll H$. Clearly $Rad(M) \subseteq K + Rad(H)$. Since M is a weak* Rad$-\oplus-$supplemented module, then for semi simple submodule L of M there exists $G \subseteq M$ such that $M = K + Rad(H) + L$, $(K + Rad(H)) \cap L \subseteq Rad(L)$ and $M = L \oplus G$. Since $Rad(H) \ll H$, we have $M = K + L, K \cap L \subseteq Rad(L)$ and $M = L \oplus G$. It follows that $N = K + (L \cap N)$ and $K \cap (L \cap N) \subseteq Rad(L \cap N)$. As N is a fully invariant submodule of M, we have $N = (L \cap N) \oplus (G \cap N)$ and $K \cap (L \cap N) \subseteq Rad(L \cap N)$. Therefore, N is weak* Rad$-\oplus-$supplemented.

Proposition 2.4. *For a finitely generated semi simple module M, the following statements are equivalent:*
(i) *M is $\oplus-$supplemented;*
(ii) *M is Rad$-\oplus-$supplemented;*
(iii) *M is weak* Rad$-\oplus-$supplemented;*
(iv) *M is cofinitely weak* Rad$-\oplus-$supplemented.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are clear. $(iv) \Rightarrow (i)$ assume that M is cofinitely weak* Rad$-\oplus-$supplemented and N be any submodule of M, so N is a direct summand and hence a semi simple submodule of M. Since M is finitely generated, N is a cofinite submodule of M. By assumption, there exists a direct summand K of M such that $M = N + K$ and $(N \cap K) \subseteq Rad(K)$. K is finitely generated since M is finitely generated, so $Rad(K) \subseteq K$. Thus we get $(N \cap K) \ll K$. Hence K is a supplement of N in M. Therefore, M is $\oplus-$supplemented.

Lemma 2.5. *Every radical module M is cofinitely weak* Rad$-\oplus-$supplemented.*

Proof. Let N be a cofinite semi simple submodule of M. Then M/N is finitely generated. As M is a radical, $N \subseteq Rad(M) = M$. Thus M is a trivial weak* Rad$-\oplus-$supplement of N in M. Hence M is cofinitely weak* Rad$-\oplus-$supplemented.

Corollary 2.6. *The largest radical submodule $P(M)$ of a module M is cofinitely weak* Rad$-\oplus$ $-$supplemented.*

Remark 2.7. It is easily seen that a module M is $w-$local if and only if $Rad(M)$ is a maximal submodule of M. Also, direct summand of a $w-$local module M is either a radical or $w-$local.

Lemma 2.8. *Every $w-$local module M is cofinitely weak* Rad$-\oplus-$supplemented.*

Proof. Let N be a cofinite semi simple submodule of M. Then M/N is finitely generated. By definition of $w-$local module, $Rad(M)$ is the unique maximal submodule of M, i.e., $M/Rad(M) \subseteq M/N$ which gives $N \subseteq Rad(M)$. Thus M is a trivial weak* Rad$-\oplus-$supplement of N in M. Hence M is cofinitely weak* Rad$-\oplus-$supplemented.

Corollary 2.9. *Every direct summand of a $w-$local module M is cofinitely weak* Rad$-\oplus$ $-$supplemented.*

Proof. As mentioned in Remark 2.7, the direct summand of a $w-$local module M is either a radical or $w-$local. Applying Lemma 2.5 and Lemma 2.8, we get the required result.

Proposition 2.10. *Let M be an indecomposable module. Then M is cofinitely weak* Rad$-\oplus$ $-$supplemented if and only if $M = Rad(M)$ or M is $w-$local.*

Proof. Assume that M is cofinitely weak* Rad$-\oplus-$supplemented with $M \neq Rad(M)$. Let N be a semi simple maximal submodule of M. Then by assumption, there exists a direct summand K of M such that $M = N + K$ and $(N \cap K) \subseteq Rad(K)$. Since M is indecomposable, we have $K = M$ which implies that $N \ll M$, i.e., $N \subseteq Rad(M)$ and hence $N = Rad(M)$ is the unique maximal submodule of M. Therefore, M is a $w-$local module. The converse is clear by Lemma 2.5 and Lemma 2.8.

Proposition 2.11. *Let K be a $w-$local direct summand of a module M. Then K is a weak* Rad$-\oplus-$supplement of N in M, where N is a proper cofinite semi simple submodule of M with $K + N = M$.*

Proof. Assume that N is a proper cofinite semi simple submodule of M such that $K + N = M$. Then $M/N = (K + N)/N$ is finitely generated. We know that $(K + N)/N \cong K/(K \cap N)$, so $K/(K \cap N) \neq 0$. Since K is a $w-$local direct summand of a module M, it has a unique maximal submodule $Rad(K)$ of K. Also $K/(K \cap N)$ has a maximal sumodule; hence it follows that $(K \cap N) \subseteq Rad(K)$. Therefore, K is a weak* Rad$-\oplus-$supplement of N in M.

Lemma 2.12. *Let K, L and N be semi simple submodules of a module M such that $K+L+N = M$. If K is a weak* Rad$-\oplus-$supplement of $L+N$ in M and L is a weak* Rad$-\oplus-$supplement of $K + N$ in M, then $K + L$ is a weak* Rad$-\oplus-$supplement of N in M.*

Proof. Assume that K is a weak* Rad$-\oplus-$supplement of $L + N$ in M; so, $K \cap (L + N) \subseteq Rad(K)$ and L is a weak* Rad$-\oplus-$supplement of $K + N$ in M so, $L \cap (K + N) \subseteq Rad(L)$. Since $(K + L) \cap N \subseteq [K \cap (N + L)] + [L \cap (N + K)]$, we have $(K + L) \cap N \subseteq Rad(K) + Rad(L) \subseteq Rad(K+L)$. Thus $(K+L) \cap N \subseteq Rad(K+L)$, which shows that $K+L$ is a weak* Rad$-\oplus-$supplement of N in M.

Proposition 2.13. *Let $K_1, K_2, ...K_r$ be $w-$local direct summands of a module M. Then for every proper cofinite semi simple submodule N of M such that $M = N + K_1 + K_2 + ... + K_r$ and $M \neq N + \sum_{i\neq j} K_i$ for each $1 \leq j \leq r$, then $K_1 + K_2 + ... + K_r$ is a weak* Rad$-\oplus-$supplement of N in M.*

Proof. The proof follows by repeated application of Proposition 2.11 and Lemma 2.12, because every submodule of M which contains N is a cofinite submodule of M.

Proposition 2.14. *Let M be a cofinitely weak* Rad$-\oplus-$supplemented module. If M contains a maximal semi simple submodule, then M contains a $w-$local direct summand.*

Proof. Assume that M is a cofinitely weak* Rad$-\oplus-$supplemented module and let N be a maximal semi simple submodule M. There exists a direct summand K of M such that $M = N + K$ and $(N \cap K) \subseteq Rad(K)$. Since N is a maximal submodule, $M/N = (N + K)/N \cong K/(N \cap K)$ is a simple module. It means $(N \cap K)$ being a maximal submodule of K, $Rad(K) \subseteq (N \cap K)$ which implies that $Rad(K) = (N \cap K)$, i.e., K is $w-$local. Therefore, K is a $w-$local direct summand of M.

Corollary 2.15. *Let M be a cofinitely weak* Rad$-\oplus-$supplemented module with $Rad(M) \ll M$. Then M contains a local direct summand.*

Proposition 2.16. *Finite direct sum of cofinitely weak* Rad$-\oplus-$supplemented modules is a cofinitely weak* Rad$-\oplus-$supplemented.*

Proof. For the proof of this result we will prove the result for only two cofinitely weak* Rad$-\oplus-$supplemented modules, which can be extended to n(finitely many) cofinitely weak* Rad$-\oplus-$supplemented module by induction. Let M_1 and M_2 be cofinitely weak* Rad$-\oplus-$supplemented modules and L be a semi simple submodule of $M = M_1 \oplus M_2$. Then $M = M_1 + M_2 + L$ has a trivial Rad$-\oplus-$supplement 0 in M. Since $M_2 \cap (M_1 + L)$ is a cofinite semi simple submodule of M_2, by assumption there exists a direct summand H of M_2 such that $M_2 = [M_2 \cap (M_1 + L)] + H$ and $(M_1 + L) \cap H \subseteq Rad(H)$. By (cf. [5, Lemma 7]), H is a weak* Rad$-\oplus-$supplement of $(M_1 + L)$ in M, i.e., $M = (M_1 + L) + H$. Since $M_1 \cap (L + H)$ is a cofinite semi simple submodule of M_1, by assumption, there exists a direct summand K of M_1 such that $M_1 = [M_1 \cap (L + H)] + K$ and $(L + H) \cap K \subseteq Rad(K)$. By (cf. [5, Lemma 7]), K is a weak* Rad$-\oplus-$supplement of $(H + L)$ in M, i.e., $M = (H + L) + K = L + (H + K)$ and $L \cap (H + K) \subseteq [H \cap (L + K)] + [K \cap (L + H)] \subseteq Rad(H) \oplus Rad(K) \subseteq Rad(H \oplus K)$, which shows that $H + K$ is a Rad$-\oplus-$supplement of L in M. Moreover, $H \oplus K$ is a direct summand of $M = M_1 \oplus M_2$. Therefore, $M = M_1 \oplus M_2$ is a cofinitely weak* Rad$-\oplus-$supplemented module.

Corollary 2.17. *Any direct sum of cofinitely weak* Rad$-\oplus-$supplemented modules is cofinitely weak* Rad$-\oplus-$supplemented.*

Proof. Let $\{M_i | i \in I\}$ be a family of cofinitely weak* Rad$-\oplus-$supplemented modules. We claim that $M = \bigoplus_{i\in I} M_i$ is cofinitely weak* Rad$-\oplus-$supplemented. Let L be a cofinite semisimple submodule of M. Then $M = L + \bigoplus_{r=1}^{n} M_{i_r}$ for a finite subset $\{i_1, i_2, ...i_r\}$ of index set I. Since $\bigoplus_{r=1}^{n} M_{i_r}$ is cofinitely weak* Rad$-\oplus-$supplemented (see Proposition 2.16) and $[\bigoplus_{r=1}^{n} M_{i_r}]/(L \cap [\bigoplus_{r=1}^{n} M_{i_r}])$ is finitely generated, there exists a direct summand K of $[\bigoplus_{r=1}^{n} M_{i_r}]$ such that $L \cap [\bigoplus_{r=1}^{n} M_{i_r}] + K = \bigoplus_{r=1}^{n} M_{i_r}$ and $(K \cap L) \subseteq Rad(K)$. As $K + L = M$, K is a weak* Rad$-\oplus-$supplement of N in M. Therefore, $M = \bigoplus_{i\in I} M_i$ is cofinitely weak* Rad$-\oplus-$supplemented.

Corollary 2.18. *Any direct sum of $w-$local modules is cofinitely weak* Rad$-\oplus-$supplemented.*

Proof. The proof follows from Lemma 2.8 and Corollary 2.17.

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