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Some theorems for Carlitz's twisted (*h*,*q*)-Euler polynomials under S_4

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Abstract. The main purpose of this paper is to give some symmetric identities for Carlitz's twisted q-Euler polynomials related to the p-adic invariant integral on $\mathbb{Z}_p$ under symmetric group of degree four denoted by S_4.

1 Introduction

Recently, many mathematicians have studied on symmetric identities of some special functions. For example, Duran et al. [3] on q-Genocchi polynomials, Duran et al. [4] on weighted q-Genocchi polynomials, Araci et al. [1] on the new family of Euler numbers and polynomials, Araci et al. [2] on q-Frobenious Euler polynomials, moreover, Dolgy et al. [5] on h-extension of q-Euler polynomials and furthermore, Kim [6] on q-Euler numbers and polynomials of higher order have worked extensively by using p-adic q-integrals on $\mathbb{Z}_p$.

Along this paper, we use the following notations: $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}^*$ denotes the set of all positive natural numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$ in which p be an any odd prime number.

The normalized absolute value with respect to the theory of p-adic analysis is given by $|p|_p = p^{-1}$. When one mentions q-extension, q can be taken to be an indeterminate, a p-adic number $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and $q^x = \exp(x \log q)$ for $|x|_p \leq 1$, or a complex number $q \in \mathbb{C}$ with $|q| < 1$. For whichever x, the q-extension of x is defined as $[x]_q = \frac{1-q^x}{1-q} = 1 + q + q^2 + \cdots q^{x-1}$. Observe that $\lim_{q\to 1} [x]_q = x$ (see [2-6, 8-10]).

For

$$f \in UD(\mathbb{Z}_p) = \{f \,|f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic invariant integral on $\mathbb{Z}_p$ is defined [6] by

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x. \tag{1.1}$$

In view of the integral (1.1), one can derive with ease that

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{l=0}^{n-1} (-1)^{n-l-1} f(l)$$

where $f_n(x)$ implies $f(x+n)$. One can take a glance at the references [1], [2], [3], [4], [5], [8].

The Euler polynomials $E_n(x)$ are defined by means of the the following Taylor series expansion at $t = 0$:

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt}, \quad (|t| < \pi). \tag{1.2}$$

Subrogating $x = 0$ in the Eq. (1.2) yields $E_n(0) := E_n$ known as n-th Euler number (see e.g., [6], [7], [8], [9],).

Let $T_p =_{N\geq 1} C_{p^N} = \lim_{N\rightarrow\infty} C_{p^N}$, in which $C_{p^N} = \left\{ w : w^{p^N} = 1 \right\}$ is the cyclic group of order p^N. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \rightarrow C_p$ the locally constant function $\ell \rightarrow w^\ell$. For $q \in C_p$ with $|1-q|_p < 1$ and $w \in T_p$, in [9], Ryoo described the Carlitz's twisted q-Euler polynomials by the following p-adic invariant integral on $\mathbb{Z}_p$:

$$\mathcal{E}_{n,q,w}(x) = \int_{\mathbb{Z}_p} w^y \left[x+y\right]_q^n d\mu_{-1}(y) \quad (n \geq 0). \tag{1.3}$$

Letting $x = 0$ into the Eq. (1.3) gives $E_{n,q,w}(0) := E_{n,q,w}$ dubbed as n-th Carlitz's twisted q-Euler numbers.

Taking $w = 1$ and $q \rightarrow 1$ in the Eq. (1.3) yields to

$$E_n(x) := \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y).$$

In the next section, we derive some novel symmetric identities of Carlitz's twisted q-Euler polynomials associated with the fermionic p-adic invariant integral on $\mathbb{Z}_p$ under S_4.

2 Symmetric Identities for $E_{n,q,w}(x)$ under S_4

Let $w_i \in \mathbb{N}$ be a natural number which satisfies the condition $w_i \equiv 1(mod 2)$, in which $i \in \mathbb{Z}$ lies in $1 \leq i \leq 4$. From the Eqs. (1.1) and (1.3), we consider

$$\begin{aligned}
&\int_{\mathbb{Z}_p} w^{w_1w_2w_3y} e^{[w_1w_2w_3y+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t} d\mu_{-1}(y) \\
&= \lim_{N\rightarrow\infty} \sum_{y=0}^{p^N-1} (-1)^y w^{w_1w_2w_3y} e^{[w_1w_2w_3y+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t} \\
&= \lim_{N\rightarrow\infty} \sum_{l=0}^{w_4-1} \sum_{y=0}^{p^N-1} (-1)^{l+y} w^{w_1w_2w_3(l+w_4y)} \\
&\quad \times e^{[w_1w_2w_3(l+w_4y)+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t}.
\end{aligned}$$

Taking

$$\sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} (-1)^{i+j+k} w^{w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k}$$

on the both sides of the above equation yields

$$\begin{aligned}
I &= \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} (-1)^{i+j+k} w^{w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k} \qquad (2.1)\\
&\quad \times \int_{\mathbb{Z}_p} w^{w_1w_2w_3y} e^{[w_1w_2w_3y+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t} d\mu_{-1}(y) \\
&= \lim_{N\rightarrow\infty} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \sum_{l=0}^{w_4-1} \sum_{y=0}^{p^N-1} (-1)^{i+j+k+y+l} w^{w_1w_2w_3(l+w_4y)+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k} \\
&\quad \times e^{[w_1w_2w_3(l+w_4y)+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t}.
\end{aligned}$$

Notice that Eq. (2.1) is invariant for any permutation $\sigma \in S_4$. Therefore, we present the following theorem.

Theorem 2.1. *Let $w_i \in \mathbb{N}$ be a natural number which satisfies the condition $w_i \equiv 1(mod 2)$, in which $i \in \mathbb{Z}$ lies in $1 \leq i \leq 4$ and $n \geq 0$. Then the following*

$$\begin{aligned} I &= \sum_{i=0}^{w_{\sigma(1)}-1}\sum_{j=0}^{w_{\sigma(2)}-1}\sum_{k=0}^{w_{\sigma(3)}-1}(-1)^{i+j+k}w^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k} \\ &\quad\times\int_{\mathbb{Z}_p} w^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}(l+w_{\sigma(4)}y)} \\ &\quad\times e^{\left[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}y+w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}x+w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k\right]_q t}d\mu_{-1}(y)\end{aligned}$$

holds true for any $\sigma\in S_4$.

From the definition of q-number $[x]_q$ we readily derive that

$$\begin{aligned} &\left[w_1w_2w_3y+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k\right]_q \\ &= [w_1w_2w_3]_q\left[y+w_4x+\frac{w_4}{w_1}i+\frac{w_4}{w_2}j+\frac{w_4}{w_3}k\right]_{q^{w_1w_2w_3}},\end{aligned} \tag{2.2}$$

which gives

$$\begin{aligned} &\int_{\mathbb{Z}_p} w^{w_1w_2w_3y}e^{[w_1w_2w_3y+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t}d\mu_{-1}(y) \\ &=\sum_{n=0}^{\infty}[w_1w_2w_3]_q^n\left(\int_{\mathbb{Z}_p} w^{w_1w_2w_3y}\left[y+w_4x+\frac{w_4}{w_1}i+\frac{w_4}{w_2}j+\frac{w_4}{w_3}k\right]^n_{q^{w_1w_2w_3}}d\mu_{-1}(y)\right)\frac{t^n}{n!} \\ &=\sum_{n=0}^{\infty}[w_1w_2w_3]_q^n\,\mathcal{E}_{m,q^{w_1w_2w_3},w^{w_1w_2w_3}}\left(w_4x+\frac{w_4}{w_1}i+\frac{w_4}{w_2}j+\frac{w_4}{w_3}k\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

From Eq. (2.3), we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} w^{w_1w_2w_3y}\left[w_1w_2w_3y+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k\right]_q d\mu_{-1}(y) \\ &=[w_1w_2w_3]_q^n\,\mathcal{E}_{m,q^{w_1w_2w_3},w^{w_1w_2w_3}}\left(w_4x+\frac{w_4}{w_1}i+\frac{w_4}{w_2}j+\frac{w_4}{w_3}k\right),\ \text{for } n\geq 0.\end{aligned} \tag{2.4}$$

Hereby, from the Theorem 2.1 and Eq. (2.4), we procure the following theorem.

Theorem 2.2. *Let $w_i\in\mathbb{N}$ be a natural number which satisfies the condition $w_i\equiv 1(mod2)$, in which $i\in\mathbb{Z}$ lies in $1\leq i\leq 4$ and $n\geq 0$. For $n\geq 0$, the following*

$$\begin{aligned} I &= \left[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}\right]_q^n\sum_{i=0}^{w_{\sigma(1)}-1}\sum_{j=0}^{w_{\sigma(2)}-1}\sum_{k=0}^{w_{\sigma(3)}-1}(-1)^{i+j+k} \\ &\quad\times w^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k} \\ &\quad\times\mathcal{E}_{m,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}},w^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}\left(w_{\sigma(4)}x+\frac{w_{\sigma(4)}}{w_{\sigma(1)}}i+\frac{w_{\sigma(4)}}{w_{\sigma(2)}}j+\frac{w_{\sigma(4)}}{w_{\sigma(3)}}k\right)\end{aligned}$$

holds true for any $\sigma\in S_4$.

By using the definitions of $[x]_q$ and binomial theorem:

$$\begin{aligned} &\left[y+w_4x+\frac{w_4}{w_1}i+\frac{w_4}{w_2}j+\frac{w_4}{w_3}k\right]^n_{q^{w_1w_2w_3}} \\ &=\sum_{m=0}^{n}\binom{n}{m}\left(\frac{[w_4]_q}{[w_1w_2w_3]_q}\right)^{n-m}[w_2w_3i+w_1w_3j+w_1w_2k]_{q^{w_4}}^{n-m} \\ &\quad\times q^{m(w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k)}[y+w_4x]^m_{q^{w_1w_2w_3}}.\end{aligned} \tag{2.5}$$

Appliying $\int_{\mathbb{Z}_p} w^{w_1w_2w_3y} d\mu_{-1}(y)$ on the both sides of the above gives

$$\begin{aligned} &\int_{\mathbb{Z}_p} w^{w_1w_2w_3y} \left[y + w_4x + \frac{w_4}{w_1}i + \frac{w_4}{w_2}j + \frac{w_4}{w_3}k\right]^n_{q^{w_1w_2w_3}} d\mu_{-1}(y) \\ =& \sum_{m=0}^{n} \binom{n}{m} \left(\frac{[w_4]_q}{[w_1w_2w_3]_q}\right)^{n-m} [w_2w_3i + w_1w_3j + w_1w_2k]^{n-m}_{q^{w_4}} \\ &\times q^{m(w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k)} \int_{\mathbb{Z}_p} w^{w_1w_2w_3y} [y + w_4x]^m_{q^{w_1w_2w_3}} d\mu_{-1}(y) \\ =& \sum_{m=0}^{n} \binom{n}{m} \left(\frac{[w_4]_q}{[w_1w_2w_3]_q}\right)^{n-m} [w_2w_3i + w_1w_3j + w_1w_2k]^{n-m}_{q^{w_4}} \\ &\times q^{m(w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k)} \mathcal{E}_{m,q^{w_1w_2w_3},w^{w_1w_2w_3}}(w_4x). \end{aligned} \tag{2.6}$$

By the Eq. (2.6), we procure

$$\begin{aligned} &[w_1w_2w_3]^n_q \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1} (-1)^{i+j+k} w^{w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k} \\ &\times \int_{\mathbb{Z}_p} w^{w_1w_2w_3y} \left[y + w_4x + \frac{w_4}{w_1}i + \frac{w_4}{w_2}j + \frac{w_4}{w_3}k\right]^n_{q^{w_1w_2w_3}} d\mu_{-1}(y) \\ =& \sum_{m=0}^{n} \binom{n}{m} [w_1w_2w_3]^m_q [w_4]^{n-m}_q \mathcal{E}_{n,q^{w_1w_2w_3},w^{w_1w_2w_3}}(w_4x) \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1} \\ &\sum_{k=0}^{w_3-1} (-1)^{i+j+k} w^{w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k} q^{m(w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k)} \\ &\times [w_2w_3i + w_1w_3j + w_1w_2k]^{n-m}_{q^{w_4}} \\ =& \sum_{m=0}^{n} \binom{n}{m} [w_1w_2w_3]^m_q [w_4]^{n-m}_q \mathcal{E}_{n,q^{w_1w_2w_3},w^{w_1w_2w_3}}(w_4x)\, U_{n,q^{w_4},w^{w_4}}(w_1,w_2,w_3 \mid m), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} &U_{n,q,w}(w_1,w_2,w_3 \mid m) \\ =& \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1} (-1)^{i+j+k} w^{w_2w_3i+w_1w_3j+w_1w_2k} \\ &\times q^{m(w_2w_3i+w_1w_3j+w_1w_2k)} [w_2w_3i + w_1w_3j + w_1w_2k]^{n-m}_q . \end{aligned} \tag{2.8}$$

Last of all, from Eqs. (2.7) and (2.8), we obtain the following theorem.

Theorem 2.3. *Let $w_i \in \mathbb{N}$ be a natural number which satisfies the condition $w_i \equiv 1(mod2)$, in which $i \in \mathbb{Z}$ lies in $1 \leq i \leq 4$ and $n \geq 0$. Hence, the following expression*

$$\begin{aligned} &\sum_{m=0}^{n} \binom{n}{m} \left[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}\right]^m_q \left[w_{\sigma(4)}\right]^{n-m}_q \\ &\times \mathcal{E}_{n,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}},w^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}\left(w_{\sigma(4)}x\right) U_{n,q^{w_{\sigma(4)}},w^{w_{\sigma(4)}}}\left(w_{\sigma(1)},w_{\sigma(2)},w_{\sigma(3)} \mid m\right) \end{aligned}$$

holds true for some $\sigma \in S_4$.

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