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SOME FUNCTIONAL IDENTITIES WITH GENERALIZED SKEW DERIVATION ON $\star$-PRIME RINGS

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Abstract Let R be a ring with involution $\star$ and centre $Z(R)$. An additive mapping $D : R \to R$ is called a skew derivation if there exists a map $g : R \to R$ such that $D(xy) = D(x)g(y) + xD(y) = D(x)y + g(x)D(y)$ for all $x,\ y \in R$. An additive mapping $F : R \to R$ is called a generalized skew derivation associated with a skew derivation D such that $F(xy) = F(x)y+g(x)D(y) = D(x)g(y)+xF(y)$ for all $x,\ y \in R$. In the present paper, some well known results concerning skew derivation of prime rings are extended to generalized skew derivation of $\star$- prime rings.

1 Introduction

Throughout the paper, R will denote an associative ring with centre $Z(R)$. A ring R is said to be prime (resp. semiprime) if $aRb = (0)$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = (0)$ implies that $a = 0$). A ring R is said to be n-torsion free if $nx = 0$ for all $x \in R$ implies $x = 0$. For each pair of elements $x, y \in R$ we shall write $[x, y]$ the commutator $xy-yx$. An additive mapping $\star : R \to R$ is called an involution if $(xy)^\star = y^\star x^\star$ and $(x^\star)^\star = x$ for all $x,\ y \in R$. A ring R equipped with an involution is called a ring with involution or $\star$-ring. A ring R with an involution is said to be $\star$-prime if $xRy = x^\star Ry = (0)$ (or equivalently $xRy = xRy^\star = (0)$) implies that $x = 0$ or $y = 0$. Every prime ring with an involution $\star$ is $\star$- prime but the converse need not hold general. An example due to Oukhtite [6] justifies the above statement; let R be a prime ring, $S = R \times R^\circ$, where R° is the opposite ring of R. Define an involution $\star$ on S as $\star(x, y) = (y, x)$. S is $\star$-prime ring and from this point of view $\star$-prime rings constitute a more general class of prime rings. In all that follows the symbol $Sa_\star(R)$, first introduced by Oukhtite, will denote the set of symmetric and skew symmetric elements of R, i.e., $Sa_\star(R) = \{x \in R \mid x^\star = \pm x\}$. An additive map $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x,\ y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. The study of derivations in prime rings was initiated by Posner in [10]. Recently, Bresar defined the following notation in [1]; An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that

$$F(xy) = F(x)y + xd(y) \text{ for all } x,\ y \in R.$$

Basic examples are derivations and generalized inner derivations (i.e., maps of type $x \to ax+xb$ for some $a, b \in R$). Several authors consider the structure of a prime ring in the case that the derivation d replaced by a generalized derivation. Generalized derivations have been primarily studied on operator algebras.

An additive mapping $D : R \to R$ is called a skew derivation if there exists a map $g : R \to R$ such that $D(xy) = D(x)g(y) + xD(y) = D(x)y + g(x)D(y)$ for all $x,\ y \in R$. In case g is an identity map of R, then all skew derivations associated with g are merely ordinary derivations. An additive mapping $F : R \to R$ is called a generalized skew derivation if there exists a skew derivation $D : R \to R$ associated with a map $g : R \to R$ such that $F(xy) = F(x)y + g(x)D(y) = D(x)g(y) + xF(y)$ for all $x,\ y \in R$.

The study of such mappings was initiated by Posner in [10]. A famous result due to Herstein [4] states that if R is a prime ring of characteristic not 2 which admits a nonzero derivation

d such that $[d(x), a] = 0$ for all $x \in R$, then $a \in Z(R)$. Also, Herstein showed that if $d(R) \subseteq Z(R)$, then R must be commutative. On the other hand, in [2], Daif and Bell proved that if R is a semiprime ring, I a non zero ideal of R and d a derivation of R such that either $d([x, y]) = [x, y]$ for all $x,\ y \in I$, or $d([x, y]) = -[x, y]$ for all $x,\ y \in I$, then R is commutative. Many authors have studied commutativity of prime and semiprime rings admitting derivations, generalized derivations and skew derivations which satisfy appropriate algebraic conditions on suitable subsets of the rings. Recently, some well known results concerning prime rings have been proved for $\star$-prime ring by Oukhtite et al. (see, [5-8], where further references can be found). In [11], Tiwari et. al. studied the structure of additive mappings and structure of prime rings. More precisely, they studied the following identities: (i) $G(xy) \pm F(x)F(y) \pm yx \in Z(R)$; (ii) $G(xy) \pm F(y)F(x) \pm yx \in Z(R)$; (iii) $G(xy) \pm F(x)F(y) \pm [x, y] \in Z(R)$; (iv) $G(xy) \pm F(y)F(x) \pm [x, y] \in Z(R)$; for all $x,\ y \in I$, where F, G are generalized derivations and I a non zero ideal of R. Further, in 2017 [12], Tiwari et. al studied the above identities to the case when F and G are multiplicative (generalized)-derivations on semiprime ring. In the present paper our objective is to generalize the afore mentioned results of Tiwari et. al. [11,12] for generalized skew derivations of a $\star$-prime ring. Some other results on prime ring are also extended for $\star$-prime rings.

2 Preliminary Result

Throughout the paper, R will be $\star$-prime ring and D be a skew derivation associated with a map g of R such that D commutes with $\star$. Also, we will make some extensive use of the basic commutator identities:

$$[x, yz] = y[x, z] + [x, y]z$$

$$[xy, z] = [x, z]y + x[y, z]$$

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$

$$(xy) \circ z) = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z].$$

This Lemma is essential to prove our Theorems.

Lemma 2.1. *Let R be a $\star$-prime ring and $a \in R$. If R admits a skew derivation D of R such that $aD(x) = 0$ (or $D(x)a = 0$) for all $x \in R$, then either $a = 0$ or $D = 0$.*

Proof. We have

$$aD(xy) = 0 \ \text{ for all } x,\ y \in R.$$

Using definition of D, we find

$$aD(x)g(y) + axD(y) = 0 \ \text{ for all } x,\ y \in R,$$

and so

$$aRD(y) = (0), \ \text{ for all } y \in R.$$

Replacing y by $y^\star$ in this equation and using $\star D = D\star$, we find that

$$aRD(y)^\star = (0), \ \text{ for all } y \in R.$$

Since R is a $*$-prime ring, we have either $a = 0$ or $D = 0$. Similarly, holds case $D(x)a = 0$.□

3 Main Results

Theorem 3.1. *Let R be a $\star$-prime ring, F a nonzero generalized skew derivation of R associated with a skew derivation D and a map g associated with D. Then either g is a homomorphism on R or F acts as a multiplier.*

Proof. We have

$$F(z(x+y)) = D(z)g(x+y) + zF(x+y) \text{ for all } x,\ y,\ z \in R.$$

This implies that

$$F(z(x+y)) = D(z)g(x+y) + zF(x) + zF(y) \text{ for all } x,\ y,\ z \in R. \tag{3.1}$$

On the other hand,

$$F(z(x+y)) = F(zx+zy) \text{ for all } x,\ y,\ z \in R.$$

This gives that

$$F(z(x+y)) = F(zx) + F(zy) \text{ for all } x,\ y,\ z \in R.$$

That is

$$F(z(x+y)) = D(z)g(x) + zF(x) + D(z)g(y) + zF(y) \text{ for all } x,\ y,\ z \in R. \tag{3.2}$$

Comparing (3.1) and (3.2), we arrive at

$$D(z)(g(x+y) - g(x) - g(y)) = 0 \text{ for all } x,\ y,\ z \in R.$$

Using Lemma 2.1, we obtain that either $D(z) = 0$ for all $z \in R$ or $g(x+y) - g(x) - g(y) = 0$ for all $x,\ y \in R$. If $D(z) = 0$, then F acts as a multiplier. On the other hand if $g(x+y) = g(x) + g(y)$ for all $x,\ y \in R$. Again we have

$$F((xy)z) = F(xy)z + g(xy)D(z) \text{ for all } x,\ y,\ z \in R.$$

This implies that

$$F(x(yz)) = F(x)yz + g(x)D(y)z + g(xy)D(z) \text{ for all } x,\ y,\ z \in R. \tag{3.3}$$

Also

$$F(x(yz)) = F(x)yz + g(x)D(yz) \text{ for all } x,\ y,\ z \in R.$$

This gives that

$$F(x(yz)) = F(x)yz + g(x)D(y)z + g(x)g(y)D(z) \text{ for all } x,\ y,\ z \in R. \tag{3.4}$$

Comparing (3.3) and (3.4), we get

$$(g(xy) - g(x)g(y))D(z) = 0 \text{ for all } x,\ y,\ z \in R.$$

Using Lemma 2.1, we have either $D(z) = 0$ for all $z \in R$ or $g(xy) - g(x)g(y) = 0$ for all $x,\ y \in R$. If $D(z) = 0$, then F acts as a multiplier. In the later case, g acts as a homomorphism on R.□

Theorem 3.2. *Let R be a $\star$-prime ring, F a nonzero generalized skew derivation of R associated with a skew derivation D and a surjective map g associated with D. If $F(R) \subseteq Z(R)$, then F acts as a multiplier or R is commutative .*

Proof. Assume that

$$F(xy) \in Z(R) \text{ for all } x,\ y \in R. \tag{3.5}$$

This gives that

$$D(x)g(y) + xF(y) \in Z(R) \text{ for all } x,\ y \in R.$$

Commuting with x, we have

$$[D(x)g(y), x] + [xF(y), x] = 0 \text{ for all } x,\ y \in R.$$

Using hypothesis, we obtain

$$[D(x)g(y), x] = 0 \text{ for all } x,\ y \in R.$$

Since g is surjective map on R, we have

$$[D(x)y, x] = 0 \text{ for all } x,\ y \in R. \tag{3.6}$$

Replacing y by yr, we get

$$[D(x)yr, x] = 0 \text{ for all } x,\ y,\ r \in R.$$

This implies that

$$D(x)y[r, x] + [D(x)y, x]r = 0 \text{ for all } x,\ y,\ r \in R.$$

By equation (3.6), we have

$$D(x)y[r, x] = 0 \text{ for all } x,\ y, r \in R. \tag{3.7}$$

Let $t \in R$. Since $x = t - t^\star \in R$, $x \in Sa_\star(R)$, we have

$$D(x)R([r, x])^\star = (0) \text{ for all } r \in R.$$

Using $\star$-primeness of R, we find that either $D(x) = 0$ or $[r, x] = 0$ for all $r \in R$ and $x \in Sa_\star(R)$. But $D(x) = 0$ implies that $D(t - t^\star) = 0$, and hence we obtain $D(t) = D(t^\star) = (D(t))^\star$. Therefore $D(t)R[r, t] = (0) = (D(t))^\star R[r, t]$, for all $t,\ r \in R$. This implies that either $D(t) = 0$ or $[r, t] = 0$ for all $t,\ r \in R$. Consequently, either $D(t) = 0$ or $t \in Z(R)$. If $[r, x] = 0$ for all $r \in R$ and $x \in Sa_\star(R)$, we find that $[r, t] = [r, t^\star]$ for all $r,\ t \in R$. Hence $D(t)R[r, t] = (0) = D(t)R([r, t])^\star$. Again using $\star$-primeness of R, we get either $D(t) = 0$ or $t \in Z(R)$.
In conclusion, for each $t \in R$, either $D(t) = 0$ or $t \in Z(R)$. Let us consider $A = \{t \in R \mid D(t) = 0\}$ and $B = \{t \in R \mid t \in Z(R)\}$. It is clear that A and B are additive subgroups of R such that $R = A \cup B$. But a group can not be the set-theoretic union of two proper subgroups, hence either $R = A$ or $R = B$. If $R = A$, then $D(t) = 0$ for all $t \in R$ implies that F acts as a multiplier. On the other hand, if $R = B$, then $t \in Z(R)$ for all $t \in R$ implies that $R = Z(R)$ i.e., R is commutative.□

Theorem 3.3. *Let R be a $\star$-prime ring, F a generalized skew derivation of R associated with a skew derivation D and a surjective map g associated with D such that $[F(x), x] = 0$ for all $x \in R$, then R is commutative or F acts as a multiplier.*

Proof. Suppose

$$[F(x), x] = 0 \quad \text{for all } x \in R. \tag{3.8}$$

Linearizing (3.8), we have

$$[F(x), y] + [F(y), x] = 0 \quad \text{for all } x,\ y \in R. \tag{3.9}$$

Substituting yx in place of y, we get

$$[F(x), yx] + [F(yx), x] = 0 \quad \text{for all } x,\ y \in R.$$

This implies that

$$[F(x), y]x + y[F(x), x] + [F(y)x + g(y)D(x), x] = 0 \quad \text{for all } x,\ y \in R.$$

Using hypothesis, we find that

$$([F(x), y] + [F(y), x])x + [g(y)D(x), x] = 0 \quad \text{for all } x,\ y \in R.$$

By equation (3.9), we get

$$[g(y)D(x), x] = 0 \quad \text{for all } x,\ y \in R. \tag{3.10}$$

Since g is surjective map on R, we have

$$[yD(x), x] = 0 \quad \text{for all } x,\ y \in R. \tag{3.11}$$

This implies that

$$[y, x]D(x) + y[D(x), x] = 0 \quad \text{for all } x,\ y \in R. \tag{3.12}$$

Substituting ry in place of y, we get

$$[ry, x]D(x) + ry[D(x), x] = 0 \quad \text{for all } x,\ y, r \in R. \tag{3.13}$$

This gives that

$$r[y, x]D(x) + [r, x]yD(x) + ry[D(x), x] = 0 \quad \text{for all } x,\ y, r \in R. \tag{3.14}$$

Now multiplying (3.12) by r from left and subtracting from (3.14), we obtain

$$[r, x]RD(x) = (0) \text{ for all } x,\ r \in R.$$

Let $t \in R$. Since $x = t - t^\star \in R$, $x \in Sa_\star(R)$, we have

$$([r, x])^\star RD(x) = (0) \text{ for all } r \in R.$$

Using $\star$-primeness of R, we find that either $[r, x] = 0$ or $D(x) = 0$ for all $r \in R$ and $x \in Sa_\star(R)$. But $D(x) = 0$ implies that $D(t - t^\star) = 0$, and hence we obtain $D(t) = D(t^\star) = (D(t))^\star$. Therefore $[r, t]RD(t) = (0) = [r, t]R(D(t))^\star$, for all $r,\ t \in R$. This implies that either $[r, t] = 0$ or $D(t) = 0$ for all $r,\ t \in R$. Consequently, either $t \in Z(R)$ or $D(t) = 0$. If $[r, x] = 0$ for all $r \in R$ and $x \in Sa_\star(R)$, we find that $[r, t] = [r, t^\star]$ for all $r,\ t \in R$. Hence $[r, t]RD(t) = (0) = ([r, t])^\star RD(t)$. Again using $\star$-primeness of R, we get either $t \in Z(R)$ or $D(t) = 0$.
In conclusion, for each $t \in R$, either $t \in Z(R)$ or $D(t) = 0$. Let us consider $A = \{t \in R \mid t \in Z(R)\}$ and $B = \{t \in R \mid D(t) = 0\}$. It is clear that A and B are additive subgroups of R such that $R = A \cup B$. But a group can not be the set-theoretic union of two proper subgroups, hence either $R = A$ or $R = B$. If $R = A$, then $t \in Z(R)$ for all $t \in R$ implies that $R = Z(R)$ i.e., R is commutative. On the other hand, if $R = B$, then $D(t) = 0$ for all $t \in R$ implies that F acts as a multiplier.□

Theorem 3.4. *Let R be a $\star$-prime ring. If R admits a non zero generalized skew derivation F associated with a skew derivation D and an onto endomorphism g on R associated with D such that $F([x, y]) = 0$ for all $x,\ y \in R$, then F acts as a multiplier or R is commutative.*

Proof. We have

$$F([x,y]) = 0 \quad \text{for all } x,\ y \in R. \tag{3.15}$$

Replacing y by xy in (3.15), we have

$$F(x[x,y]) = 0 \quad \text{for all } x,\ y \in R.$$

This implies that

$$D(x)g([x,y]) + xF([x,y]) = 0 \quad \text{for all } x,\ y \in R.$$

Using hypothesis, we get

$$D(x)g([x,y]) = 0 \quad \text{for all } x,\ y \in R.$$

Since g is an endomorphism on R, we find

$$D(x)[g(x), g(y)] = 0 \quad \text{for all } x,\ y \in R.$$

Since g is surjective map on R, we get

$$D(x)[g(x), y] = 0 \quad \text{for all } x,\ y \in R. \tag{3.16}$$

Substituting yz for y and using (3.16), we obtain that

$$D(x)y[g(x), z] = 0 \quad \text{for all } x,\ y,\ z \in R.$$

This implies that

$$D(x)R[x,z] = (0), \quad \text{for all } x,\ z \in R. \tag{3.17}$$

Let $t \in R$. Since $x = t - t^\star \in R$, $x \in Sa_\star(R)$, we have

$$D(x)R([x,z])^\star = (0) \text{ for all } r \in R.$$

Using $\star$-primeness of R, we find that either $D(x) = 0$ or $[x,z] = 0$ for all $z \in R$ and $x \in Sa_\star(R)$. But $D(x) = 0$ implies that $D(t - t^\star) = 0$, and hence we obtain $D(t) = D(t^\star) = (D(t))^\star$. Therefore $D(t)R[t,z] = (0) = (D(t))^\star R[t,z]$, for all $t,\ z \in R$. This implies that either $D(t) = 0$ or $[t,z] = 0$ for all $t,\ z \in R$. Consequently, either $D(t) = 0$ or $t \in Z(R)$. If $[x,z] = 0$ for all $z \in R$ and $x \in Sa_\star(R)$, we find that $[t,z] = [t,z^\star]$ for all $z,\ t \in R$. Hence $D(t)R[t,z] = (0) = D(t)R([t,z])^\star$. Again using $\star$-primeness of R, we get either $D(t) = 0$ or $t \in Z(R)$.
In conclusion, for each $t \in R$, either $D(t) = 0$ or $t \in Z(R)$. Let us consider $A = \{t \in R \mid D(t) = 0\}$ and $B = \{t \in R \mid t \in Z(R)\}$. It is clear that A and B are additive subgroups of R such that $R = A \cup B$. But a group can not be the set-theoretic union of two proper subgroups, hence either $R = A$ or $R = B$. If $R = A$, then $D(t) = 0$ for all $t \in R$ implies that F acts as a multiplier. On the other hand, if $R = B$, then $t \in Z(R)$ for all $t \in R$ implies that $R = Z(R)$ i.e., R is commutative.□

Theorem 3.5. *Let R be a $\star$-prime ring. If R admits a non zero generalized skew derivation F associated with a skew derivation D and an onto endomorphism g on R associated with D such that $F([x,y]) = \pm[x,y]$ for all $x,\ y \in R$, then F acts as a multiplier or R is commutative.*

Proof. We have

$$F([x,y]) = \pm[x,y] \quad \text{for all } x,\ y \in R.$$

Replacing y by xy, we have

$$F(x[x,y]) = \pm x[x,y] \quad \text{for all } x,\ y \in R.$$

This implies that

$$D(x)g([x,y]) + xF([x,y]) = \pm x[x,y] \quad \text{for all } x,\ y \in R.$$

Using hypothesis, we get

$$D(x)g([x,y]) = 0 \quad \text{for all } x,\ y \in R.$$

Using the same arguments as we have used in the last part of proof of the Theorem 3.4. we get the required result.□

Theorem 3.6. *Let R be a $\star$-prime ring. If R admits generalized skew derivation F associated with a non zero skew derivation D and an onto endomorphism g associated with D such that $F(x \circ y) + xy = 0$ for all $x,\ y \in R$, then R is commutative.*

Proof. We have

$$F(x \circ y) + xy = 0 \quad \text{for all } x,\ y \in R. \tag{3.18}$$

Substituting yx instead of y in (3.18), we have

$$F(x \circ yx) + xyx = 0 \quad \text{for all } x,\ y \in R.$$

This implies that

$$F(x \circ y)x + g(x \circ y)D(x) + xyx = 0 \quad \text{for all } x,\ y \in R.$$

This gives that

$$(F(x \circ y) + xy)x + g(x \circ y)D(x) = 0 \quad \text{for all } x,\ y \in R.$$

By equation (3.18), we have

$$g(x \circ y)D(x) = 0 \quad \text{for all } x,\ y \in R. \tag{3.19}$$

Using Lemma 2.1, we find that either $g(x \circ y) = 0$ or $D(x) = 0$. By the given hypothesis $D \neq (0)$, then $g(x \circ y) = 0$. Since g is an endomorphism on R, we have

$$g(xy) + g(yx) = 0 \quad \text{for all } x,\ y \in R.$$

This implies that

$$g(x)g(y) = -g(y)g(x) \quad \text{for all } x,\ y \in R. \tag{3.20}$$

Replacing y by yz, we obtain

$$g(x)g(y)g(z) = -g(y)g(z)g(x) \quad \text{for all } x,\ y, z \in R.$$

By equation (3.20), we get

$$g(y)g(x)g(z) = g(y)g(z)g(x) \quad \text{for all } x,\ y, z \in R.$$

This implies that $g(y)[g(x), g(z)] = 0$ for all $x,\ y, z \in R$. Using hypothesis, we find $y[g(x), g(z)] = 0$. Now replacing y by yr, we get $yr[g(x), g(z)] = 0$ for all $x,\ y, z, r \in R$. Again replacing y by $y^\star$, we have $y^\star R([g(x), g(z)]) = (0)$, for all $x,\ y, z \in R$. Using $*$-primeness of R, we find that $[g(x), g(z)] = 0$. By hypothesis $[x, z] = 0$ for all $x, z \in R$. Hence R is commutative.□

Theorem 3.7. *Let R be a $\star$-prime ring. If R admits a generalized skew derivation F associated with a non zero skew derivation D and an onto endomorphism g on R associated with D such that $F(x \circ y) + x \circ y = 0$ for all $x,\ y \in R$, then R is commutative.*

Proof. We have

$$F(x \circ y) + x \circ y = 0 \quad \text{for all } x,\ y \in R. \tag{3.21}$$

Replacing y by yx in (3.21), we have

$$F((x \circ y)x) + (x \circ y)x = 0 \quad \text{for all } x,\ y \in R.$$

This implies that

$$F(x \circ y)x + g(x \circ y)D(x) + (x \circ y)x = 0 \quad \text{for all } x,\ y \in R.$$

This gives that

$$(F(x \circ y) + x \circ y)x + g(x \circ y)D(x) = 0 \quad \text{for all } x,\ y \in R.$$

By equation (3.21), we have

$$g(x \circ y)D(x) = 0 \quad \text{for all } x,\ y \in R. \tag{3.22}$$

Arguing in the similar manner as in Theorem 3.6, we find that R is commutative.□

Theorem 3.8. *Let R be a $\star$-prime ring. If R admits a non zero generalized skew derivation F associated with a skew derivation D and a map g associated with D such that $[F(x), y] + xy = 0$ for all $x,\ y \in R$, then $F(R) \subseteq Z(R)$.*

Proof. We have

$$[F(x), y] + xy = 0 \quad \text{for all } x,\ y \in R. \tag{3.23}$$

Replacing y by yr in (3.23), we have

$$[F(x), y]r + y[F(x), r] + xyr = 0 \quad \text{for all } x,\ y,\ r \in R.$$

This gives that

$$([F(x), y] + xy)r + y[F(x), r] = 0 \quad \text{for all } x,\ y,\ r \in R.$$

Using (3.23), we find

$$y[F(x), r] = 0 \quad \text{for all } x,\ y,\ r \in R. \tag{3.24}$$

Again substituting $[F(x), r]y$ for y in (3.24), we have

$$[F(x), r]y[F(x), r] = 0 \quad \text{for all } x,\ y,\ r \in R.$$

But since every $\star$-prime ring is semiprime, in above yields that $[F(x), r] = 0$ for all $x,\ r \in R$ i.e., $F(R) \subseteq Z(R)$.□

Theorem 3.9. *Let R be a $\star$-prime ring. If R admits a non zero generalized skew derivation F associated with a skew derivation D of R and an onto endomorphism g on R associated with D such that $F(xy) + [F(x), y] = 0$ for all $x,\ y \in R$, then R is commutative or $[[F(x), x], D(x)] = 0$ for all $x \in R$.*

Proof. We have

$$F(xy) + [F(x), y] = 0 \quad \text{for all } x,\ y \in R. \tag{3.25}$$

Replacing yx instead of y in (3.25), we have

$$F(xyx) + [F(x), yx] = 0 \quad \text{for all } x,\ y \in R.$$

This implies that

$$F(xy)x + g(xy)D(x) + y[F(x), x] + [F(x), y]x = 0 \quad \text{for all } x,\ y \in R.$$

This gives that

$$(F(xy) + [F(x), y])x + g(xy)D(x) + y[F(x), x] = 0 \quad \text{for all } x,\ y \in R.$$

By equation (3.25), we have

$$g(xy)D(x) + y[F(x), x] = 0 \quad \text{for all } x,\ y \in R. \tag{3.26}$$

Again substituting yr instead of y in (3.26), we have

$$g(xyr)D(x) + yr[F(x), x] = 0 \quad \text{for all } x,\ y,\ r \in R. \tag{3.27}$$

Using hypothesis as g is an endomorphism, we obtain

$$g(xy)g(r)D(x) + yr[F(x), x] = 0 \quad \text{for all } x,\ y,\ r \in R.$$

Since g is an onto map on R, we take $g(r) = r$.

$$g(xy)rD(x) + yr[F(x), x] = 0 \quad \text{for all } x,\ y,\ r \in R. \tag{3.28}$$

Replacing rs instead of r in (3.28), we get

$$g(xy)rsD(x) + yrs[F(x), x] = 0 \quad \text{for all } x,\ y,\ r,\ s \in R. \tag{3.29}$$

Multiplying (3.28) by s from right and subtracting from (3.29), we get

$$g(xy)r[D(x), s] + yr[[F(x), x], s] = 0 \quad \text{for all } x,\ y,\ r,\ s \in R. \tag{3.30}$$

Now, substituting s by $D(x)$ in (3.30), we obtain

$$yr[[F(x), x], D(x)] = 0 \quad \text{for all } x,\ y,\ r \in R.$$

Replacing $[z, t]$ by y, we have

$$[z, t]r[[F(x), x], D(x)] = 0 \quad \text{for all } x,\ r,\ t,\ z \in R. \tag{3.31}$$

This implies that

$$([z, t])^{\star}R[[F(x), x], D(x)] = (0), \quad \text{for all } x,\ t,\ z \in R. \tag{3.32}$$

Using $\star$-primness of R, we have either $[z, t] = 0$ or $[[F(x), x], D(x)] = 0$ for all $x,\ t,\ z \in R$. If $[z, t] = 0$ implies R is commutative. On the other hand $[[F(x), x], D(x)] = 0$ for all $x \in R$.□

Theorem 3.10. Let R be a $\star$-prime ring. If R admits a non zero generalized skew derivation F associated with a skew derivation D of R and an onto endomorphism g on R associated with D such that $F(xy) + [D(x), y] = 0$ for all $x,\ y \in R$, then R is commutative or $[[D(x), x], D(x)] = 0$ for all $x \in R$.

Proof. We have

$$F(xy) + [D(x), y] = 0 \quad \text{for all } x,\ y \in R. \tag{3.33}$$

Replacing y by yx in (3.33), we have

$$F(xyx) + [D(x), yx] = 0 \quad \text{for all } x,\ y \in R.$$

This implies that

$$F(xy)x + g(xy)D(x) + y[D(x), x] + [D(x), y]x = 0 \quad \text{for all } x,\ y \in R.$$

This gives that

$$(F(xy) + [D(x), y])x + g(xy)D(x) + y[D(x), x] = 0 \quad \text{for all } x,\ y \in R.$$

By equation (3.33), we have

$$g(xy)D(x) + y[D(x), x] = 0 \quad \text{for all } x,\ y \in R. \tag{3.34}$$

Again substituting yr for y in (3.34), we have

$$g(xyr)D(x) + yr[D(x), x] = 0 \quad \text{for all } x,\ y,\ r \in R. \tag{3.35}$$

As g is an endomorphism on R, we find

$$g(xy)g(r)D(x) + yr[D(x), x] = 0 \quad \text{for all } x,\ y,\ r \in R.$$

Now by hypothesis, we have

$$g(xy)rD(x) + yr[D(x), x] = 0 \quad \text{for all } x,\ y,\ r \in R. \tag{3.36}$$

Substituting rs for r in (3.36), we get

$$g(xy)rsD(x) + yrs[D(x), x] = 0 \quad \text{for all } x,\ y,\ r,\ s \in R. \tag{3.37}$$

Multiplying (3.36) by s from right and subtracting from (3.37), we obtain

$$g(xy)r[D(x), s] + yr[[D(x), x], s] = 0 \quad \text{for all } x,\ y,\ r,\ s \in R. \tag{3.38}$$

Replacing s by $D(x)$ in (3.38), we get

$$yr[[D(x), x], D(x)] = 0 \quad \text{for all } x,\ y,\ r \in R. \tag{3.39}$$

Again substitute y by $[s, y]$ in (3.39), we get

$$[s, y]r[[D(x), x], D(x)] = 0 \quad \text{for all } x,\ y,\ r,\ s \in R.$$

This implies that

$$([s, y])^{\star}R[[D(x), x], D(x)] = (0), \quad \text{for all } x,\ y,\ s \in R.$$

Using ⋆-primeness of R, we get either $[s, y] = 0$ or $[[D(x), x], D(x)] = 0$ for all $x,\ y,\ s \in R$. If $[s, y] = 0$ implies R is commutative. On the other hand $[[D(x), x], D(x)] = 0$ for all $x \in R$.□

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