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A note on generalized derivations of prime and semiprime rings

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Abstract In this manuscript, we investigate the differential identities on (semi)-prime rings involving generalized derivations. Further, we obtain the structure of rings and information about the form of generalized derivations on prime rings in terms of the multiplication by the specific element from the extended centroid

1 Preliminaries and Motivation

Throughout this paper, $\mathcal{R}$ is a (semi)-prime ring with the center $\mathcal{Z}(\mathcal{R})$, $\mathcal{Q}$ is the Martindale quotient ring of $\mathcal{R}$ and $\mathcal{U}$ is the Utumi quotient ring of $\mathcal{R}$. The center of $\mathcal{U}$ denoted by C is called the extended centroid of $\mathcal{R}$. For more details we refer to the reader [3]. For any $x, y \in \mathcal{R}$, the symbol $[x, y]$ and $x \circ y$ denote the commutator $xy - yx$ and anti-commutator $xy + yx$ respectively. Given $x, y \in \mathcal{R}$ we set $x \circ_0 y = x$, $x \circ_1 y = x \circ y = xy + yx$, and inductively $x \circ_m y = (x \circ_{m-1} y) \circ y$ for $m > 1$.

Let us remind some basic notations and definitions for the sake of completeness. A ring $\mathcal{R}$ is said to be prime if $x\mathcal{R}y = (0)$ implies that $x = 0$ or $y = 0$ and $\mathcal{R}$ is semiprime ring if $x\mathcal{R}x = (0)$ implies that $x = 0$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in \mathcal{R}$. In particular, d is an inner derivation induced by an element $q \in \mathcal{R}$, if $d(x) = [q, x]$ holds for all $x \in R$. An additive map $F : R \to R$ is called a generalized derivation associated with a derivation $d : \mathcal{R} \to \mathcal{R}$ if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in \mathcal{R}$. Familiar examples of generalized derivations are derivations and the map $F : \mathcal{R} \to \mathcal{R}$ of the from $F(x) = ax + xb$ for fixed $a, b \in \mathcal{R}$, is known generalized inner derivation of $\mathcal{R}$.

Let us introduce the brief background of our motivation. In 2002, Ashraf and Rehman [1, Theorem 4.1] studied about the structure of prime rings satisfying skew commutator identity involving derivation. More precisely, they proved that: "*If $\mathcal{R}$ is a prime ring, I is a nonzero ideal of $\mathcal{R}$ and d is a derivation of $\mathcal{R}$ such that $d(x \circ y) = (x \circ y)$ for all $x, y \in I$, then $\mathcal{R}$ is commutative*". Further, Quadri et al. [16, Theorem 2.3] discussed the commutativity of prime rings involving generalized derivations and they studied $F(x \circ y) = (x \circ y)$. Namely, they demonstrated that: "*If $\mathcal{R}$ is a prime ring, I is a nonzero ideal of $\mathcal{R}$ and F is a generalized derivation associated with a nonzero derivation d of $\mathcal{R}$ such that $F(x \circ y) = (x \circ y)$ for all $x, y \in I$, then $\mathcal{R}$ is commutative*". In 2009, Argac and Inceboz [2, Theorem 1] generalized Ashraf and Rehman [1, Theorem 4.1] result by investigating the differential identity $d(x \circ y)^n = (x \circ y)$. More exactly, they established the following: "*Let $\mathcal{R}$ be a prime ring, I be a nonzero ideal of $\mathcal{R}$, d be a derivation of $\mathcal{R}$ and n be a fixed positive integer. Suppose that $\mathcal{R}$ satisfies $d(x \circ y)^n = (x \circ y)$ for all $x, y \in I$, then $\mathcal{R}$ is commutative*". In 2012, Huang [9] extended Quadri et al. [16, Theorem 2.3] result and he proved that: "*if $\mathcal{R}$ is a prime ring, I is a nonzero ideal of $\mathcal{R}$, n a fixed positive integer and F a generalized derivation with associated nonzero derivation d such that $F(x \circ y)^n = x \circ y$ for all $x, y \in I$, then $\mathcal{R}$ is commutative*". Further more related results see [5, 17, 18, 19, 20] and references therein.

The present paper is motivated by the above mentioned results and we continue this line of investigation by examining the following identity involving $F(x \circ_k y)^n = (x \circ_k y)$. In particular, we generalized the Huang [9] results and proved it for more general case.

2 Main Results

In order to prove our main results in this section, we begin with the following remark:

Remark 2.1. ([4, Lemma 7.1]). Let ${}_{\mathcal{D}}\mathcal{V}$ be a left vector space over a division ring $\mathcal{D}$ with $dim_{\mathcal{D}}\mathcal{V} \geq 2$ and $\mathcal{T} \in End(\mathcal{V})$. If x and $x\mathcal{T}$ are $\mathcal{D}$-dependent for every $x \in \mathcal{V}$, then there exists $\lambda \in \mathcal{D}$ such that $x\mathcal{T} = \lambda x$ for all $x \in \mathcal{M}$.

Now, we facilitate our discussion with the following theorem:

Theorem 2.2. *Let $\mathcal{R}$ be a prime ring with characteristic different from 2, I be a nonzero ideal of $\mathcal{R}$ and $n, k \in \mathbb{Z}^+$. If $\mathcal{R}$ admits a generalized derivation F associated with a derivation d such that $F(x \circ_k y)^n = (x \circ_k y)$ for all $x, y \in I$, then either $\mathcal{R}$ is commutative or $d = 0$ and there exists $q \in C$, extended centroid of $\mathcal{R}$ such that $F(x) = qx$ for $x \in \mathcal{R}$.*

Proof. By the given hypothesis and Lee [14, Theorem 3], we can write

$$(a(x \circ_k y) + d(x \circ_k y))^n = (x \circ_k y) \tag{2.1}$$

for all $x, y \in I$, which can be rewritten as

$$\begin{aligned}(a(x \circ_k y) \quad &+ \sum_{m=0}^{k} \binom{k}{m} (\sum_{i+j=m-1} y^i d(y) y^j) x y^{k-m} \\ &+ \sum_{m=0}^{k} \binom{k}{m} y^m d(x) y^{k-m} \\ &+ \sum_{m=0}^{k} \binom{k}{m} y^m x (\sum_{r+s=k-m-1} y^r d(y) y^s)^n = (x \circ_k y)\end{aligned}$$

for all $x, y \in I$. Firstly, we assume that d is $\mathcal{U}$-outer derivation. By Kharchenko's theorem [11], I satisfies the generalized polynomial identity

$$\begin{aligned}(a(x \circ_k y) \quad &+ \sum_{m=0}^{k} \binom{k}{m} (\sum_{i+j=m-1} y^i z y^j) x y^{k-m} \\ &+ \sum_{m=0}^{k} \binom{k}{m} y^m w y^{k-m} \\ &+ \sum_{m=0}^{k} \binom{k}{m} y^m x (\sum_{r+s=k-m-1} y^r z y^s))^n = (x \circ_k y)\end{aligned}$$

for all $x, y, z, w \in I$. In particular, I satisfies $\left(\sum_{m=0}^{k} \binom{k}{m} y^m w y^{k-m}\right)^n = 0$. That is $(w \circ_k y)^n = 0$ for all $w, y \in I$. Using the same techniques as used in [17], we get the desired conclusion.

Secondly, we assume that d is $\mathcal{U}$-inner derivation, there exists a noncentral $q \in \mathcal{U}$ such that $d(x) = [q, x]$ for all $x \in \mathcal{R}$. Therefore from (2.1), we have

$$(a(x \circ_k y) + [q, x \circ_k y])^n = (x \circ_k y)$$

for all $x, y \in I$. By Chuang [6, Theorem 2], I and $\mathcal{U}$ satisfy the same generalized polynomial identity, thus we have $(a(x \circ_k y) + [q, x \circ_k y])^n = (x \circ_k y)$ for all $x, y \in \mathcal{U}$. By hypothesis, we have $((a + q)(x \circ_k y) - (x \circ_k y)q)^n = (x \circ_k y)$ for all $x, y \in \mathcal{U}$ as argued before. If, now, $a + q \in C$, then $((x \circ_k y)a)^n = (x \circ_k y)$ for all $x, y \in \mathcal{U}$. Since $q \notin C$, we have $a \notin C$, and thus the last identity is a nontrivial generalized polynomial identity (GPI) for $\mathcal{U}$. If, on the other hand, $a + q \notin C$, then the first identity above is the nontrivial GPI for $\mathcal{U}$. Therefore, in any case $\mathcal{U}$ is a prime GPI ring. We also note that, in the case when C is infinite field, we have

$((a+q)(x \circ_k y) - (x \circ_k y)q)^n = (x \circ_k y)$ for all $x, y \in \mathcal{U} \otimes_C \overline{C}$, where $\overline{C}$ is algebraic closure of C. Since both $\mathcal{U}$ and $\mathcal{U} \otimes_C \overline{C}$ are prime and centrally closed [8, Theorems 2.5 and 3.5], we may replace $\mathcal{R}$ by $\mathcal{U}$ or $\mathcal{U} \otimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that $\mathcal{R}$ is centrally closed over C (*i.e.*, $\mathcal{R}C = \mathcal{R}$) which is either finite or algebraically closed and $((a+q)(x\circ_k y)-(x\circ_k y)q)^n = (x\circ_k y)$ for all $x, y \in \mathcal{R}$. By Martindale [15, Theorem 3], $\mathcal{R}C$ (and so $\mathcal{R}$) is a primitive ring having nonzero socle H with $\mathcal{D}$ as the associated division ring. Hence by Jacobson's theorem [10, p.75], $\mathcal{R}$ is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over $\mathcal{D}$ and H consists of the finite rank linear transformations in $\mathcal{R}$. If $\mathcal{V}$ is a finite dimensional over $\mathcal{D}$, then the density of R on $\mathcal{V}$ implies that $R \cong \mathcal{M}_s(\mathcal{D})$, where $s = dim_{\mathcal{D}}\mathcal{V}$.

Assume first that $dim_{\mathcal{D}}\mathcal{V} \geq 3$ such that v and qv are linearly $\mathcal{D}$-independent for all $v \in \mathcal{V}$. By density of $\mathcal{R}$, there exists $u \in \mathcal{V}$ such that $v,\ qv,\ u$ are linearly $\mathcal{D}$-independent and $x,\ y \in \mathcal{R}$ such that

$$xv = v, \quad xqv = 0, \quad xu = -v$$
$$yv = 0 \quad yqv = u, \quad yu = u$$

This gives that

$$v = ((a+q)(x \circ_k y) - (x \circ_k y)q)^n - (x \circ_k y) = 0$$

a contradiction. So, we conclude that $\{v, qv\}$ is linearly is $\mathcal{D}$-dependent, for all $v \in \mathcal{V}$. Thus, by the Remark 2.1, there exists $\lambda \in \mathcal{D}$ such that $qv = v\lambda$ for all $v \in \mathcal{V}$. Now let $r \in \mathcal{R}$, $v \in \mathcal{V}$. Since $qv = v\lambda$ $[q, r]v = (qr)v - (rq)v = q(rv) - r(qv) = (rv)\lambda - r(v\lambda) = 0$, that is, $[q, \mathcal{R}]\mathcal{V} = 0$. Since $\mathcal{V}$ is a faithful irreducible $\mathcal{R}$-module, hence $[q, \mathcal{R}] = 0$, i.e., $q \in \mathcal{Z}(\mathcal{R})$, and hence $d = 0$.

Now suppose that $dim_{\mathcal{D}}\mathcal{V} \leq 2$. In this case $\mathcal{R}$ is a simple GPI-ring with 1 and so it is a central simple algebra finite dimensional over its center. By Lanski[12, Lemma 2], it follows that there exists a suitable field $\mathbb{F}$ such that $R \subseteq \mathcal{M}_t(\mathbb{F})$ the ring of $t \times t$ matrices over $\mathbb{F}$ and moreover, $\mathcal{M}_t(\mathbb{F})$ satisfy the same GPI as $\mathcal{R}$. Assume $m \geq 3$, then by the same argument as above we get a contradiction. Obviously if $m = 1$, then $\mathcal{R}$ is commutative. Thus we may assume that $m = 2$, i.e., $\mathcal{R} \subseteq \mathcal{M}_2(\mathbb{F})$, where $\mathcal{M}_2(\mathbb{F})$ satisfies $((a+q)(x \circ_k y) - (x \circ_k y)q)^n = (x \circ_k y)$. Denote by e_{ij} the usual unit matrix with 1 at (i, j)-entry and zero elsewhere. By putting $x = e_{ij}, y = e_{ii}$ in the above identity and then right multiplying by e_{ij}, one can easily get $q_{ji} = 0$. Similarly we can get $q_{ij} = 0$. Thus in all, we see that q is a diagonal matrix in $\mathcal{M}_2(\mathbb{F})$.

Let $\phi \in Aut(\mathcal{M}_2(\mathbb{F}))$. Since $(\phi(a+q)(\phi(x)\circ_k\phi(y))-(\phi(x)\circ_k\phi(y))\phi(q))^n-(\phi(x)\circ_k\phi(y)) = 0$, $\phi(q)$ must be a diagonal matrix in $\mathcal{M}_2(\mathbb{F})$. In particular, let $\phi(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$. Then $\phi(q) = q + (q_{ii} - q_{jj})e_{ij}$, that is $q_{ii} = q_{jj}$ for $i \neq j$. This implies that q is central in $\mathcal{M}_2(\mathbb{F})$, which leads to $d = 0$. This completes the proof of the theorem. □

Immediately, we can write the following corollary:

Corollary 2.3. *([9, Theorem A]). Let $\mathcal{R}$ be a prime ring, I be a nonzero ideal of $\mathcal{R}$ and n be a fixed positive integer. If $\mathcal{R}$ admits a generalized derivation F associated with a nonzero derivation d such that $F(x \circ y)^n = x \circ y$ for all $x, y \in I$, then $\mathcal{R}$ is commutative.*

Now, we are going to proof our result for semiprime case. From now on, $\mathcal{R}$ is a semiprime ring and $\mathcal{U}$ is the left Utumi quotient ring of $\mathcal{R}$. In order to prove the main result of for semiprime ring, we will make use of the following remarks:

Remark 2.4. ([3, Proposition 2.5.1]). Any derivation of a semiprime ring $\mathcal{R}$ can be uniquely extended to a derivation of its left Utumi quotient ring $\mathcal{U}$, and so any derivation of $\mathcal{R}$ can be defined on the whole $\mathcal{U}$.

Remark 2.5. ([7, p.38]). If $\mathcal{R}$ is semiprime then so is its left Utumi quotient ring. The extended centroid C of a semiprime ring coincides with the center of its left Utumi quotient ring.

Remark 2.6. ([7, p.42]). Let B be the set of all the idempotents in C, the extended centroid of $\mathcal{R}$. Assume $\mathcal{R}$ is a B-algebra orthogonal complete. For any maximal ideal P of B, $P\mathcal{R}$ forms a minimal prime ideal of $\mathcal{R}$, which is invariant under any nonzero derivation of $\mathcal{R}$.

We begin our investigation with the following theorem:

Theorem 2.7. *Let $\mathcal{R}$ be a semiprime ring with characteristic different from 2, $\mathcal{U}$ be the left Utumi quotient ring of $\mathcal{R}$ and $n, k \in \mathbb{Z}^+$. If $\mathcal{R}$ admits a generalized derivation F with associated derivation d such that $F(x \circ_k y)^n = (x \circ_k y)$ for all $x, y \in R$, then there exists a central idempotent e of $\mathcal{U}$ such that on the direct sum decomposition $\mathcal{U} = e\mathcal{U} \bigoplus (1-e)\mathcal{U}$, d vanishes identically on $e\mathcal{U}$ and the ring $(1-e)\mathcal{U}$ is commutative.*

Proof. Since $\mathcal{R}$ is semiprime and F is a generalized derivation of $\mathcal{R}$, by Lee [14, Theorem 3], $F(x) = ax + d(x)$ for some $a \in \mathcal{U}$ and a derivation d on $\mathcal{U}$ we are given that

$$(a(x \circ_k y) + d(x \circ_k y))^n = x \circ_k y$$

for all $x, y \in \mathcal{R}$. By the Remark 2.5, $Z(\mathcal{U}) = C$, the extended centroid of $\mathcal{R}$, and by the Remark 2.4, the derivation d can be uniquely extended on $\mathcal{U}$. In view of Lee[13], $\mathcal{R}$ and $\mathcal{U}$ satisfy the same differential identities, Then

$$(a(x \circ_k y) + d(x \circ_k y))^n = (x \circ_k y), \tag{2.2}$$

for all $x, y \in \mathcal{U}$. Let B be the Bolean algebra of idempotents in C and M be the maximal ideal of B. By Chuang [7], $\mathcal{U}$ is orthogonally complete B-algebra, and by the Remark 2.6, $M\mathcal{U}$ is a prime ideal of $\mathcal{U}$ which is d-invariant. Denote $\overline{U} = \mathcal{U}/M\mathcal{U}$ and $\overline{d}$ is a derivation induced by d on $\mathcal{U}$, i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in \mathcal{U}$.

$$(\overline{a}(\overline{x} \circ_k \overline{y}) + \overline{d}(\overline{x} \circ_k \overline{y}))^n = (\overline{x} \circ_k \overline{y}). \tag{2.3}$$

It is obvious that $\overline{\mathcal{U}}$ is prime. Therefore by Theorem 2.2, we have either $\overline{\mathcal{U}}$ is commutative or $\overline{d} = 0$ in $\overline{\mathcal{U}}$. That is , either $d(\mathcal{U}) \subseteq M\mathcal{U}$ or $[\mathcal{U}, \mathcal{U}] \subseteq M\mathcal{U}$. In any case $d(\mathcal{U})[\mathcal{U}, \mathcal{U}] \subseteq \bigcap_M M\mathcal{U} = 0$, we obtain that $d(\mathcal{U})[\mathcal{U}, \mathcal{U}] = 0$.

By using the theory of orthogonal completion of semiprime rings [3, Chapter 3], it is clear that there exists a central idempotent e in $\mathcal{U}$ such that on the direct sum decomposition $\mathcal{U} = e\mathcal{U} \bigoplus (1-e)\mathcal{U}$, d vanishes identically on $e\mathcal{U}$ and the ring $(1-e)\mathcal{U}$ is commutative. This completes the proof of the theorem. □

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