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On Almost Pseudo Symmetric K$\ddot{a}$hler Manifolds

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Abstract. We define almost pseudo symmetry, almost pseudo Bochner symmetry, almost pseudo Ricci symmetry and almost pseudo Bochner Ricci symmetry of K$\ddot{a}$hler manifolds and obtain results pertaining to Ricci solitons.

1 Introduction

We know, symmetric spaces perform a leading role in differential geometry. In the late twenties, Cartan [3] originated Riemannian symmetric spaces, in particular, obtained a classification of those spaces. Let (M, g) be an n-dimensional Riemannian manifold with the metric g and the Levi-Civita connection ∇. A Riemannian manifold is called locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M, g). The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature. During the last five decades the notion of locally symmetric manifolds has been studied by many authors in several ways to a different extent such as conformally symmetric manifolds [5], semi-symmetric manifolds [15], pseudo symmetric manifolds [4, 9], weakly symmetric manifolds [17] and almost pseudo concircularly symmetric manifolds [8] etc.
A non-flat Riemannian manifold (M, g) is said to be almost pseudo symmetric manifold [8] if its curvature tensor satisfies the condition

$$
\begin{aligned}
(\nabla_X R)(Y, Z, U, W) &= [A(X) + B(X)]R(Y, Z, U, W) + A(Y)R(X, Z, U, W) \\
&+A(Z)R(Y, X, U, W) + A(U)R(Y, Z, X, W) + A(W)R(Y, Z, U, X),
\end{aligned}
\tag{1.1}
$$

where A, B are two non zero 1-forms defined by

$$
g(X, \rho) = A(X), \;\; g(X, Q) = B(X),
\tag{1.2}
$$

for all vector fields X, ∇ denotes the operator of covariant differentiation with respect to the metric g. The 1-forms A and B are called the associated 1-forms. If $A = B$ in (1.1) then the manifold reduces to a pseudo symmetric manifold, introduced by Chaki [4]. If $A = B = 0$ in (1.1) then the manifold reduces to a symmetric manifold in the sense of Cartan [3].
In a paper [6] Chaki and Kawaguchi introduced a type of non-flat Riemannian manifold (M, g) whose Ricci tensor S of type $(0, 2)$ satisfies the condition

$$
(\nabla_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X),
\tag{1.3}
$$

where A, B and ∇ have the meaning already stated. Such a manifold was called an almost pseudo Ricci symmetric manifold.
It is to be noted that the notion of pseudo symmetry in the sense of Chaki [4] is different from that of Deszcz [7]. It may be mentioned that an almost pseudo symmetric manifold is not a particular case of a weakly symmetric manifold introduced by Tamassy and Binh [17]. Tamassy, De and Binh [18] found interesting results on weakly symmetric and weakly Ricci symmetric K$\ddot{a}$hler manifolds in 2000. The authors Shaikh, Hui and Bagewadi [16] discussed on quasi-conformlly flat almost pseudo Ricci symmetric manifolds in 2010. Also we studied some symmetric conditions by using some curvature tensors on generalized complex space form [12, 1].

2 Preliminaries

A Kähler manifold is an n(even)-dimensional manifold, with a complex structure J and a positive-definite metric g which satisfies the following conditions;

$$J^2(X) = -X,\ \ g(JX, JY) = g(X, Y)\ \ and\ \ (\nabla_X J)(Y) = 0, \tag{2.1}$$

where ∇ means covariant derivative according to the Levi-Civita connection.
The formulae [2]

$$R(X,Y) = R(JX, JY), \tag{2.2}$$
$$S(X,Y) = S(JX, JY), \tag{2.3}$$
$$S(X, JY) + S(JX, Y) = 0,$$

are well known for a Kähler manifold.

Definition 2.1. A Kähler manifold is called almost pseudo Bochner symmetric manifold if its Bochner curvature tensor D of type (0,4) is not zero and satisfies the condition

$$\begin{aligned}(\nabla_X D)(Y,Z,U,W) &= [A(X)+B(X)]D(Y,Z,U,W)+A(Y)D(X,Z,U,W)\\ &+A(Z)D(Y,X,U,W)+A(U)D(Y,Z,X,W)+A(W)D(Y,Z,U,X),\end{aligned} \tag{2.4}$$

where A, B are 1-forms(not simultaneously zero) and D is given by [19],

$$\begin{aligned}D(X,Y,Z,U) &= R(X,Y,Z,U) - \frac{1}{2n+4}[g(Y,Z)S(X,U) - S(X,Z)g(Y,U)\\ &+ g(JY,Z)S(JX,U) - S(JX,Z)g(JY,U) + S(Y,Z)g(X,U)\\ &- g(X,Z)S(Y,U) + S(JY,Z)g(JX,U) - g(JX,Z)S(JY,U)\\ &- 2S(Y,JX)g(JZ,U) - 2S(JZ,U)g(JX,Y)]\\ &+ \frac{r}{(2n+2)(2n+4)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\\ &+ g(JY,Z)g(JX,U) - g(JX,Z)g(JY,U) - 2g(JX,Y)g(JZ,U)].\end{aligned} \tag{2.5}$$

Definition 2.2. A Kähler manifold is called almost pseudo Bochner Ricci symmetric manifold if its Bochner Ricci tensor K of type $(0,2)$ is not identically zero and satisfies the condition

$$(\nabla_X K)(Y,Z) = [A(X)+B(X)]K(Y,Z) + A(Y)K(X,Z) + A(Z)K(Y,X) \tag{2.6}$$

where A, B are nowhere vanishing 1-forms and K is given by,

$$K(Y,Z) = \frac{n}{2n+4}[S(Y,Z) - \frac{r}{2(n+1)}g(Y,Z)]. \tag{2.7}$$

Also we need the notion of Ricci solitons. It is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g). A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$L_V g + 2S + 2\lambda g = 0, \tag{2.8}$$

where S is Ricci tensor of M and L_V denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively [10].
Suppose (M, g) is a Kähler manifold and (g, V, λ) is a Ricci soliton in (M, g). If V is conformal killing vector field, then

$$(L_V g) = \alpha g. \tag{2.9}$$

where α is some scalar function.

3 Almost Pseudo Symmetric K$\ddot{a}$hler Manifolds

In this section we study almost pseudo symmetric K$\ddot{a}$hler manifold. We have

$$R(JY, JZ, U, W) = R(Y, Z, U, W). \tag{3.1}$$

Taking the covariant derivative of (3.1), we get

$$(\nabla_X R)(JY, JZ, U, W) = (\nabla_X R)(Y, Z, U, W). \tag{3.2}$$

Using (1.1) in (3.2) and by virtue of (3.1) we get

$$\begin{aligned} A(Y)R(X, Z, U, W) + A(Z)R(Y, X, U, W) &= A(JY)R(X, JZ, U, W) \\ &+ A(JZ)R(JY, X, U, W). \end{aligned} \tag{3.3}$$

Putting $Z = U = e_i$ in (3.3) after simplification, we get

$$A(Y)S(X, W) - R(Y, X, W, \rho) = A(JY)S(X, JW) + R(JY, X, W, J\rho). \tag{3.4}$$

And by replacing $Y = \rho = e_i$ $(1 \leq i \leq n)$ in (3.4) and summing over i, we infer

$$(n-2)S(X, W) = 0. \tag{3.5}$$

This implies

$$S(X, W) = 0. \tag{3.6}$$

Thus we can state the following;

Theorem 3.1. *Let M be an almost pseudo symmetric Kähler manifold then it is Ricci flat.*

Using equation (3.6) in (2.8), we get

$$(L_V g)(X, W) + 2\lambda g(X, W) = 0. \tag{3.7}$$

Putting $X = W = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i $(1 \leq i \leq n)$, we get

$$(L_V g)(e_i, e_i) + 2\lambda g(e_i, e_i) = 0. \tag{3.8}$$

This implies

$$divV + \lambda n = 0. \tag{3.9}$$

If V is solenoidal then $divV = 0$. Therefore the equation (3.9) can be reduced to

$$\lambda = 0. \tag{3.10}$$

Hence we gain the following

Corollary 3.2. *The Ricci soliton (g, V, λ) in an almost pseudo symmetric Kähler manifold is steady if and only if V is solenoidal.*

Equation (3.7) can be write

$$(L_V g)(X, W) = -2\lambda g(X, W), \tag{3.11}$$

comparing equation (3.11) and (2.9) we write the following.

Corollary 3.3. *If (g, V, λ) be a Ricci soliton in an almost pseudo symmetric Kähler manifold then V is conformal killing.*

4 Almost Pseudo Bochner Symmetric Kähler Manifolds

Using equations (2.2) and (2.5) we find

$$D(JY, JZ, U, W) = D(Y, Z, U, W). \tag{4.1}$$

In this section we suppose that (M, g) is an almost pseudo Bochner symmetric and Kähler manifold. Then (2.4), (2.1) and (4.1) we find

$$(\nabla_X D)(JY, JZ, U, W) = (\nabla_X D)(Y, Z, U, W). \tag{4.2}$$

Using (2.4) in (4.2), we get

$$\begin{aligned} A(Y)D(X, Z, U, W) + D(Z)R(Y, X, U, W) &= A(JY)D(X, JZ, U, W) \\ &+ A(JZ)D(JY, X, U, W). \end{aligned} \tag{4.3}$$

Putting $Z = U = e_i$ in (4.3), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i $(1 \leq i \leq n)$, we get

$$A(Y)K(X, W) - A(D(Y, X)W) = -A(JY)K(JX, W) - A(D(JY, X)JW). \tag{4.4}$$

Again putting $Y = \rho = e_i$ in (4.4) and taking sum over $i(1 \leq i \leq n)$, we get

$$(n - 2)K(X, W) = 0. \tag{4.5}$$

This can be written as

$$K(X, W) = 0. \tag{4.6}$$

The above equation in (2.7) gives

$$S(X, W) = \frac{r}{2(n+1)} g(X, W). \tag{4.7}$$

Thus we can state the following;

Theorem 4.1. *If M is an almost pseudo Bochner symmetric Kähler manifold then it is an Einstein manifold.*

Using equation (4.7) in (2.8), we get

$$(L_V g)(X, W) + 2[\frac{r}{2(n+1)} + \lambda]g(X, W) = 0. \tag{4.8}$$

Setting $X = W = e_i$ in (4.8) and then taking summation over i $(1 \leq i \leq n)$, we obtain

$$(L_V g)(e_i, e_i) + 2\frac{r}{2(n+1)} g(e_i, e_i) + 2\lambda g(e_i, e_i) = 0. \tag{4.9}$$

This implies

$$divV + \frac{r}{2(n+1)} n + \lambda n = 0. \tag{4.10}$$

If V is solenoidal then $divV = 0$. Therefore the equation (4.10) can be reduced to

$$\lambda = -\frac{r}{2(n+1)}. \tag{4.11}$$

Hence we have the following

Corollary 4.2. *Let (g, V, λ) be a Ricci soliton in an almost pseudo Bochner symmetric Kähler manifold. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the sign of scalar curvature.*

5 Almost Pseudo Ricci Symmetric Kähler Manifolds

If the manifold M is an almost pseudo Ricci symmetric Kähler Manifold, then from (1.3), (2.1) and (2.3) we find

$$(\nabla_X S)(JY, JZ) = (\nabla_X S)(Y, Z). \tag{5.1}$$

Using (1.3) in (5.1), we get

$$A(JY)S(X, JZ) + A(JZ)S(JY, X) = A(Y)S(X, Z) + A(Z)S(Y, X). \tag{5.2}$$

By substituting $Y = \rho = e_i$ $(1 \leq i \leq n)$ in (5.2) and summing over i, we get

$$(n + 2)S(X, Z) = 0. \tag{5.3}$$

This implies

$$S(X, Z) = 0. \tag{5.4}$$

Thus we can write the following;

Theorem 5.1. *Let M be an almost pseudo Ricci-symmetric Kähler manifold then it is Ricci flat.*

Using equation (5.4) in (2.8), we get

$$(L_V g)(X, Z) + 2\lambda g(X, Z) = 0. \tag{5.5}$$

Let $\{e_i : i = 1, 2, ..., n\}$ be an orthonormal basis of the tangent space at each point of the manifold. Then setting $X = Z = e_i$ in (5.5) and then taking summation over i, we obtain

$$(L_V g)(e_i, e_i) + 2\lambda g(e_i, e_i) = 0. \tag{5.6}$$

This implies

$$divV + \lambda n = 0. \tag{5.7}$$

If V is solenoidal then $divV = 0$. Therefore the equation (5.7) can be reduced to

$$\lambda = 0. \tag{5.8}$$

Then we can state the following result

Corollary 5.2. *Ricci soliton (g, V, λ) in an almost pseudo Ricci symmetric Kähler manifold is steady if and only if V is solenoidal.*

6 Almost Pseudo Bochner Ricci Symmetric Kähler Manifolds

If the manifold M is an almost pseudo Bochner Ricci symmetric Kähler manifold, then we can easily write

$$K(JY, JZ) = K(Y, Z). \tag{6.1}$$

Taking the covariant derivative of (6.1), we get

$$(\nabla_X K)(JY, JZ) = (\nabla_X K)(Y, Z). \tag{6.2}$$

Using (2.6) in (6.2), we get

$$A(JY)K(X, JZ) + A(JZ)K(JY, X) = A(Y)K(X, Z) + A(Z)K(Y, X). \tag{6.3}$$

Putting $Y = \rho = e_i$ in (6.3), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i $(1 \leq i \leq n)$, we get

$$(n + 2)K(X, Z) = 0. \tag{6.4}$$

This implies

$$K(X, Z) = 0. \tag{6.5}$$

The above equation in (2.7) we write

$$S(X, Z) = \frac{r}{2(n + 1)} g(X, Z). \tag{6.6}$$

Hence from the above equation, we conclude;

Theorem 6.1. *An almost pseudo Bochner Ricci-symmetric Kähler manifold M is an Einstein manifold.*

Using equation (6.6) in (2.8), we get

$$(L_V g)(X,Z) + 2\frac{r}{2(n+1)} + 2\lambda g(X,Z) = 0. \tag{6.7}$$

Setting $X = Z = e_i$ in (6.7) and then taking summation over i $(1 \le i \le n)$, we obtain

$$(L_V g)(e_i,e_i) + 2\frac{r}{2(n+1)}g(e_i,e_i) + 2\lambda g(e_i,e_i) = 0. \tag{6.8}$$

This implies

$$divV + \frac{r}{2(n+1)}n + \lambda n = 0. \tag{6.9}$$

If V is solenoidal then $divV = 0$. Therefore the equation (6.9) can be reduced to

$$\lambda = -\frac{r}{2(n+1)}. \tag{6.10}$$

Thus we have the following

Corollary 6.2. *Let (g,V,λ) be a Ricci soliton in an almost pseudo Bochner Ricci symmetric Kähler manifold. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the scalar curvature is positive, zero and negative respectively.*

Equation (6.7) can be written as

$$(L_V g)(X,Z) = -2[\frac{r}{2(n+1)} + \lambda]g(X,Z), \tag{6.11}$$

comparing equation (6.11) and (2.9) we can write the following

Theorem 6.3. *If (g,V,λ) is a Ricci soliton in an almost pseudo Ricci symmetric Kähler manifold then V is conformal killing.*

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