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On Semi-Continuity of the Metric Co-Projection

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Abstract. In this paper, we discuss the semi-continuity (upper semi-continuity, lower semi-continuity, upper (K)-semi-continuity and lower (K)-semi-continuity) properties of the maps associated with best coapproximation. The underlying spaces are metric spaces and metric linear spaces. The results proved in the paper generalize and extend several known results on the subject.

1 Introduction and Preliminaries

A kind of approximation, called best coapproximation was introduced in normed linear spaces by C. Franchetti and M. Furi [1] to obtain some characterizations of real Hilbert spaces among real Banach spaces. This study was subsequently taken up by many researchers in normed linear spaces and Hilbert spaces (see e.g. [3],[4],[6],[11]-[13], and references cited therein). But only a few have taken up this study in more general abstract spaces (see e.g. [7]-[10]). The theory of best coapproximation is very less developed as compared to the theory of best approximation. The present paper is also a step in this direction. In this paper, we discuss the semi-continuity (upper semi-continuity, lower semi-continuity, upper (K)-semi-continuity and lower (K)-semi-continuity) properties of the maps associated with best coapproximation. The underlying spaces are metric spaces and metric linear spaces. The results proved in the paper generalize and extend various known results including those proved in [3] and [13]. Some of the proved results are similar to those for best approximation maps proved in [2],[5] and [14] for normed linear spaces. We start with a few definitions.

Let G be a non-empty subset of a metric space (X, d). An element $g_0 \in G$ is called a **best coapproximation (best approximation)** to $x \in X$ if

$$d(g_0, g) \leq d(x, g)\ (d(x, g_0) \leq d(x, g))$$

for all $g \in G$. The set of all such $g_0 \in G$ is denoted by $R_G(x)(P_G(x))$. The set G is called **co-proximinal (proximinal)** if $R_G(x)$ $(P_G(x))$ contains at least one element for every $x \in X$. If for each $x \in X$, $R_G(x)(P_G(x))$ has exactly one element, then the set G is called **co-Chebyshev (Chebyshev)**.

We shall denote the set $\{x \in X : R_G(x) \neq \phi\}(\{x \in X : P_G(x) \neq \phi\})$ by $D(R_G)(D(P_G))$ and the set $\{x \in X : g_0 \in R_G(x)\}(\{x \in X : g_0 \in P_G(x)\})$ by $R_G^{-1}(g_0)(P_G^{-1}(g_0))$. For a subset A of X, the set $R_G(A)(P_G(A))$ is defined as $R_G(A) = \bigcup_{x\in A} R_G(x)(P_G(A) = \bigcup_{x\in A} P_G(x))$.

The set-valued mapping $R_G(P_G) : X \to 2^G \equiv$ the collection of all subsets of G defined by $R_G(x) = \{g_0 \in G : d(g_0, g) \leq d(x, g)\ for\ every\ g \in G\}$ $(P_G(x) = \{g_0 \in G : d(x, g_0) \leq d(x, g)\ for\ every\ g \in G\})$ is called **metric co-projection (metric projection)**.

Remarks:

(i) A proximinal subset of a metric space need not be co-proximinal: In the Euclidian plane $X = \mathbb{R}^2$, the set $G = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, being compact subset of $\mathbb{R}^2$ is proximinal. However, G is not co-proximinal as $(0, 0) \in \mathbb{R}^2$ does not have any best coapproximation in G.

(ii) A co-proximinal subset of a metric space need not be proximinal: Let $X = \mathbb{R} - \{1\}$ and $M = (1, 2]$, then M is a co-proximinal subset of X but is not proximinal.

(iii) A Chebyshev subset of a metric space need not be co-Chebyshev: Let $X = \mathbb{R}$ and $G = [1, 2]$, then G is Chebyshev but not co-Chebyshev.

(iv) A co-Chebyshev subset of a metric space need not be Chebyshev: Let $X = \mathbb{R}^2$ with the metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ and $G = \{(x, y) \in \mathbb{R}^2 : x = y\}$. Then G is a proximinal subset of (X, d). We have $P_G(x, y) = \{\alpha(x, x)+(1-\alpha)(y, y) : 0 \leq \alpha \leq 1\}$ i.e. G is not Chebyshev, but $R_G(x, y) = \{(\frac{x+y}{2}, \frac{x+y}{2})\}$ i.e. G is co-Chebyshev.

(v) The set $R_G(x)$ $(P_G(x))$ is closed if G is closed.

(vi) Every co-proximinal (proximinal) set is closed.

(vii) The set $R_G^{-1}(g_0)(P_G^{-1}(g_0))$ is a closed set for every $g_0 \in G$.

(viii) If G is subspace of a metric linear space (X, d), then $g_0 \in R_G(x)$ $(g_0 \in P_G(x))$ if and only if $x-g_0 \in R_G^{-1}(0)$ $(x - g_0 \in P_G^{-1}(0))$ and $R_G(x+g) = R_G(x)+g,\ P_G(x+g) = P_G(x)+g$ for every $g \in G$.

For metric spaces X and Y, a mapping $u : X \to 2^Y$ is called
(i) **upper (K)-semi-continuous (u.(K)-s.c.)** if $x_n \to x,\ y_n \in u(x_n),\ y_n \to y$ imply $y \in u(x)$.
(ii) **lower (K)-semi-continuous (l.(K)-s.c.)** if $x_n \to x,\ y \in u(x)$ imply the existence of a sequence $\{y_n\}$ such that $y_n \in u(x_n)$ and $y_n \to y$.
(iii) **upper semi-continuous (lower semi-continuous)** if the set

$$H = \{x \in X : u(x) \bigcap N \neq \phi\}$$

is closed(open) for every closed(open) subset $N \subseteq Y$.

2 Main Results

It is known (see [7]) that for a closed subset G of a metric space (X, d), the map R_G is u.(K)-s.c. on $D(R_G)$. Using this result, we prove our first theorem.

Theorem 2.1. *Let G be a closed subset of a metric space (X, d) then*
(i) For each compact subset $A \subseteq D(R_G)$, the subset $R_G(A)$ is closed.
(ii) The relations $\{x_n\} \to x,\ y \in R_G(x_n)$ $(n = 1, 2, 3, ...)$ imply $y \in R_G(x)$.

Proof. (i) Let A be a compact subset of $D(R_G)$ and g_0 a limit point of $R_G(A)$. Then there exist a sequence $\{g_n\} \subseteq R_G(A)$ such that $g_n \to g_0$. Now, $g_n \in R_G(A)$ implies that there exist $x_n \in A$ such that $g_n \in R_G(x_n)$ $(n = 1, 2, 3, ...)$. Since A is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\} \to x$. Now, $\{x_{n_k}\} \to x,\ g_{n_k} \in R_G(x_{n_k})$ *and* $\{g_{n_k}\} \to g_0$, so using upper (K)-semi-continuity of R_G, we have $g_0 \in R_G(x) \subseteq R_G(A)$ and therefore $R_G(A)$ is closed.
(ii) follows from the definition of upper (K)-semi-continuity.

A non-empty subset G of a metric space (X, d) is said to be **boundedly compact** if every bounded sequence in G has a convergent subsequence in X.

The following result shows that the map R_G is u.s.c. if the set $R_G^{-1}(0)$ is boundedly compact.

Theorem 2.2. *Let G be a closed linear subspace of a metric linear space (X, d) and $R_G^{-1}(0)$ is boundedly compact then*
(i)$R_G(x)$ is compact for each $x \in D(R_G)$.
(ii)R_G is u.s.c. on $D(R_G)$.
(iii) $R_G(A)$ is compact for each compact subset A of $D(R_G)$.

Proof. (i) and (ii) have been proved in [8].
(iii) Let $\{g_n\}$ be a sequence in $R_G(A)$ then $g_n \in R_G(x_n)$ $(n = 1, 2, 3, ...)$ for some $x_n \in A$. Since A is compact, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that $\{x_{n_i}\} \to x \in A$. As $g_{n_i} \in$

$R_G(x_{n_i})$, $x_{n_i} - g_{n_i} \in R_G^{-1}(0)$. Consider $d(x_{n_i} - g_{n_i}, 0) = d(x_{n_i}, g_{n_i}) \le d(x_{n_i}, 0) + d(0, g_{n_i}) \le 2d(x_{n_i}, 0)$. This implies that $\{x_{n_i} - g_{n_i}\}$ is a bounded sequence in $R_G^{-1}(0)$ as the sequence $\{x_{n_i}\}$ is bounded.

Since $R_G^{-1}(0)$ is boundedly compact and closed, $\{x_{n_i} - g_{n_i}\}$ has a subsequence $\{x_{n_{i_m}} - g_{n_{i_m}}\} \to y \in R_G^{-1}(0)$. Also $\{x_{n_{i_m}}\} \to x \in A$. Therefore, we have $\{g_{n_{i_m}}\} \to x - y$. Since $d(g_{n_{i_m}}, g) \le d(x_{n_{i_m}}, g)$ for every $g \in G$ we get, $x - y \in R_G(x) \subseteq R_G(A)$. Hence $R_G(A)$ is compact.

Replacing the bounded compactness of $R_G^{-1}(0)$ by bounded compactness of G, the above result was proved in [7] for metric spaces.

Remarks:

(i) If we take G to be a subset instead of a subspace, then bounded compactness of $R_G^{-1}(0)$ need not imply compactness of $R_G(x)$.
Example: Let $X = \mathbb{R}$ with the usual metric and $G = (-1, 1]$, then $R_G^{-1}(0) = (-\infty, -2] \bigcup [2, \infty)$ is a boundedly compact set but $R_G(-2) = (-1, 0]$ is not compact.

(ii) The converse parts of (i) and (ii) of Theorem 2.2 are not true i.e. for a subspace G of a metric linear space (X, d), if
(a)$R_G(x)$ is compact for each $x \in D(R_G)$ then $R_G^{-1}(0)$ need not be boundedly compact.
(b)R_G is u.s.c. on $D(R_G)$ then $R_G^{-1}(0)$ need not be boundedly compact.
Example: Let $X = l_1$ and $G = \{(g, 0, 0, 0, ...) : g \in \mathbb{R}\} \subseteq X$, then G is boundedly compact and so R_G is u.s.c. and $R_G(x)$ is compact for each $x \in D(R_G)$. But $R_G^{-1}(0) = \{x \in l_1 : ||0 - g|| \le ||x - g||\ for\ all\ g \in G\} = \{x \in l_1 : |g_1| \le |x_1 - g_1| + |x_2| + |x_3| + ...\}$ is not boundedly compact as the sequence $\{(0, 1, 0, 0, ...), (0, 0, 1, 0, 0, 0, ...), (0, 0, 0, 1, 0, 0, 0...)...\}$ is a bounded sequence in $R_G^{-1}(0)$ having no convergent subsequence.

(iii) There exist subsets G of a metric space (X, d) such that
(i) R_G is u.s.c. on $D(R_G)$ but G is not boundedly compact.
(ii) $R_G(x)$ is compact for each $x \in D(R_G)$ but G is not boundedly compact.
Example: Let $G = X = l_1$ then G being an infinite dimensional space, is not boundedly compact but $R_G^{-1}(0) = \{0\}$ is a boundedlly compact set and so R_G is u.s.c.
Example: Let $X = [1, 2)$ with the usual metric d and $G = [1 + \varepsilon, 2)$, $0 \le \varepsilon < 0.5$. Let $x \in X \backslash G$ be such that $d(x, 1 + \varepsilon) = \delta$, then $R_G(x) = [1 + \varepsilon, 1 + \varepsilon + \delta]$ i.e. $R_G(x)$ is compact for each $x \in X$ but G is not boundedly compact since the bounded sequence $\{2 - \frac{1}{n} : n \in \mathbb{N}\}$ has no subsequence converging in X.

Let G be a closed linear subspace of a metric linear space (X, d). Then the set-valued mapping v_G of $D(R_G)/G$ into $2^{R_G^{-1}(0)}$ is defined as:

$$v_G(x + G) = \{x - R_G(x)\} = \{x - g_0 : g_0 \in R_G(x)\}.$$

If $x + G = y + G$, then $x - y \in G$. This gives, $R_G(x) = R_G(y + (x - y)) = R_G(y) + (x - y)$ i.e. $R_G(x) = R_G(y) + x - y$ and so $x - R_G(x) = y - R_G(y)$. Hence v_G is well defined. It is easy to see that $v_G(x + G) \in 2^{R_G^{-1}(0)}$.

Since for a closed linear subspace G of a metric linear space (X, d), $R_G(x)$ is closed, we have $x - R_G(x)$ is closed i.e. $v_G(x + G)$ is closed.

Concerning the upper (K)-semi-continuity of the map v_G, we have

Theorem 2.3. *Let G be a closed linear subspace of a metric linear space (X, d), then the mapping v_G is u.(K)-s.c. on $D(R_G)/G$.*

Proof. Let $x_n + G \to x + G\ in\ D(R_G)/G,\ y_n \in v_G(x_n + G)$ such that $y_n \to y$.

Now, $x_n + G \to x + G$ implies the existence of a sequence $\{g_n\}$ in G such that $\{x_n + g_n\} \to x$. Since $y_n \in v_G(x_n + G)$, we have $x_n - y_n \in R_G(x_n)$, $y_n \to y$. Therefore, we have $\{x_n + g_n\} \to x$, $x_n - y_n + g_n \in R_G(x_n + g_n)\ and\ \{x_n - y_n + g_n\} \to x - y$. Since R_G is upper (K)-semi-continuous, we have $x - y \in R_G(x)$ and so $y \in v_G(x + G)$. Hence v_G is u.(K)-s.c.

Analogous to Theorem 2.1., we have

Theorem 2.4. *Let G be a closed linear subspace of a metric linear space (X, d) then*
(i) For each compact subset $A \subseteq D(R_G)/G$, the subset $v_G(A)$ is closed in $R_G^{-1}(0)$.
(ii) The relation $x_n + G \to x + G$ in $D(R_G)/G$, $y \in v_G(x_n + G)$ $(n = 1, 2, 3, ...)$ imply $y \in v_G(x + G)$.

If G is a closed linear subspace of a metric linear space (X, d), the canonical mapping w_G of X onto X/G is defined as $w_G(x) = x + G,\ x \in X$.

For a coproximinal subspace G, we have

Lemma 2.5. *Let G be a co-proximinal subspace of a metric linear space (X, d) then*

$$v_G(x + G) = w_G^{-1}(x + G) \bigcap R_G^{-1}(0)$$

and so $v_G = (w_G \mid_{R_G^{-1}(0)})^{-1}$.

Proof. Let $z \in v_G(x + G) = x - R_G(x)$, then $z = x - g'$ for some $g' \in R_G(x)$. Now, $g' \in R_G(x)$ implies that $x - g' \in R_G^{-1}(0)$ i.e. $0 \in R_G(x - g') = R_G(z)$ and so $z \in R_G^{-1}(0)$. Hence $v_G(x + G) \subseteq R_G^{-1}(0)$. Now $w_G(z) = w_G(x - g') = x + G$ implies that $z \in w_G^{-1}(x + G)$ i.e. $v_G(x + G) \subseteq w_G^{-1}(x + G)$. Consequently,

$$v_G(x + G) \subseteq w_G^{-1}(x + G) \bigcap R_G^{-1}(0). \tag{2.1}$$

Now suppose $z \in w_G^{-1}(x + G) \bigcap R_G^{-1}(0)$. Then $z = x + g$ for some $g \in G$ and $z \in R_G^{-1}(0)$. Since $z \in R_G^{-1}(0)$, $d(0, g'') \leq d(z, g'')$ for every $g'' \in G$. Put $g'' = g + g'$; $g, g' \in G$, we get $d(-g, g') \leq d(z - g, g')$ for every $g' \in G$. This gives $-g \in R_G(z - g) = R_G(x)$ and therefore $z = x - (-g) \in x - R_G(x) = v_G(x + G)$ i.e.

$$w_G^{-1}(x + G) \bigcap R_G^{-1}(0) \subseteq v_G(x + G). \tag{2.2}$$

From (2.1) and (2.2), we get $v_G(x + G) = w_G^{-1}(x + G) \bigcap R_G^{-1}(0)$.

Since by definition, for any $x + G \in X/G$, we have

$$(w_G \mid_{R_G^{-1}(0)})^{-1}(x + G) = \{z \in R_G^{-1}(0) : w_G(z) = x + G\}$$

$$= \{z \in R_G^{-1}(0) : z \in w_G^{-1}(x + G)\} = w_G^{-1}(x + G) \bigcap R_G^{-1}(0) = v_G(x + G).$$

Consequently, we have $v_G = (w_G \mid_{R_G^{-1}(0)})^{-1}$.

Lemma 2.6. *Let G be a co-proximinal subspace of a metric linear space (X, d). Then for any set $A \subseteq X$, we have*

$$w_G(A) = \{x + G \in X/G : w_G^{-1}(x + G) \bigcap A \neq \phi\}. \tag{2.3}$$

Consequently, for any set $A \subseteq R_G^{-1}(0)$, we have

$$w_G \mid_{R_G^{-1}(0)} (A) = \{x + G : v_G(x + G) \bigcap A \neq \phi\}.$$

Proof. Let $x + G \in w_G(A)$. Then $x + G = w_G(z)$ for some $z \in A$ which implies that $z \in w_G^{-1}(x + G)$. Now, $z \in A$, $z \in w_G^{-1}(x + G)$ imply $z \in w_G^{-1}(x + G) \bigcap A$ and so $w_G^{-1}(x + G) \bigcap A \neq \phi$.

Conversely, if $z \in w_G^{-1}(x + G) \bigcap A$ then $x + G = w_G(z) \in w_G(A)$. Thus, we have (2.3).

Now, if $A \subseteq R_G^{-1}(0)$, then $A = R_G^{-1}(0) \bigcap A$ and so from (2.3)

$$w_G \mid_{R_G^{-1}(0)} (A) = \{x + G : w_G^{-1}(x + G) \bigcap R_G^{-1}(0)) \bigcap A \neq \phi\}.$$

But, $v_G(x + G) = w_G^{-1}(x + G) \bigcap R_G^{-1}(0)$. Therefore,

$$w_G \mid_{R_G^{-1}(0)} (A) = \{x + G : v_G(x + G) \bigcap A \neq \phi\}.$$

Using Lemma 2.6, we obtain

Theorem 2.7. *Let G be a co-proximinal subspace of a metric linear space (X, d). Then following statements are equivalent:*
(i) v_G is u.s.c.
(ii) $w_G \mid_{R_G^{-1}(0)}$ carries closed sets onto closed sets.

Proof. (i)⇒ (ii) Since v_G is u.s.c., $H = \{x+G \in D(R_G)/G : v_G(x+G) \bigcap A \neq \phi\}$ is closed for any closed subset A of $R_G^{-1}(0)$. Using Lemma 2.6, $w_G \mid_{R_G^{-1}(0)} (A) = \{x+G : v_G(x+G) \bigcap A \neq \phi\}$ is closed for any closed subset A of $R_G^{-1}(0)$. Hence $w_G \mid_{R_G^{-1}(0)}$ carries closed sets onto closed sets.
(ii)⇒ (i) Since $w_G \mid_{R_G^{-1}(0)}$ carries closed sets onto closed sets, $w_G \mid_{R_G^{-1}(0)} (A) = \{x + G : v_G(x + G) \bigcap A \neq \phi\}$ is closed whenever A is closed in $R_G^{-1}(0)$. Hence v_G is u.s.c.

Remark: For the best approximation map, Lemmas 2.5, 2.6 and Theorem 2.7 are given in [14] for normed linear spaces.

The following theorem also deals with the upper semi-continuity of the map v_G.

Theorem 2.8. *Let G be a closed linear subspace of a metric linear space (X, d) such that for every compact subset A of $D(R_G)/G$, the subset $v_G(A)$ is compact, then v_G is u.s.c.*

Proof. Let N be a closed subset of $R_G^{-1}(0)$ and $B = \{y + G \in D(R_G)/G : v_G(y + G) \bigcap N \neq \phi\}$. Let $x_0 + G$ be a limit point of B, then there exist a sequence $\{x_n + G\}$ in B such that $\{x_n + G\} \to x_0 + G$. Since $\{x_n + G\} \to x_0 + G$, there exist a sequence $\{y_n\}$ in G such that $\{x_n + y_n\} \to x_0$. For each $x_n + G$, choose $g_n \in v_G(x_n + G)$. Let $A = \{x_n + G\} \bigcup \{x_0 + G\}$. Then A is a compact subset of $D(R_G)/G$ and so $v_G(A)$ is compact. Since $\{g_n\} \subseteq v_G(A)$ and $v_G(A)$ is compact, there exist a subsequence $\{g_{n_i}\}$ of $\{g_n\}$ such that $\{g_{n_i}\} \to g_0 \in v_G(A)$. Since $x_{n_i} - g_{n_i} \in R_G(x_{n_i})$, we have $x_{n_i} - g_{n_i} + y_{n_i} \in R_G(x_{n_i} + y_{n_i})$ i.e.

$$d(x_{n_i} - g_{n_i} + y_{n_i}, g) \leq d(x_{n_i} + y_{n_i}, g) \; for \; every \; g \in G. \tag{2.4}$$

Since $\{x_{n_i} + y_{n_i}\} \to x_0$, $\{g_{n_i}\} \to g_0 \; imply \; \{x_{n_i} - g_{n_i} + y_{n_i}\} \to x_0 - g_0$. Therefore, using (2.4) $d(x_0 - g_0, g) \leq d(x_0, g)$ for every $g \in G$ i.e. $x_0 - g_0 \in R_G(x_0)$, $g_0 \in v_G(x_0 + G) \bigcap N$. Hence $x_0 + G \in B$ and v_G is u.s.c.

If we take G to be boundedly compact then we have

Theorem 2.9. *Let G be a boundedly compact, closed linear subspace of a metric linear space (X, d). Then*
(i) For each compact subset A of $(D(R_G))/G$, $v_G(A)$ is compact.
(ii)v_G is u.s.c. on $D(R_G)/G$.

Proof. (i) Let $\{g_n\}$ be a sequence in $v_G(A)$. Then there exist $x_n + G \in A$ such that $g_n \in v_G(x_n + G)$ i.e. $\{x_n + G\}$ is a sequence in a compact set A and so it has a subsequence $\{x_{n_k} + G\} \to x + G \in A$. Since $\{x_{n_k} + G\} \to x + G$ implies that there exist a sequence $\{y_{n_k}\}$ in G such that $\{x_{n_k} + y_{n_k}\} \to x$. Now, $g_{n_k} \in v_G(x_{n_k} + G)$ implies that $x_{n_k} - g_{n_k} \in R_G(x_{n_k})$ and so $x_{n_k} - g_{n_k} + y_{n_k} \in R_G(x_{n_k} + y_{n_k})$. We claim that $\{x_{n_k} - g_{n_k} + y_{n_k}\}$ is a bounded sequence in G. Consider

$$d(x_{n_k} - g_{n_k} + y_{n_k}, 0) \leq d(x_{n_k} - g_{n_k} + y_{n_k}, g) + d(g, 0) \; for \; every \; g \in G$$

$$\Rightarrow d(x_{n_k} - g_{n_k} + y_{n_k}, 0) \leq d(x_{n_k} + y_{n_k}, g) + d(g, 0)\ for\ every\ g \in G$$

$$i.e.\ d(x_{n_k} - g_{n_k} + y_{n_k}, 0) \leq d(x_{n_k} + y_{n_k}, 0)$$

$$\Rightarrow \sup_{n_k \in N} d(x_{n_k} - g_{n_k} + y_{n_k}, 0) \leq \sup_{n_k \in N} d(x_{n_k} + y_{n_k}, 0) < \infty.$$

Since G is a boundedly compact, closed subspace, $\{x_{n_k} - g_{n_k} + y_{n_k}\}$ has a subsequence $\{x_{n_{k_m}} - g_{n_{k_m}} + y_{n_{k_m}}\} \to z \in G$. Also $\{x_{n_{k_m}} + y_{n_{k_m}}\} \to x$. Therefore, $g_{n_{k_m}} \to x - z$. As $d(x_{n_k} - g_{n_k} + y_{n_k}, g) \leq d(x_{n_k} + y_{n_k}, g)$ for every $g \in G$ implies that $d(z, g) \leq d(x, g)\ for\ every\ g \in G$, $z \in R_G(x)$ and so $x - z \in v_G(x + G) \in v_G(A)$. Hence $v_G(A)$ is compact.
(ii) follows from (i) and Theorem 2.8.

Assuming $R_G^{-1}(0)$ to be boundedly compact, we have

Theorem 2.10. *Let G be closed linear subspace of a metric linear space (X, d). If $R_G^{-1}(0)$ is boundedly compact, then*
(i)v_G is u.s.c. on $D(R_G)/G$.
(ii) $v_G(x + G)$ is compact for each $x + G \in D(R_G)/G$.

Proof. (i) Let N be a closed subset of $R_G^{-1}(0)$ and $B = \{y+G \in D(R_G)/G : v_G(y+G) \bigcap N \neq \phi\}$. Let $x_0 + G$ be a limit point of B, then there exist a sequence $\{x_n + G\}$ in B such that $\{x_n + G\} \to x_0 + G$. Since $\{x_n + G\} \to x_0 + G$, there exist a sequence $\{y_n\}$ in G such that $\{x_n + y_n\} \to x_0$. For each $x_n + G$, choose $g_n \in v_G(x_n + G)$. Then $x_n - g_n \in R_G(x_n)$ and so $g_n \in R_G^{-1}(0)$. Since $d(g_n, 0) \leq 2d(x_n+y_n, 0)$ and $\{x_n+y_n\}$ is a convergent sequence, $\{g_n\}$ is a bounded sequence. The bounded compactness of $R_G^{-1}(0)$ implies that there exist a subsequence $\{g_{n_i}\} \to g_0$. Since $x_{n_i} - g_{n_i} \in R_G(x_{n_i})$, we have $x_{n_i} - g_{n_i} + y_{n_i} \in R_G(x_{n_i} + y_{n_i})$ i.e.

$$d(x_{n_i} - g_{n_i} + y_{n_i}, g) \leq d(x_{n_i} + y_{n_i}, g)\ for\ every\ g \in G. \tag{2.5}$$

Since $\{x_{n_i} + y_{n_i}\} \to x_0$, $\{g_{n_i}\} \to g_0\ imply\ \{x_{n_i} - g_{n_i} + y_{n_i}\} \to x_0 - g_0$. Therefore using (2.5), $d(x_0 - g_0, g) \leq d(x_0, g)$ for every $g \in G$ i.e. $x_0 - g_0 \in R_G(x_0)$, $g_0 \in v_G(x_0 + G) \bigcap N$. Hence $x_0 + G \in B$ and v_G is u.s.c.
(ii) Let $\{x - g_n\}$ be a sequence in $v_G(x + G)$. Since $v_G(x + G)$ is a bounded set, $\{x - g_n\}$ is a bounded sequence in $x - R_G(x)$. Now, $x - g_n \in x - R_G(x)\ i.e.\ g_n \in R_G(x)$ and so $x - g_n \in R_G^{-1}(0)$. Hence $\{x - g_n\}$ is a bounded sequence in $R_G^{-1}(0)$. Since $R_G^{-1}(0)$ is boundedly compact and closed, there exist a subsequence $\{x - g_{n_i}\} \to x - g_0 \in R_G^{-1}(0)$. This gives $g_0 \in R_G(x)$ and so $x - g_0 \in v_G(x + G)$. Hence $v_G(x + G)$ is compact.

We shall be needing the following result of Singer [14] for our next theorems.

Lemma 2.11. *If X and Y are metric spaces then a mapping $u : X \to 2^Y$ is l.s.c. if and only if u is l.(K)-s.c.*

Using this lemma, we prove the following:

Theorem 2.12. *Let G be a closed linear subspace of a metric linear space (X, d). Then R_G is l.s.c. if and only if v_G is l.s.c.*

Proof. Assume R_G is l.s.c., then using above lemma, we have R_G is l.(K)-s.c. We prove that v_G is l.s.c. Let $u_n + G \to u + G\ in\ D(R_G)/G$, $y \in v_G(u + G)$. Since $u_n + G \to u + G$ implies that there exist a sequence$\{g_n\}$ in G such that $\{u_n + g_n\} \to u$. Also $y \in v_G(u + G)$ implies that $u - y \in R_G(u)$. Since, $\{u_n + g_n\} \to u$, $u - y \in R_G(u)\ and\ R_G$ is l.(K)-s.c., there exist a sequence $\{z_n\}$ such that $z_n \in R_G(u_n + g_n)$ and $\{z_n\} \to u - y$. Put $u_n + g_n = t_n$, then there exist a sequence $\{z_n\}$ such that $t_n - z_n \in t_n - R_G(t_n)$ and $z_n \to u - y$. Now, $t_n \to u$, $z_n \to u - y \Rightarrow \{t_n - z_n\} \to y$ i.e there exist a sequence $\{t_n - z_n\}$, $t_n - z_n \in v_G(t_n)$ such that $\{t_n - z_n\} \to y$ and so v_G is l.(K)-s.c. and hence l.s.c.

Conversely, assume that v_G is l.s.c. Let $u_n \to u\ in\ D(R_G)$ and $y \in R_G(u)$. As $y \in R_G(u) \Rightarrow u - y \in v_G(u+G)$ and $u_n \to u \Rightarrow u_n + G \to u + G$. Since v_G is l.(K)-s.c., there exist a sequence $\{z_n\}$, $z_n \in v_G(u_n + G)$ such that $\{z_n\} \to u - y$. Now, $z_n \in v_G(u_n + G)$ implies that $u_n - z_n \in R_G(u_n)$. Since $u_n \to u$ *and* $z_n \to u - y$, $\{u_n - z_n\} \to y$. Therefore, there exist a sequence $\{u_n - z_n\}$ such that $u_n - z_n \in R_G(u_n)$ and $\{u_n - z_n\} \to y$. This implies that R_G is l.(K)-s.c. and hence l.s.c.

Remark: For normed linear spaces, Theorem 2.12 has been given in [13].

By taking G to be a co-proximinal subspace, we have:

Theorem 2.13. *For a co-proximinal subspace G of a metric linear space (X,d), the following statements are equivalent:*
(i) R_G is l.s.c.
(ii) v_G is l.s.c.
(iii)$w_G\mid_{R_G^{-1}(0)}$ is an open mapping.

Proof. (i)⇔ (ii) follows from Theorem 2.12.
(ii)⇒ (iii) Since v_G is l.s.c., the set $H = \{x + G \in X/G : v_G(x + G) \bigcap A \neq \phi\}$ is open for any open subset A of $R_G^{-1}(0)$.
Using Lemma 2.6, we have

$$w_G\mid_{R_G^{-1}(0)}(A) = \{x + G : v_G(x + G) \bigcap A \neq \phi\} \tag{2.6}$$

i.e. if A is open then by (2.6), $w_G\mid_{R_G^{-1}(0)}(A)$ is also open and so $w_G\mid_{R_G^{-1}(0)}$ is an open mapping.
(iii)⇒(ii) Since $w_G\mid_{R_G^{-1}(0)}$ is an open mapping, $w_G\mid_{R_G^{-1}(0)}(A)$ is open for every open subset $A \subseteq R_G^{-1}(0)$. Using Lemma 2.6, we get

$$w_G\mid_{R_G^{-1}(0)}(A) = \{x + G : v_G(x + G) \bigcap A \neq \phi\}$$

is open for any open subset A of $R_G^{-1}(0)$ i.e. v_G is l.s.c.

Taking the subspace G to be co-Chebyshev, we have

Theorem 2.14. *For a co-Chebyshev subspace G of a metric linear space (X,d), the following are equivalent:*
(i) R_G is continuous.
(ii)$w_G\mid_{R_G^{-1}(0)}$ is a homeomorphism of $R_G^{-1}(0)$ onto X/G.
(iii) $v_G = (w_G\mid_{R_G^{-1}(0)})^{-1}$ is continuous.

Proof. (i) ⇔(ii) has been proved in [10].
(iii) ⇒ (ii) Let $x_n \to x$ in $R_G^{-1}(0)$. Then $x_n + G \to x + G$ i.e. $w_G(x_n) \to w_G(x)$. Hence $w_G\mid_{R_G^{-1}(0)}$ is continuous. Also $(w_G\mid_{R_G^{-1}(0)})^{-1}$ is continuous. Since G is co-Chebyshev, $w_G\mid_{R_G^{-1}(0)}$ is one-one and so is a homeomorphism of $R_G^{-1}(0)$ onto X/G.
(ii)⇒ (iii) is trivial.

Remark: For the best approximation map, Theorems 2.13 and 2.14 have been proved in [14] for normed linear spaces.

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