Palestine Journal of Mathematics

Vol. 7(Special Issue: I, 2018) , 43–49

ON (m, n)-REGULAR le-SEMIGROUPS AND (m, n)-IDEAL ELEMENTS

Noor Mohammad Khan, Ahsan Mahboob and Abdus Salam

Communicated by Mohammad Ashraf

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.

Keywords and phrases: le-semigroup, (m, n)-ideal element, (m, n)-quasi-ideal element.

Abstract After characterizing (m, n)-ideal elements of (m, n)-regular $\vee e$-semigroups in terms of its $(m, 0)$-ideal elements and $(0, n)$-ideal elements as well as the minimality of (m, n)-ideal elements of (m, n)-regular le-semigroups in terms of its minimal $(m, 0)$-ideal elements and minimal $(0, n)$-ideal elements, the minimality of (m, n)-quasi-ideal elements in le-semigroups has been studied. We, then, use $(m, 0)$-ideal elements, $(0, n)$-ideal elements, (m, n)-ideal elements and (m, n)-quasi–ideal elements of le-semigroups to characterize (m, n)-regular le-semigroups.

1 Introduction and Preliminaries

The notion of the left (resp. right) ideal element of an ordered semigroup was introduced in [1] as an element a of an ordered semigroup S such that $xa \leq a$ (resp. $ax \leq a$) for each $x \in S$. In [6, 7], Kehayopulu studied left (resp. right) duo regular, left-regular and left duo poe-semigroups. In [11], kehayopulu characterized the idempotent ideal elements of le-semigroups in terms of semisimple elements while intra-regular le-semigroup in terms of prime and semiprime ideal elements. In [12], kehayopulu Characterized left simple ve-semigroups in terms of left ideal elements. The concept of (m, n)-ideals in semigroups was introduced by S. Lajos. Thereafter, several authors studied (m, n)-ideals in various algebraic structures such as rings, semirings and ordered semigroups etc. Kehayopulu [3], studied (m, n)-ideal elements and (m, n)-quasi-ideal elements in poe-semigroups and le-semigroups.

Definition 1.1. Let S be a non-empty set, the triplet $(S, ., \leq)$ is called po-semigroup if $(S, .)$ is a semigroup and $(S, \leq)$ is a partially ordered set such that

$$a \leq b \Rightarrow ac \leq bc \text{ and } ca \leq cb$$

for all $a, b, c \in S$. A po-semigroup with a greatest element “e" (i.e. for each $a \in S, a \leq e$) is said to be a poe-semigroup.

Let S be a poe-semigroup. An element $a \in S$ is called a *subsemigroup element* if $a^2 \leq a$ and a is called a *left (resp. a right) ideal* element of S if $ea \leq a$ (resp. $ae \leq a$). It is called an *ideal-element* of S if it is both a left and a right-ideal element of S. A subsemigroup element a is called a *bi-ideal element* if $aea \leq a$. If we drop the subsemigroup condition from bi-ideal element, then it is called a *generalized bi-ideal element*. An element a of S is called an *idempotent-element* if $a = a^2$. An element a is called a *quasi-ideal element* if $ae \wedge ea$ exists and $ae \wedge ea \leq a$. An element $z \in S$ is called a *zero-element* of S if $za = az = z$ and $z \leq a$ for each $a \in S$. The zero element, if exists, is unique. we shall denote it, in whatever follows, by the symbol 0. A poe-semigroup S is called *regular (left-regular, right-regular)* if $a \leq aea$ $(a \leq ea^2, a \leq a^2e)$ for each $a \in S$. A poe-semigroup S is said to be *commutative* if $ab = ba$ for each $a, b \in S$.

Definition 1.2. A poe-semigroup S is said to be $\vee e$-semigroup if it is an upper semilattice under $\vee$ and

$$c(a \vee b) = ca \vee cb \text{ and } (a \vee b)c = ac \vee bc$$

for all $a, b, c \in S$. A $\vee e$-semigroup which is also a lattice is said to be an le-semigroup.

It is well known that in a po-semigroup S,

$$a \leq b \Leftrightarrow a \wedge b = a \text{ and } a \vee b = b$$

for all $a, b \in S$.

Definition 1.3. Let S be a poe-semigroup and m, n be non-negative integers. An element a of S is called an (m, n)-ideal element of S if $a^m e a^n \leq a$.

Remark 1.4. Throughout the paper, we shall use the convention $a^0 b = b a^0 = b$, for each $a, b \in S$. In particular for $m = 0,\ n = 1$ $(resp.\ m = 1,\ n = 0\ and\ m = 1 = n)$, a is a left-ideal element (resp. a right-ideal element and generalized bi-ideal element). Clearly each left-ideal element (resp. each right-ideal element and each generalized bi-ideal element) is a $(0, n)$-ideal element for each positive integer n (resp. $(m, 0)$-ideal element for each positive integer m and (m, n)-ideal element for each positive integers m, n). Therefore the concept of a $(0, n)$-ideal element (resp. $(m, 0)$-ideal element and (m, n)-ideal element) is the generalization of a left-ideal element (resp. a right-ideal element and a generalized bi-ideal element).

Remark 1.5. For each non-negative integers m and n, any $(m, 0)$-ideal element (resp. $(0, n)$-ideal element) is an (m, n)-ideal element as $a^m e a^n \leq a^m e \leq a$ $(a^m e a^n \leq e a^n \leq a)$.

Definition 1.6. An element q of a poe-semigroup S is called an (m,n)-quasi-ideal element of S (m, n positive integers) if $q^m e \wedge e q^n$ exists and $q^m e \wedge e q^n \leq q$. Clearly every quasi-ideal element is an (m, n)-quasi-ideal element for each positive integers m and n such that $q^m e \wedge e q^n$ exists.

We denote by (a), $< a >_{<m,n>}$ and $(a)_{<m,n>}$, the ideal-element, (m, n)-ideal element and (m, n)-quasi-ideal element of S generated by the element a of S i.e. the least ideal-element, the least (m, n)-ideal element and the least (m, n) quasi-ideal element of S greater than the element a and given by [3, 11] as follows:

$$\begin{aligned} (a) &= a \vee ea \vee ae \vee eae \\ < a >_{<m,n>} &= a \vee a^m e a^n \\ (a)_{<m,n>} &= a \vee (a^m e \wedge e a^n). \end{aligned}$$

Thus $a \in S$, is an ideal (resp. (m, n)-ideal, (m, n) quasi-ideal) element if and only if $(a) = a$ (resp. $< a >_{<m,n>} = a$, $(a)_{<m,n>} = a$).

2 Minimality of quasi-ideal element

Let S be a $\vee e$-semigroup and m, n be positive integers. An (m, n)-ideal element a is said to be minimal if for each (m, n)-ideal element b of S such that

$$b \leq a \Rightarrow a = b.$$

Similarly, we may define minimal (m, n)-quasi-ideal.

Proposition 2.1. *Let S be a $\vee e$-semigroup and m,n be positive integers. If S is (m, n)-regular, then for each (m, n)-ideal element a of S there exists $(m, 0)$-ideal element b and $(0, n)$-ideal element c of S such that $a = bc$.*

Proof. Let a be an (m, n)-ideal element of S. Then $a^m e a^n \leq a$. As S is (m, n)-regular, $a \leq a^m e a^n$. Therefore $a = a^m e a^n$. Since S is (m, n)-regular, $< a >_{<m,0>} = a^m e$ and $< a >_{<0,n>} = e a^n$. Thus we have

$$\begin{aligned} < a >_{<m,0>} < a >_{<0,n>} &= a^m e e a^n \\ &= a^m e a^n \qquad \text{(as } S \text{ is } (m, n)\text{-regular, } ee = e) \\ &= a. \end{aligned}$$

□

Remark 2.2. Let S be a poe-semigroup and m, n be positive integers. If S is (m, n)-regular, then the $(m, 0)$ and the (0,n)-ideal elements of S are idempotents.

Proposition 2.3. *Let S be a $\vee e$-semigroup and m,n be positive integers. If S is (m, n)-regular, then for each $(m, 0)$-ideal element a and for each element b of S, the element ab is an (m, n)-ideal element of S*

Proof. Let a be an $(m, 0)$-ideal element of S and b be an element of S. Now

$$\begin{aligned}
&(ab)^m e(ab)^n \\
&= \underbrace{(ab)(ab)\dots(ab)}_{m\text{-times}}\, e\, \underbrace{(ab)(ab)\dots(ab)}_{n\text{-times}} \\
&= \underbrace{(ab)\big((ab)\dots(ab)}_{m\text{-times}}\, e\, \underbrace{(ab)(ab)\dots\big)(ab)}_{n\text{-times}} \\
&\leq (ab)e(ab) \\
&= a(bea)b \\
&\leq aeb \\
&= a^m eb \qquad \text{(by Remark 2.2)} \\
&\leq ab.
\end{aligned}$$

□

By Propositions 2.1 and 2.3, we have following:

Theorem 2.4. *Let S be a $\vee e$-semigroup and m,n be positive integers. If S is (m, n)-regular, then an element a of S is an (m, n)-ideal element of S if and only if there exist $(m, 0)$-ideal b and $(0, n)$-ideal element c of S such that $a = bc$.*

Lemma 2.5. *Let S be a $\vee e$-semigroup and m,n be positive integers. If S is (m, n)-regular, then for each $a \in S$, $< a >_{<m,n>} = < a >_{<m,0>} < a >_{<0,n>}$.*

Proof. As S is (m, n)-regular, $< a >_{<m,n>} = a^m e a^n, < a >_{<m,0>} = a^m e$ and $< a >_{<0,n>} = e a^n$. Now

$$\begin{aligned}
< a >_{<m,n>} &= a^m e a^n \\
&= a^m e e a^n \qquad (ee = e \text{ as } S \text{ is } (m, n)\text{-regular}) \\
&= < a >_{<m,0>} < a >_{<0,n>},
\end{aligned}$$

as required. □

Theorem 2.6. *[15] Let S be an le-semigroup and m,n be non-negative integers. Then S is (m, n)-regular if and only if $a \wedge b = a^m b^n$ (or by Remark 2.2, $a \wedge b = ab$) for each $(m, 0)$-ideal element a and for each $(0, n)$-ideal element b of S.*

Theorem 2.7. *Let S be an le-semigroup and m,n be positive integers. If S is (m, n)-regular, then an element a of S is a minimal (m, n)-ideal element of S if and only if a=bc for some minimal $(m, 0)$-ideal element b and $(0, n)$-ideal element c of S.*

Proof. Let a be a minimal (m, n)-ideal element of S. As S is (m, n)-regular $< a >_{<m,n>} = a$. Thus, by Lemma 2.5, $a = < a >_{<m,0>} < a >_{<0,n>}$. Next we show that $< a >_{<m,0>}$ is a minimal $(m, 0)$-ideal element of S. For this, assume that a' be any $(m, 0)$-ideal element of S such that $a' \leq < a >_{<m,0>}$. As S is (m, n)-regular, by Theorem 2.6, $< a >_{<m,0>} \wedge < a >_{<0,n>} = < a >_{<m,0>} < a >_{<0,n>}$. Again, by Theorem 2.6, $a' < a >_{<0,n>} = a' \wedge < a >_{<0,n>} \leq < a >_{<m,0>} \wedge < a >_{<0,n>} = < a >_{<m,0>} < a >_{<0,n>} = a$. By Proposition 2.3, $a' < a >_{<0,n>}$ is an (m, n)-ideal element of S. Since $a' < a >_{<0,n>} \leq a$, by minimality of (m, n)-ideal element a of S, we have $a' < a >_{<0,n>} = a$. Therefore $< a >_{<m,0>} \wedge < a >_{<0,n>} = a' \wedge < a >_{<0,n>}$. As $a \leq < a >_{<m,0>} \wedge < a >_{<0,n>}$, $a \leq a' \wedge < a >_{<0,n>}$ implies that $a \leq a'$. Thus $< a >_{<m,0>} \leq a'$. Therefore $a' = < a >_{<m,0>}$. So $< a >_{<m,0>}$ is a minimal $(m,0)$-ideal element of S. Similarly $< a >_{<0,n>}$ is a minimal $(0,n)$-ideal element of S.

Conversely assume that a be an element of S such that $a = bc$ for some minimal $(m,0)$-ideal element b and $(0,n)$-ideal element c of S. By Theorem 2.4, a is an (m,n)-ideal element of S. To show that a is a minimal (m,n)-ideal element of S, let a' be an (m,n)-ideal element of S such that $a' \leq a$. Then $a'^m e \leq a^m e \leq (bc)^m e = (bc)(bc)...(bc)e \leq be \leq b^m eb^n e \leq b^m e \leq b$. As $a'^m e$ is a $(m,0)$-ideal element of S and b is a minimal $(m,0)$-ideal element, $a'^m e = b$. Similarly $ea'^n = c$. By hypothesis, $a = bc = a'^m eea'^n = \leq a'^m ea'^n \leq a'$. Hence a is a minimal (m,n)-ideal element of S. □

Lemma 2.8. *Let x be any minimal (m,n)-quasi-ideal element of an le-semigroup S and $a \in S$ be such that $a \leq x$. Then $x = a^m e \wedge ea^n$.*

Proof. Assume that x is a minimal (m,n)-quasi-ideal element of S and $a \in S$ be such that $a \leq x$. Since $a^m e$ and ea^n are $(m,0)$-right and $(0,n)$-ideal elements of S respectively, $a^m e \wedge ea^n$ is an (m,n)-quasi-ideal element of S. Now $a^m e \wedge ea^n \leq x$. So, by minimality of x, $a^m e \wedge ea^n = x$, as required. □

Proposition 2.9. *Each minimal (m,n)-quasi-ideal element of an le-semigroup S is the infimum of a minimal $(m,0)$-ideal element and a minimal $(0,n)$-ideal element of S.*

Proof. Let x be any minimal (m,n)-quasi-ideal element of S. Then, by Lemma 2.8, $x = a^m e \wedge ea^n$ for each $a \in S$ such that $a \leq x$. Now to complete the proof, it is sufficient to show that $a^m e$ and ea^n are, respectively, minimal $(m,0)$ and minimal $(0,n)$-ideal elements of S. To show that $a^m e$ is a minimal $(m,0)$-ideal element of S, take any $(m,0)$-ideal element $c \in S$ such that $c \leq a^m e$. Then $c \wedge ea^n \leq a^m e \wedge ea^n = x$. Since $c \wedge ea^n$ is a (m,n)-quasi-ideal element of $S, c \wedge ea^n = x$. Therefore $x \leq c$ implies that $a^m e \leq x^m e \leq c^m e \leq c$. So $a^m e = c$ i.e. $a^m e$ is a minimal $(m,0)$-ideal element of S. Similarly we may show that ea^n is a minimal $(0,n)$-ideal element of S. □

Proposition 2.10. *Let S be an le-semigroup and x be any (m,n)-quasi-ideal element of S. If $x = a \wedge b$, where a and b are minimal $(m,0)$-ideal element and minimal $(0,n)$-ideal element of S respectively, then x is a minimal (m,n)-quasi-ideal element of S.*

Proof. Assume that $x = a \wedge b$ for any minimal $(m,0)$-ideal element a and minimal $(0,n)$-ideal element b of S respectively. Therefore $x \leq a$ and $x \leq b$. Let y be any (m,n)-quasi-ideal element of S such that $y \leq x$. Then $y^m e \leq x^m e \leq a^m e \leq a$ and $ey^n \leq ex^n \leq eb^n \leq b$. As $y^m e$ and ey^n are $(m,0)$-ideal element and $(0,n)$-ideal element of S respectively, by minimality of a and b, $y^m e = a$ and $ey^n = b$. Therefore $x = a \wedge b = y^m e \wedge ey^n \leq y$. So $x = y$. Hence x is a minimal (m,n)-quasi-ideal element of S. □

Now combining Propositions 2.9 and 2.10, we get:

Theorem 2.11. *Let S be an le-semigroup and x be a (m,n)-quasi-ideal element of S. Then x is a minimal (m,n)-quasi-ideal element of S if and only if x is the infimum of a minimal $(m,0)$-ideal element and a minimal $(0,n)$-ideal element of S.*

Corollary 2.12. *Let S be an le-semigroup. Then S has at least one minimal (m,n)-quasi-ideal element if and only if S has at least one minimal $(m,0)$-ideal element and at least one minimal $(0,n)$-ideal element.*

Theorem 2.13. *[3] Let S be an le-semigroup, $a \in S$ and m, n be non-negative integers. Then the following are equivalent:*

(1) S is $< m,n >$-regular;

(2) $a^m ea^n = a$ for each (m,n)-ideal element of S;

(3) $q^m eq^n = a$ for each (m,n)-quasi-ideal element of S;

(4) $(< a >_{<m,n>})^m e(< a >_{<m,n>})^n =< a >_{<m,n>}$ *for each* $a \in S$*;*

(5) $((a)_{<m,n>})^m e((a)_{<m,n>})^n = (a)_{<m,n>}$ *for each* $a \in S$*.*

Lemma 2.14. *[3] Let* S *be a* $\vee e$*-semigroup,* $a \in S$ *and* $m, n, k \geq 0$ *be integers. Then*

(1) $(a \vee a^m e a^k)^m e = a^m e$*;*

(2) $e(a \vee a^k e a^n)^n = e a^n$*;*

(3) $< a >_{<m,n>}$ *exists and* $< a >_{<m,n>}= a \vee a^m e a^n$*.*

In whatever follows, $I_{(m,0)}, I_{(0,n)}, I_{(m,n)}$ and $Q_{(m,n)}$ denote the sets of all $(m, 0)$, $(0, n)$, (m, n)-ideal elements and (m, n)-quasi-ideal elements of the le-semigroup S respectively

Theorem 2.15. *Let* S *be an* le*-semigroup and* $m,\ n$ *be non-negative integers. Then* S *is* (m, n)*-regular if and only if* $b \wedge c \leq b^m c^n$ *for each* $b \in I_{(m,n)}$ *and for each* $c \in I_{(0,n)}$*.*

Proof. The statement is trivially true for $m = 0 = n$. If $m = 0$ and $n \neq 0$ or $m \neq 0$ and $n = 0$, then the result follows by Theorem 2.13. So, let $m \neq 0,\ n \neq 0$, $B \in I_{(m,n)}$ and $L \in I_{(0,n)}$. As S is (m, n)-regular, we have

$$\begin{aligned} &(b \wedge c)\\ &\leq (b \wedge c)^m e(b \wedge c)^n\\ &\leq b^m e c^n\\ &\leq b^m c\\ &= b^m cc \ \ \text{(by Remark 2.2)}\\ &= b^m ccc \ \ \text{(by Remark 2.2)}\\ &\vdots\\ &\leq b^m c^n. \end{aligned}$$

Therefore $b \wedge c \leq b^m c^n$.

Conversely assume that $b \wedge c \leq b^m c^n$ for each $b \in I_{(m,n)}$ and for each $c \in I_{(0,n)}$. Take any $a \in S$. As $< a >_{<m,n>}\in I_{(m,n)}$ and $< a >_{<0,n>}\in I_{(0,n)}$, we have

$$\begin{aligned} < a >_{<m,n>} =< a >_{<m,n>} \wedge e &\leq (< a >_{<m,n>})^m e^n \quad \text{(by hypothesis)}\\ &\leq (< a >_{<m,n>})^m e = a^m e \qquad \text{(by Lemma 2.14).} \end{aligned}$$

Similarly $< a >_{<0,n>}\leq e a^n$. As $a^m e \in I_{(m,n)}$ and $e a^n \in I_{(0,n)}$, by hypothesis

$$\begin{aligned} \{a\} &\leq< a >_{<m,n>} \wedge < a >_{<0,n>}\leq a^m e \wedge e a^n\\ &= (a^m e)^m (e a^n)^n\\ &\leq a^m e a^n. \end{aligned}$$

Hence S is (m, n)-regular. □

Similarly we may prove the following:

Theorem 2.16. *Let* S *be an* le*-semigroup and* $m,\ n$ *be non-negative integers. Then* S *is* (m, n)*-regular if and only if* $a \wedge b \leq a^m b^n$ *for each* $b \in I_{(m,n)}$ *and for each* $a \in I_{(m,0)}$*.*

Lemma 2.17. *[3] Let* S *be an* le*-semigroup,* $a \in S$ *and* $m,\ n$ *be any positive integers.*

(1) $(a \vee (a^m e \wedge e a^n))^m e \leq a^m e$*;*

(2) $e(a \vee (a^m e \wedge e a^n))^n \leq e a^n$*;*

(3) $(a)_{<m,n>}$ *exists and* $(a)_{<m,n>} = a \vee (a^m e \wedge e a^n)$*.*

Theorem 2.18. *Let S be an le-semigroup and $m,\ n$ be non-negative integers. Then S is (m,n)-regular if and only if $q \wedge c \leq q^m c^n$ for each $q \in Q_{(m,n)}$ and for each $c \in I_{(0,n)}$.*

Proof. Statement follows by Theorem 2.15 because each (m,n)-quasi-ideal element of S is an (m,n)-ideal element of S.

Conversely assume that $qc \leq q^m c^n$ for each (m,n)-quasi-ideal element q and for each $(0,n)$-ideal element c of S. Let $a \in S$. As $(a)_{<m,n>}$ is an (m,n)-quasi-ideal element and e is a $(0,n)$-hyperideal of S, we have

$$\begin{aligned}(a)_{<m,n>} &= (a)_{<m,n>} \wedge e \leq ((a)_{<m,n>})^m e^n \qquad \text{(by hypothesis)}\\ &\leq ((a)_{<m,n>})^m e = a^m e \qquad \text{(by Lemma 2.17)}.\end{aligned}$$

Similarly $< a >_{<0,n>} \leq ea^n$. As $(a)_{<m,n>}$ and ea^n are (m,n)-quasi-ideal element and $(0,n)$-ideal element of S, by hypothesis, we have

$$\begin{aligned}\{a\} &\leq (a)_{<m,n>} \wedge < a >_{<0,n>} \leq a^m e \wedge ea^n\\ &= (a^m e)^m (ea^n)^n\\ &\leq a^m ea^n.\end{aligned}$$

Hence, S is (m,n)-regular. □

Similarly we may prove the following:

Theorem 2.19. *Let S be an le-semigroup and $m,\ n$ be non-negative integers. Then S is (m,n)-regular if and only if $a \wedge q \leq a^m q^n$ for each (m,n)-quasi-ideal element q and for each $(m,0)$-ideal element a of S.*

Theorem 2.20. *Let S be an le-semigroup and $m,\ n$ be non-negative integers. Then S is (m,n)-regular if and only if $a \wedge c = a^m c \wedge ac^n$ for each $(m,0)$-ideal element a and for each $(0,n)$-ideal element c of S.*

Proof. The statement is trivially true for $m = 0 = n$. If $m \neq 0$ and $n = 0$, then we have to show that S is $(m,0)$-regular if and only if $a = a^m e$ which directly follows by Theorem 2.13. Similarly when $m = 0$ and $n \neq 0$, then the result follows by Theorem 2.13. So, let $m \neq 0,\ n \neq 0$, a be any $(m,0)$-ideal element and c be any $(0,n)$-ideal element of S. As S is (m,n)-regular, By Remark 2.2 and Theorem 2.6, $a \wedge c = a^m c^n = a^m c$ and $a \wedge c = a^m c^n = ac^n$. So $a \wedge c = a^m c \wedge ac^n$.

Conversely assume that $a \wedge c = a^m c \wedge ac^n$ for each $(m,0)$-ideal element a and for each $(0,n)$-ideal element c of S. Let $t \in S$. As $< t >_{<m,0>}$ is an $(m,0)$-ideal element and e is a $(0,n)$-ideal element of S, we have

$$\begin{aligned}< t >_{<m,0>} &=< t >_{<m,0>} \wedge e = (< t >_{<m,0>})^m e \wedge < t >_{<m,0>} e^n \qquad \text{(by hypothesis)}\\ &\leq (< t >_{<m,0>})^m e = t^m e \qquad \text{(by Lemma 2.14)}.\end{aligned}$$

Similarly $< t >_{<0,n>} \leq et^n$. As $t^m e$ and et^n are $(m,0)$-ideal element and $(0,n)$-ideal element of S respectively, by hypothesis, we have

$$\begin{aligned}&< t >_{<m,0>} \wedge < t >_{<0,n>}\\ &\leq t^m e \wedge et^n\\ &= (t^m e)^m et^n \wedge t^m e (et^n)^n\\ &= \underbrace{t^m e t^m e \ldots t^m e}_{m\text{-times}} et^n \wedge t^m e \underbrace{et^n et^n \ldots et^n}_{n\text{-times}}\\ &\leq \underbrace{t^m e t^m e \ldots t^m e}_{m-1\text{-times}} t^m eet^n \wedge t^m et^n \underbrace{et^n et^n \ldots et^n}_{n-1\text{-times}}\\ &\leq \underbrace{t^m e t^m e \ldots t^m e}_{m-1\text{-times}} t^m et^n \wedge t^m et^n \underbrace{et^n et^n \ldots et^n}_{n-1\text{-times}}\\ &\vdots\\ &\leq t^m et^n.\end{aligned}$$

Hence S is (m,n)-regular. □

Corollary 2.21. *Let S be an le-semigroup and $m,\ n$ be non-negative integers. Then the following are equivalent:*

(1) S *is* (m, n)*-regular;*

(2) $b \wedge c \leq b^m c^n$ *for each* $b \in I_{(m,n)}$ *and for each* $c \in I_{(0,n)}$*;*

(3) $q \wedge c \leq q^m c^n$ *for each* $q \in Q_{(m,n)}$ *and for each* $c \in I_{(0,n)}$*;*

(4) $a \wedge b \leq a^m b^n$ *for each* $b \in I_{(m,n)}$ *and for each* $a \in I_{(m,0)}$*;*

(5) $a \wedge q \leq a^m q^n$ *for each* $q \in Q_{(m,n)}$ *and for each* $a \in I_{(m,0)}$*;*

(6) $c \wedge a = a^m c \wedge ac^n$ *for each* $a \in I_{(m,0)}$ *and for each* $c \in I_{(0,n)}$*.*

References

[1] G. Birkhoff, Lattice theory, Amer. Math. Soc. Coll. Publ., Vol. XXV, Providence, (1967).

[2] J. M. Howie, An Introduction to Semigroup Theory, *Academic Press*, (1976).

[3] N. Kehayopulu, Generalized ideal elements in poe-semigroups, *Semigroup Forum* **25**, 213-222 (1982).

[4] N. Kehayopulu, On intra-regular $\vee e$-semigroups, *Semigroup Forum* **19**, 111-121 (1980).

[5] N. Kehayopulu, A characterization of regular le-semigroups, *Semigroup Forum* **23**, 85-86 (1981).

[6] N. Kehayopulu, On regular and intra-regular poe-semigroups, *Semigroup Forum* **29**, 255-257 (1984).

[7] N. Kehayopulu, On left (resp. right) duo regular poe-semigroups, *Semigroup Forum* **31**, 123-125 (1985).

[8] N. Kehayopulu, On Left Regular and Left Duo poe-Semigroups, *Semigroup Forum* **44**, 306-313 (1992).

[9] N. Kehayopulu. On prime, weakly prime ideals in ordered semigroups, *Semigroup Forum* **44**, 341-346 (1992).

[10] N. Kehayopulu, On regular le-semigroups, *Semigroup Forum* **49**, 267-269 (1994).

[11] N. Kehayopulu, On le-semigroups, Turk J Math. **40**, 310-316 (2016).

[12] N. Kehayopulu, The role of the ideal elements in studying the structure of some ordered semigroups, Turk J Math. **41**, 1144-1154 (2017).

[13] N. M. Khan, A. Mahboob and M. A. Khan, On characterizations of generalized Green-Kehayopulu type (m,n)-relations on le-semigroups, (Communicated).

[14] S. Lajos, Generalized ideals in semigroups, *Acta Sci. Math.* **2**, 217-222 (1961).

[15] A. Mahboob and N. M. Khan, (m, n)-ideal elements and Ordered semigroups, (Communicated).

[16] A. Mahboob and N. M. Khan, ideal elements and Ordered semigroups, (Communicated).

[17] P. Petro and E. Pasku, The Green-Kehayopulu relation $\mathcal{H}$ in le-semigroups, *Semigroup Forum* **65**, 33-42 (2002).

[18] E. Pasku and P. Petro, The Relation $\mathcal{B}$ in le-semigroups, *Semigroup Forum* **75**, 427-437 (2007).

Author information

Noor Mohammad Khan, Ahsan Mahboob and Abdus Salam, Department of Mathematics, Aligarh Muslim University, Aligarh, 202002, India.
E-mail: noormohammad.khan@gmail.com