Palestine Journal of Mathematics

Vol. 6(Special Issue: II) (2017) , 314–319

NEW INFORMATION INEQUALITIES IN TERMS OF RELATIVE ARITHMETIC- GEOMETRIC DIVERGENCE AND RENYI'S ENTROPY

K.C. Jain and Praphull Chhabra

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 94A17; Secondary 26D15.

Keywords and phrases: New generalized f- divergence measure, Relative Arithmetic- Geometric divergence, Renyi's entropy, Several means.

Abstract. New information inequalities on new generalized f- divergence measure in terms of Relative Arithmetic- Geometric divergence and Renyi's entropy have been derived for comparing two discrete probability distributions and further, some results for the Triangular discrimination, Chi- square divergence and Relative J- divergence have been obtained.

1 Introduction

Without essential loss of insight, we have restricted ourselves to discrete probability distributions, so let $\Gamma_n = \{P = (p_1, p_2, p_3, ..., p_n) : p_i > 0, \sum_{i=1}^{n} p_i = 1\}$, $n \geq 2$ be the set of all complete finite discrete probability distributions. If we take $p_i \geq 0$ for some $i = 1, 2, 3..., n$, then we have to suppose that $0f(0) = 0f\left(\frac{0}{0}\right) = 0$.
For real, continuous, convex function $f : (0, \infty) \to (-\infty, \infty)$ and $P = (p_1, p_2, ..., p_n), Q = (q_1, q_2, ..., q_n) \in \Gamma_n$, Jain and Saraswat [5] introduced the following new generalized f- divergence measure

$$S_f(P, Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i + q_i}{2q_i}\right), \tag{1.1}$$

where p_i and q_i are probabilities. The advantage of this generalized divergence is that many divergence measures can be obtained from this generalized measure by suitably defining the function f.
Now we are stating the followings theorems for evaluating the new information inequalities in the next section.

Theorem 1.1. *(Cerone etc. all [1]) Let $f : [a, b] \subset (0, \infty) \to (-\infty, \infty)$ be an absolutely continuous function on $[a, b]$ with $b > a > 0$. Then for any $x \in [a, b]$, we have*

$$\left| \frac{f(x)}{x}(b-a) - \int_a^b \frac{f(t)}{t} dt \right| \leq 2\left[\log\frac{x}{B} + \frac{A-x}{x}\right] \|f'l - f\|_\infty, \tag{1.2}$$

where $B \equiv B(a, b) = \sqrt{ab}$ and $A \equiv A(a, b) = \frac{a+b}{2}$ are Geometric and Arithmetic mean of a and b respectively, l is the identity function, i.e., $l(x) = x \; \forall \; x \in [a, b]$ and

$$\|f'l - f\|_\infty = ess \sup_{t \in [a,b]} |(f'l - f)(t)| < \infty.$$

The constant 2 is best possible.

Theorem 1.2. *(Dragomir [3]) Let $f : [a, b] \to (-\infty, \infty)$ be continuous function on $[a, b]$ and differentiable on (a, b) with $b > a$ and $[a, b]$ not containing 0. Then for any $x \in [a, b]$, we have*

$$\left| \frac{f(x)}{x} A - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{b-a}{|x|}\left[\frac{1}{4} + \left(\frac{x-A}{b-a}\right)^2\right] \|f'l - f\|_\infty. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible.

Now we extend the work on $S_f(P,Q)$ and derive the new information inequalities for comparing two discrete probability distributions in section 2, and also obtain interesting results in section 3 by using these new inequalities. Several means, like: Arithmetic mean, Geometric mean, Harmonic mean, Logarithmic mean, Centroidal mean, Root mean square, and Identric mean are being used for summarize the calculations only.

2 New information inequalities

Now, we derive new information inequalities in terms of the Relative Arithmetic- Geometric divergence and Renyi's entropy separately by using theorems 1.1 and 1.2 respectively. The results are on the similar lines to the results presented by (Cerone etc. all [1]) and (Dragomir [3]) respectively.

Proposition 2.1. *Let $f:[\alpha,\beta]\subset(0,\infty)\to(-\infty,\infty)$ be an absolutely continuous and convex function on $[\alpha,\beta]$ with $0<\alpha\le 1\le\beta<\infty, \alpha\neq\beta$. Then we have the following inequality involving $S_f(P,Q)$ and $G(P,Q)$ between probability distributions $P,Q\in\Gamma_n$:*

$$\left|S_f(P,Q)-\frac{1}{\beta-\alpha}\int_\alpha^\beta\frac{f(t)}{t}dt\right|\le\frac{2}{\beta-\alpha}\left[G(Q,P)-\log B+A-1\right]\|f'l-f\|_\infty, \tag{2.1}$$

where $S_f(P,Q)$ is defined by (1.1) and

$$G(P,Q)=\sum_{i=1}^n\left(\frac{p_i+q_i}{2}\right)\log\frac{p_i+q_i}{2p_i} \tag{2.2}$$

is the Relative Arithmetic- Geometric divergence (Taneja [8]).

Proof: Put $a=\alpha, b=\beta$ such that $0<\alpha\le1\le\beta<\infty$ with $\alpha\neq\beta$ and $x=\frac{p_i+q_i}{2q_i}$, $i=1,2...,n$ in inequality (1.2), we obtain

$$\left|f\left(\frac{p_i+q_i}{2q_i}\right)\frac{2(\beta-\alpha)q_i}{p_i+q_i}-\int_\alpha^\beta\frac{f(t)}{t}dt\right|\le2\left[\log\frac{p_i+q_i}{2q_i}-\log B+\frac{2Aq_i-p_i-q_i}{p_i+q_i}\right]\|f'l-f\|_\infty.$$

Now multiply the above expression by $\frac{p_i+q_i}{2(\beta-\alpha)}$ for $i=1,2...,n$, we obtain

$$\begin{aligned}&\left|q_if\left(\frac{p_i+q_i}{2q_i}\right)-(p_i+q_i)\frac{1}{2(\beta-\alpha)}\int_\alpha^\beta\frac{f(t)}{t}dt\right|\\&\le\frac{1}{\beta-\alpha}\left[(p_i+q_i)\log\left(\frac{p_i+q_i}{2q_i}\right)-(p_i+q_i)\log B+2Aq_i-p_i-q_i\right]\|f'l-f\|_\infty.\end{aligned}$$

Now sum over all from $i=1$ to n and consider $\sum_{i=1}^n p_i=\sum_{i=1}^n q_i=1$, we get the desired inequality (2.1) in terms of the Relative Arithmetic- Geometric divergence.

Proposition 2.2. *Let $f:[\alpha,\beta]\subset(0,\infty)\to(-\infty,\infty)$ be continuous convex function on $[\alpha,\beta]$ and differentiable on (α,β) with $0<\alpha\le1\le\beta<\infty,\alpha\neq\beta$. Then we have the following inequality involving $S_f(P,Q)$ and $R_2(P,Q)$ between probability distributions $P,Q\in\Gamma_n$:*

$$\begin{aligned}&\left|S_f(P,Q)-\frac{2}{\beta^2-\alpha^2}\int_\alpha^\beta f(t)\,dt\right|\\&\le\frac{\beta-\alpha}{2(\alpha+\beta)}\left[1+\frac{1}{(\beta-\alpha)^2}\left\{R_2(P,Q)+4(A-1)^2-1\right\}\right]\|f'l-f\|_\infty,\end{aligned} \tag{2.3}$$

where

$$R_2(P,Q)=\sum_{i=1}^n\frac{p_i^2}{q_i} \tag{2.4}$$

is the Renyi's entropy of second order (Renyi [7]).

Proof: Put $a = \alpha, b = \beta$ such that $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$ and $x = \frac{p_i+q_i}{2q_i}$, $i = 1, 2..., n$ in inequality (1.3), we obtain

$$\left| f\left(\frac{p_i+q_i}{2q_i}\right)\frac{2Aq_i}{p_i+q_i} - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)\,dt\right|$$
$$\leq \frac{2(\beta-\alpha)q_i}{p_i+q_i}\left[\frac{1}{4} + \frac{1}{4(\beta-\alpha)^2 q_i^2}(p_i+q_i-2Aq_i)^2\right]\|f'l-f\|_\infty.$$

Now multiply the above expression by $\frac{p_i+q_i}{2A}$ for $i = 1, 2..., n$, we obtain

$$\left| q_i f\left(\frac{p_i+q_i}{2q_i}\right) - (p_i+q_i)\frac{1}{2A(\beta-\alpha)}\int_\alpha^\beta f(t)\,dt\right|$$
$$\leq \frac{\beta-\alpha}{4A}\left[q_i + \frac{1}{(\beta-\alpha)^2}\left(\frac{p_i^2}{q_i} + q_i + 2p_i + 4A^2 q_i - 4Ap_i - 4Aq_i\right)\right]\|f'l-f\|_\infty.$$

Now sum over all from $i = 1$ to n and consider $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$, we get the desired inequality (2.3) in terms of the Renyi's entropy of second order.

3 Results by using obtained new inequalities

In this section, we obtain new results on existing divergence measures; Triangular discrimination, Chi- square divergence and Relative J- divergence, in terms of Relative Arithmetic Geometric divergence and Renyi's entropy of second order separately.

Result 3.1. *For $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$, we have*

$$\left|\Delta(P,Q) - 2\left(1 + \frac{1}{B^2} - 2L\right)\right| \leq 4F_1 \sup_{t\in[\alpha,\beta]} g_1(t). \tag{3.1}$$

$$\left|\Delta(P,Q) - \frac{2}{A}(A + L - 2)\right| \leq F_2 \sup_{t\in[\alpha,\beta]} g_1(t), \tag{3.2}$$

where $L \equiv L(\alpha,\beta) = \frac{\log\beta - \log\alpha}{\beta-\alpha}$ is the Logarithmic mean of α and β with $\alpha \neq \beta$, also

$$F_1 \equiv \frac{1}{\beta-\alpha}\left[G(Q,P) - \log B + A - 1\right]$$

and

$$F_2 \equiv \frac{\beta-\alpha}{\alpha+\beta}\left[1 + \frac{1}{(\beta-\alpha)^2}\left\{R_2(P,Q) + 4(A-1)^2 - 1\right\}\right]$$

and $\Delta(P,Q)$, $\sup_{t\in[\alpha,\beta]} g_1(t)$ are evaluated below in the proof.

Proof: Let us consider

$$f(t) = \frac{(t-1)^2}{t}, t \in (0,\infty), f(1) = 0, f'(t) = \frac{t^2-1}{t^2} \text{ and } f''(t) = \frac{2}{t^3}.$$

Since $f''(t) > 0\ \forall\ t > 0$ and $f(1) = 0$, so $f(t)$ is strictly convex and normalized function respectively. Now for $f(t)$, we obtain

$$S_f(P,Q) = \frac{1}{2}\sum_{i=1}^n \frac{(p_i-q_i)^2}{p_i+q_i} = \frac{1}{2}\Delta(P,Q), \tag{3.3}$$

where $\Delta(P,Q)$ is the Triangular discrimination (Dacunha- Castelle etc. all [2]). Also

$$\begin{aligned}\int_\alpha^\beta f(t)\,dt &= \int_\alpha^\beta \frac{(t-1)^2}{t}dt = \int_\alpha^\beta \frac{t^2-2t+1}{t}dt = \left[\frac{t^2}{2} - 2t + \log t\right]_\alpha^\beta \\ &= \frac{1}{2}\left(\beta^2-\alpha^2\right) - 2(\beta-\alpha) + (\log\beta - \log\alpha).\end{aligned} \tag{3.4}$$

$$\begin{aligned}\int_\alpha^\beta \frac{f(t)}{t}dt &= \int_\alpha^\beta \frac{(t-1)^2}{t^2}dt = \int_\alpha^\beta \frac{t^2-2t+1}{t^2}dt = \left[t - \frac{1}{t} - 2\log t\right]_\alpha^\beta \\ &= (\beta-\alpha) + \frac{\beta-\alpha}{\alpha\beta} - 2(\log\beta - \log\alpha).\end{aligned} \tag{3.5}$$

Let

$$g_1(t) = |(f'l - f)(t)| = \left|\frac{t^2-1}{t^2}t - \frac{(t-1)^2}{t}\right| = \frac{2}{t}|t-1| = \begin{cases}\frac{2}{t}(t-1) & \text{if } t \geq 1 \\ \frac{2}{t}(1-t) & \text{if } 0 < t < 1\end{cases},$$

and

$$g_1'(t) = \begin{cases}\frac{2}{t^2} & \text{if } t \geq 1 \\ -\frac{2}{t^2} & \text{if } 0 < t < 1\end{cases}.$$

It is clear that $g_1'(t) < 0$ in $(0,1)$ and > 0 in $(1,\infty)$, i.e., $g_1(t)$ is strictly decreasing in $(0,1)$ and strictly increasing in $(1,\infty)$, so

$$\begin{aligned}\|f'l - f\|_\infty &= \sup_{t\in[\alpha,\beta]} |(f'l-f)(t)| = \sup_{t\in[\alpha,\beta]} g_1(t) \\ &= \begin{cases} max[g_1(\alpha), g_1(\beta)] = \frac{g_1(\alpha)+g_1(\beta)+|g_1(\alpha)-g_1(\beta)|}{2} & \text{if } 0<\alpha<1 \\ g_1(\beta) & \text{if } \alpha = 1\end{cases} \\ &= \begin{cases}\frac{(1-\alpha)}{\alpha} + \frac{(\beta-1)}{\beta} + \left|\frac{(1-\alpha)}{\alpha} - \frac{(\beta-1)}{\beta}\right| & \text{if } 0<\alpha<1 \\ \frac{2(\beta-1)}{\beta} & \text{if } \alpha = 1\end{cases} \\ &= \begin{cases}\frac{\beta-\alpha}{\alpha\beta} + 2\left|\frac{1}{H} - 1\right| & \text{if } 0<\alpha<1 \\ \frac{2(\beta-1)}{\beta} & \text{if } \alpha = 1\end{cases},\end{aligned} \tag{3.6}$$

where $H \equiv H(\alpha,\beta) = \frac{2\alpha\beta}{\alpha+\beta}$ is the Harmonic mean of α and β.
The results (3.1) and (3.2) are obtained after putting (3.3), (3.4), (3.5), and (3.6) in inequalities (2.1) and (2.3) respectively.

Result 3.2. *For $P, Q \in \Gamma_n$ and $0 < \alpha \leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$, we have*

$$\left|\chi^2(P,Q) - 4(A+L-2)\right| \leq 8F_1 \sup_{t\in[\alpha,\beta]} g_2(t). \tag{3.7}$$

$$\left|\chi^2(P,Q) - 4(A+R-2)\right| \leq 2F_2 \sup_{t\in[\alpha,\beta]} g_2(t), \tag{3.8}$$

where $R \equiv R(\alpha,\beta) = \frac{2(\alpha^2+\alpha\beta+\beta^2)}{3(\alpha+\beta)}$ is the Centroidal mean of α and β. Also F_1, F_2 defined earlier and $\chi^2(P,Q)$, $\sup_{t\in[\alpha,\beta]} g_2(t)$ are evaluated below in the proof.

Proof: Let us consider

$$f(t) = (t-1)^2, t\in(0,\infty), f(1) = 0, f'(t) = 2(t-1) \text{ and } f''(t) = 2.$$

Since $f''(t) > 0 \ \forall \ t > 0$ and $f(1) = 0$, so $f(t)$ is strictly convex and normalized function respectively. Now for $f(t)$, we obtain

$$S_f(P,Q) = \frac{1}{4}\sum_{i=1}^n \frac{(p_i-q_i)^2}{q_i} = \frac{1}{4}\chi^2(P,Q), \tag{3.9}$$

where $\chi^2(P,Q)$ is the Chi- square divergence (Pearson [6]). Also

$$\begin{aligned}\int_\alpha^\beta f(t)\,dt &= \int_\alpha^\beta (t-1)^2\,dt = \int_\alpha^\beta \left(t^2-2t+1\right)dt = \left[\frac{t^3}{3} - t^2 + t\right]_\alpha^\beta \\ &= \frac{1}{3}(\beta-\alpha)\left(\alpha^2+\alpha\beta+\beta^2\right) - \left(\beta^2-\alpha^2\right) + (\beta-\alpha).\end{aligned} \tag{3.10}$$

$$\int_\alpha^\beta \frac{f(t)}{t}dt = \int_\alpha^\beta \frac{(t-1)^2}{t}dt = \int_\alpha^\beta \left(t+\frac{1}{t}-2\right)dt = \left[\frac{t^2}{2}+\log t - 2t\right]_\alpha^\beta$$
$$= \frac{\beta^2-\alpha^2}{2} + (\log\beta - \log\alpha) - 2(\beta-\alpha). \tag{3.11}$$

Let

$$g_2(t) = |(f'l-f)(t)| = \left|2(t-1)t-(t-1)^2\right| = (t+1)|t-1| = \begin{cases} t^2-1 & \text{if } t \geq 1 \\ -t^2+1 & \text{if } 0<t<1 \end{cases},$$

and

$$g_2'(t) = \begin{cases} 2t & \text{if } t \geq 1 \\ -2t & \text{if } 0<t<1 \end{cases}.$$

It is clear that $g_2'(t) < 0$ in $(0,1)$ and > 0 in $(1,\infty)$, i.e., $g_2(t)$ is strictly decreasing in $(0,1)$ and strictly increasing in $(1,\infty)$, so

$$\|f'l-f\|_\infty = \sup_{t\in[\alpha,\beta]} |(f'l-f)(t)| = \sup_{t\in[\alpha,\beta]} g_2(t)$$
$$= \begin{cases} max\left[g_2(\alpha), g_2(\beta)\right] = \frac{g_2(\alpha)+g_2(\beta)+|g_2(\alpha)-g_2(\beta)|}{2} & \text{if } 0<\alpha<1 \\ g_2(\beta) & \text{if } \alpha = 1 \end{cases} \tag{3.12}$$
$$= \begin{cases} \frac{\beta^2-\alpha^2}{2} + \left|1-S^2\right| & \text{if } 0<\alpha<1 \\ \beta^2-1 & \text{if } \alpha=1 \end{cases},$$

where $S \equiv S(\alpha,\beta) = \sqrt{\frac{\alpha^2+\beta^2}{2}}$ is the Root mean square of α and β.
The results (3.7) and (3.8) are obtained after putting (3.9), (3.10), (3.11), and (3.12) in inequalities (2.1) and (2.3) respectively.

Result 3.3. *For $P,Q \in \Gamma_n$ and $0<\alpha\leq 1 \leq \beta < \infty$ with $\alpha \neq \beta$, we have*

$$\left|J_R(P,Q) - 2\log\frac{I(\alpha,\beta)}{B^L}\right| \leq 4F_1 \sup_{t\in[\alpha,\beta]} g_3(t). \tag{3.13}$$

$$\left|J_R(P,Q) - \frac{2\log I(\alpha,\beta)}{A} - \log I\left(\alpha^2,\beta^2\right)\right| \leq F_2 \sup_{t\in[\alpha,\beta]} g_3(t), \tag{3.14}$$

where $I(\alpha,\beta) = \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}}$, $\alpha\neq\beta$ is the Identric mean of α and β. Also F_1, F_2 defined earlier and $J_R(P,Q)$, $\sup_{t\in[\alpha,\beta]} g_3(t)$ are evaluated below in the proof.

Proof: Let us consider

$$f(t) = (t-1)\log t, t\in(0,\infty), f(1)=0, f'(t) = \frac{t-1}{t} + \log t \text{ and } f''(t) = \frac{t+1}{t^2}.$$

Since $f''(t) > 0 \ \forall \ t > 0$ and $f(1) = 0$, so $f(t)$ is strictly convex and normalized function respectively. Now for $f(t)$, we obtain

$$S_f(P,Q) = \frac{1}{2}\sum_{i=1}^n (p_i-q_i)\log\left(\frac{p_i+q_i}{2q_i}\right) = \frac{1}{2}J_R(P,Q), \tag{3.15}$$

where $J_R(P,Q)$ is the Relative J- divergence (Dragomir etc. all [4]). Also

$$\int_\alpha^\beta f(t)\,dt = \int_\alpha^\beta (t-1)\log t dt = \frac{1}{2}\left[(t-1)^2\log t - \frac{t^2}{2} - \log t + 2t\right]_\alpha^\beta$$
$$= \frac{1}{2}\left[\left(\beta^2\log\beta - \alpha^2\log\alpha\right) - 2(\beta\log\beta - \alpha\log\alpha) - \frac{\beta^2-\alpha^2}{2} + 2(\beta-\alpha)\right]. \tag{3.16}$$

$$\int_{\alpha}^{\beta} \frac{f(t)}{t} dt = \int_{\alpha}^{\beta} \frac{(t-1)\log t}{t} dt = \int_{\alpha}^{\beta} \left(\log t - \frac{\log t}{t}\right) dt$$
$$= (\beta \log \beta - \alpha \log \alpha) - \frac{1}{2} (\log \beta - \log \alpha) \log (\alpha\beta) - (\beta - \alpha). \quad (3.17)$$

Let

$$g_3(t) = |(f'l - f)(t)| = \left| \left(\frac{t-1}{t} + \log t \right) t - (t-1) \log t \right|$$
$$= |-1 + t + \log t| = \begin{cases} -1 + t + \log t & \text{if } t \geq 1 \\ 1 - t - \log t & \text{if } 0 < t < 1 \end{cases},$$

and

$$g_3'(t) = \begin{cases} 1 + \frac{1}{t} & \text{if } t \geq 1 \\ -1 - \frac{1}{t} & \text{if } 0 < t < 1 \end{cases}.$$

It is clear that $g_3'(t) < 0$ in $(0,1)$ and > 0 in $(1,\infty)$, i.e., $g_3(t)$ is strictly decreasing in $(0,1)$ and strictly increasing in $(1,\infty)$, so

$$\|f'l - f\|_\infty = \sup_{t\in[\alpha,\beta]} |(f'l - f)(t)| = \sup_{t\in[\alpha,\beta]} g_3(t)$$
$$= \begin{cases} max\,[g_3(\alpha), g_3(\beta)] = \frac{g_3(\alpha)+g_3(\beta)+|g_3(\alpha)-g_3(\beta)|}{2} & \text{if } 0 < \alpha < 1 \\ g_3(\beta) & \text{if } \alpha = 1 \end{cases} \quad (3.18)$$
$$= \begin{cases} |1 - \log B - A| + \frac{\log\beta - \log\alpha}{2} + \frac{\beta - \alpha}{2} & \text{if } 0 < \alpha < 1 \\ -1 + \beta + \log\beta & \text{if } \alpha = 1 \end{cases}.$$

The results (3.13) and (3.14) are obtained after putting (3.15), (3.16), (3.17), and (3.18) in inequalities (2.1) and (2.3) respectively.

References

[1] P. Cerone, S.S. Dragomir, and E. Kikianty (2015), Ostrowski and Trapezoid type inequalities related to Pompeiu's mean value theorem, Journal of Mathematical Inequalities, vol. 9, no. 3, 739- 762.

[2] D. Dacunha- Castelle, H. Heyer and B. Roynette, (1978), Ecole dâĂŹEte de probabilities de, Saint-Flour VII-1977, Berlin, vol. 678, Heidelberg, New York: Springer.

[3] S.S. Dragomir (2005), An inequality of Ostrowski type via Pompeiu's mean value theorem, J. Inequal. Pure Appl. Math., vol. 6, no. 3, Article 83, 9 pp.

[4] S.S. Dragomir, V. Gluscevic and C.E.M. Pearce (2001), Approximation for the Csiszar's f- divergence via midpoint inequalities, in Inequality Theory and Applications- Y.J. Cho, J.K. Kim, and S.S. Dragomir (Eds.), Nova Science Publishers, Inc., Huntington, New York, vol. 1, 139-154.

[5] K.C. Jain and R.N. Saraswat (2012), Some new information inequalities and its applications in information theory, International Journal of Mathematics Research, vol. 4 (3), 295-307.

[6] K. Pearson (1900), On the criterion that a given system of deviations from the probable in the case of correlated system of variables is such that it can be reasonable supposed to have arisen from random sampling, Phil. Mag., vol. 50, 157-172.

[7] A. Renyi (1961), On measures of entropy and information, In Proceedings Fourth Berkeley Symposium on Mathematics Statistics and Probability, 1, University of California Press, Berkeley, CA U.S., 547-561.

[8] I.J. Taneja (1995), New developments in generalized information measures, Advances in Electronics and Electron Physics (Edited by P.W. Hawkes), vol. 91, 37-135.

Author information

K.C. Jain and Praphull Chhabra, Department of Mathematics, Malaviya National Institute of Technology, Jaipur- 302017 (Rajasthan), India, India.
E-mail: jainkc_2003@yahoo.com, prf124@gmail.com

Received: January 12, 2016.

Accepted: September 26, 2016.