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On a field-theoretic invariant for extensions of commutative rings, II

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Abstract. This paper is a sequel. The earlier paper introduced, for any (unital) extension of (commutative unital) rings $R \subseteq T$, an invariant $\Lambda(T/R)$ defined as the supremum of the lengths of chains of intermediate fields in the extension $\mathbf{k}_R(Q \cap R) \subseteq \mathbf{k}_T(Q)$, where Q runs over the prime ideals of T. Theorem 2.5 of that earlier paper calculated $\Lambda(T/R)$ in case $R \subset T$ are (commutative integral) domains such that $R \subset T$ are "adjacent rings" (that is, in case $R \subset T$ is a minimal ring extension of domains). The statement of that Theorem 2.5 is incorrect for some adjacent rings $R \subset T$ such that R is integrally closed in T. Counterexamples are given to the original statement of Theorem 2.5. Two corrected versions of Theorem 2.5 are stated, proved and generalized from the domain-theoretic setting to the context of extensions of arbitrary rings. These results lead naturally to discussions involving the conductor $(R : T)$ arising from a normal pair (R, T) of rings.

Dedicated to the memory of Paul-Jean Cahen

1 Introduction

All rings and algebras considered below are commutative and unital; all inclusions of rings, ring extensions and algebra/ring homomorphisms are unital. If A is a ring, then $\text{Spec}(A)$ (resp., $\text{Max}(A)$; resp., $\text{Min}(A)$) denotes the set of all prime (resp., maximal; resp., minimal prime) ideals of A and $\dim(A)$ denotes the Krull dimension of A. If $A \subseteq B$ are rings and $P \in \text{Spec}(A)$, then as usual, $\mathbf{k}_A(P) := A_P/PA_P$, viewed canonically as the quotient field of A/P. If $A \subseteq B$ are rings and $P \in \text{Spec}(A)$, then $B_P := B_{A\setminus P}$. As usual, $|\mathcal{U}|$ denotes the cardinal number of a set $\mathcal{U}$; $\subset$ and $\supset$ denote proper inclusions; and X denotes an indeterminate over the ambient ring(s).

Let $A \subseteq B$ be rings. As usual, $[A, B]$ denotes the set of intermediate rings, $\{C \mid C$ is a ring such that $A \subseteq C \subseteq B\}$. If $A \neq B$, we say, following [16], that $A \subset B$ is a *minimal ring extension* if $[A, B] = \{A, B\}$; that is, if there is no ring C such that $A \subset C \subset B$. This concept has been studied using different terminology. For instance, what we have described here as "a minimal ring extension" $A \subset B$ was described in [4] by saying that $A \subset B$ "are adjacent rings."

The present note was prompted when, in rapid succession, the authors discovered the following four items: a counter-example which serves to expose an error in a published result [13, Theorem 2.5] concerning adjacent rings $A \subset B$ where A and B are (commutative integral) domains; a way to edit the statement of [13, Theorem 2.5] so that the new assertion is valid and can be proved by rewriting only one sentence in the published proof of [13, Theorem 2.5]; a variant of the corrected version of [13, Theorem 2.5] in which one identifies another property that is equivalent to the property that had purportedly been characterized in [13, Theorem 2.5]; and a generalization of the equivalences in the preceding two items, from the context of ring extensions of domains to the context of arbitrary (unital) extensions of (commutative unital) rings. The first and second of these four items are quickly addressed in Sections 2 and 3, respectively. The third and fourth of these four items constitute our main contribution here and they appear in Section 4.

The final section of the paper addresses some pertinent matters concerning normal pairs of rings (in the sense of [3]) and conducive Prüfer domains (in the sense of [11]). The most important upshots in Section 5 include Example 5.1 giving a counterexample to [13, Theorem 2.5] in which the base ring is not quasi-local, Example 5.4 using certain conducive Prüfer domains to answer an analogous question concerning normal pairs, and Remark 5.5 explaining why it would have been impossible to use a two-dimensional conducive Prüfer domain to settle the question that was resolved in Example 5.1.

We next recall the focus of [13] and the context of its erroneous result. Let $F \subseteq L$ be fields and view $[F, L]$ as a poset under inclusion. Following [12], let the cardinal number $\lambda(L/F)$ denote the supremum of the lengths of chains of intermediate fields in $[F, L]$, taking the "length" of an infinite chain to be its cardinality. Now, suppose that $A \subseteq B$ are rings, with $Q \in \text{Spec}(B)$ and $P := Q \cap A$ ($\in \text{Spec}(A)$). The canonical A-algebra homomorphism $A \to B/Q$ induces an injective A-algebra homomorphism $A/P \to B/Q$ and, hence, a (nonzero) field homomorphism $\mathbf{k}_A(P) \to \mathbf{k}_B(Q)$. It is harmless to view this field homomorphism as a field extension and, in this way, $\lambda(\mathbf{k}_B(Q)/\mathbf{k}_A(P))$ is well defined. Taking the supremum of the cardinal numbers $\lambda(\mathbf{k}_B(Q)/\mathbf{k}_A(P))$ as Q varies over $\text{Spec}(B)$, one obtains the definition of the domain-theoretic invariant $\Lambda(B/A)$ whose study was the focus of [13]. One upshot of the assertion of [13, Theorem 2.5] is that if domains $A \subset B$ are adjacent rings, then $\Lambda(B/A)$ is either 0 or 1. That upshot is correct and will be an easy consequence of Theorem 3.1. Unfortunately, the statement of [13, Theorem 2.5] is incorrect at the point where it alleges to characterize when, for $A \subset B$ as above, one has $\Lambda(B/A) = 1$. Rectifying that unfortunate situation is the main purpose of Theorem 3.1. Giving some generalizations of that rectification to the context of arbitrary rings is the purpose of our main result, Theorem 4.4.

This paragraph collects some background on minimal ring extensions that will be useful in Sections 4 and 5. If $A \subset B$ is a minimal ring extension, it follows from [16, Théorème 2.2 (i) and Lemme 1.3] that there exists $M \in \text{Max}(A)$ (called the *crucial maximal ideal* of $A \subset B$) such that the canonical injective ring homomorphism $A_M \to B_M$ can be viewed as a minimal ring extension while the canonical ring homomorphism $A_P \to B_P$ is an isomorphism for all prime ideals P of A other than M. An easy proof in [6] via globalization and a case analysis showed that, conversely, a minimal ring extension can be characterized as a ring extension for which there exists a crucial maximal ideal (in the above sense).

Any unexplained material is standard, as in [18], [20].

2 Some simple families of counterexamples to [13, Theorem 2.5]

The (erroneous) statement of [13, Theorem 2.5] was as follows. Let $R \subset T$ be adjacent domains (that is, domains such that $R \subset T$ are adjacent rings). If $(R : T) \in \text{Max}(T)$, then $\Lambda(T/R) = 1$; otherwise, $\Lambda(T/R) = 0$. Perhaps the easiest counterexample to [13, Theorem 2.5] is given by taking $R \subset T$ to be $\mathbb{Z}_{2\mathbb{Z}} \subset \mathbb{Q}$. More generally, if (R, M) is any valuation domain with $\dim(R) = 1$ and T denotes the quotient field of R, then $R \subset T$ are adjacent domains (cf. [20, Exercise 29, page 43]) and $(R : T) = 0 \in \text{Max}(T)$, but $\Lambda(T/R) = \lambda(T_0/R_0) = \lambda(T/T) = 0 \neq 1$.

Still more generally, let (R, M) be any valuation domain (of possibly infinite Krull dimension), but not a field, such that some $P \in \text{Spec}(R)$ is adjacent to M as a prime ideal (that is, such that M/P has height 1 as a prime ideal of R/P). For instance, take (R, M) to be a valuation domain of finite Krull dimension $n \geq 2$ and take P to be the prime ideal of R having height $n - 1$. Then another counterexample to [13, Theorem 2.5] is given by taking R as above and $T := R_P$. Note that, in contrast to the data in the preceding paragraph, the present data satisfy that T is not a field and $0 \neq P = PR_P = (R : T) \in \text{Max}(T)$;

$$\Lambda(T/R) = \sup_{\{q \in \text{Spec}(R) | q \subseteq P\}} \lambda(T_{qR_P}/R_q) = 0,$$

with the last displayed equality holding since, for $q \subseteq P$ as above, $T_{qR_P} = (R_P)_{qR_P} = R_q$. The conclusion that this example satisfies $\Lambda(T/R) = 0$ can also be obtained from some known results (cf. any of [13, Proposition 2.3 (a), Proposition 2.3 (b), Corollary 2.7, Theorem 2.9]) ensuring that $\Lambda(E/D) = 0$ whenever E is an overring of a Prüfer domain D.

3 The simplest way to correct [13, Theorem 2.5]

By inserting a hypothesis of integrality into one of the cases in the (erroneous) statement of [13, Theorem 2.5], one obtains the following valid result.

Theorem 3.1. *If $R \subset T$ are adjacent domains, then*

$$\Lambda(T/R) = \begin{cases} 1, & if\ T\ is\ integral\ over\ R\ and\ (R : T) \in Max(T) \\ 0, & otherwise. \end{cases}$$

Proof. The error in the statement of [13, Theorem 2.5] was due to the authors' misunderstanding of the statement of [4, Theorem 2.13.2]. Professors Mullins and Dobbs apologize for having inadvertently misrepresented the content of [4, Theorem 2.13.2]. The simplest way to correct matters, and thereby obtain a proof of the present result, is to edit the published "proof" of [13, Theorem 2.5] as follows. Replace the first sentence of the second paragraph of that "proof" (where [4, Theorem 2.13.2] had been mentioned) with a citation of either the earlier result [13, Proposition 2.3 (b)] on normal pairs or its application to adjacent extensions [13, Corollary

2.4]. This replacement permits the remainder of the proof to proceed, as published and as had originally been intended, in the context of an integral (minimal ring) extension of domains. □

4 Generalizations to the ring-theoretic setting

At several points in the proof of our main result, Theorem 4.4, the argument will involve reductions to the domain-theoretic context. Many of those steps will depend on the fact that the calculation of the Λ invariant can itself be reduced to the domain-theoretic context. We begin the section by recalling the precise statement of that fact from [13].

Proposition 4.1. *([13, Proposition 2.1]) If $A \subseteq B$ are rings, then*

$$\Lambda(B/A) = \sup_{Q_0 \in \mathrm{Min}(B)} \Lambda((B/Q_0)/(A/(Q_0 \cap A))).$$

If $F \subseteq L$ are fields, then $\Lambda(L/F) = \lambda(L_0/F_0) = \lambda(L/F)$. In particular, if $F \subseteq L$ is a minimal field extension, then $\Lambda(L/F) = 1$. More generally, we have the following result of combining Proposition 4.1 with Theorem 3.1.

Corollary 4.2. *If $A \subset B$ are adjacent rings, then $\Lambda(B/A)$ is either* 0 *or* 1.

Proof. By Zorn's Lemma, any nonzero ring has a minimal prime ideal (cf. [20, Theorems 1 and 10]). Moreover, if $Q_0 \in \mathrm{Min}(B)$, then the adjacency of $A \subset B$ implies that either $A/(Q_0 \cap A)$ (identified with $(A + Q_0)/Q_0) = B/Q_0$ or $A/(Q_0 \cap A) \subset B/Q_0$ are adjacent rings. Hence, by Proposition 4.1, we may assume that A and B are (adjacent) domains. Then an application of Theorem 3.1 completes the proof. □

To facilitate the flow of the proof of Theorem 4.4, we next collect some relatively easy facts that will be used in that proof.

Proposition 4.3. *(a) Let $A \subset B$ be rings such that $(A : B) \in \mathrm{Spec}(B)$. Then there exists $Q_0 \in \mathrm{Min}(B)$ such that $Q_0 \subseteq (A : B)$, so that, in particular, $Q_0 \in \mathrm{Spec}(A)$.*
(b) Let $A \subset B$ be rings, with J a common ideal of A and B. Put $\overline{A} := A/J$ and $\overline{B} := B/J$. Then:
(i) $A \subset B$ is an integral ring extension if and only if $\overline{A} \subset \overline{B}$ is an integral ring extension.
(ii) $A \subset B$ are adjacent rings if and only if $\overline{A} \subset \overline{B}$ are adjacent rings.
(iii) $(\overline{A} : \overline{B}) = (A : B)/J$.
(iv) If C denotes either A or B, then $\mathrm{Max}(\overline{C}) = \{Q/J \mid J \subseteq Q \in \mathrm{Max}(C)\}$.

Proof. (a) The existence of $Q_0 \in \mathrm{Min}(B)$ such that $Q_0 \subseteq (A : B)$ follows from Zorn's Lemma (cf. [20, Theorem 10]). As $(A : B) \in \mathrm{Spec}(B)$ and $(A : B) \subseteq A$, we also have $(A : B) = (A : B) \cap A \in \mathrm{Spec}(A)$.
(b) (i) This assertion follows by an easy calculation (cf. [17, Corollary 1.5 (5)]).
(ii) This assertion follows from a standard homomorphism theorem (cf. also [14, Lemma II.3]).
(iii) This assertion follows by an easy calculation.
(iv) This assertion follows from a standard homomorphism theorem. □

Theorem 4.4. *Let $A \subset B$ be adjacent rings. Then the following conditions are equivalent:*
(1) B is integral over A and $(A : B) \in \mathrm{Max}(B)$;
(2) $(A : B) \in \mathrm{Max}(A) \cap \mathrm{Max}(B)$;
(3) $\Lambda(B/A) = 1$;
(4) $\Lambda(B/A) \neq 0$.

Proof. (1) ⇒ (2): Suppose that (1) holds. Then, by [16, Théorème 2.2 (ii)], $(A : B)$ is the crucial maximal ideal of the minimal ring extension $A \subset B$. In particular, $(A : B) \in \mathrm{Max}(A)$, as desired.

(2) ⇒ (1): Suppose that (2) holds. Then $M := (A : B) \in \mathrm{Max}(A) \cap \mathrm{Max}(B)$, and our task is to prove that B is integral over A. Let N denote the crucial maximal ideal of the minimal ring extension $A \subset B$. We claim that $N = M$. Suppose, for the moment, that this claim fails. Then one can pick $r \in M \setminus N$, since M and N are distinct maximal ideals of A. In particular, $r \in (A : B) \cap (A \setminus N)$. It follows (cf. the hints for given for [20, Exercise 41, page 46]) that the canonical injective A-algebra isomorphism $A_N \to B_N$ is surjective; that is, $A_N = B_N$ canonically. However, by a fundamental property of crucial maximal ideals [16, Théorème 2.2 (i)], $A_N \subset B_N$ is a minimal ring extension. This (desired) contradiction proves the above claim; that is, $N = M$. In particular, $N \in \mathrm{Spec}(B)$ lies over the crucial maximal ideal $N \in \mathrm{Spec}(A)$. Hence, by [16, Théorème 2.2 (ii)], B is integral over A, as desired.

(3) ⇔ (4): This follows at once from Corollary 4.2.

(1) $\Rightarrow$ (4): Suppose that the assertion fails. Then, since (1) $\Rightarrow$ (2), B is integral over A and $(A : B) \in \mathrm{Max}(A) \cap \mathrm{Max}(B)$, but $\Lambda(B/A) = 0$. Hence, by Proposition 4.3 (a), we can fix some $Q_0 \in \mathrm{Min}(B)$ such that $Q_0 \subseteq (A : B)$. In particular, $Q_0 \cap A = A$. Thus, by Proposition 4.1, $\Lambda((B/Q_0)/(A/Q_0)) = 0$. Consider the domains $\overline{B} := B/Q_0$ and $\overline{A} := A/Q_0$. We have $\Lambda(\overline{B}/\overline{A}) = 0$. By parts (i) and (ii) of Proposition 4.3 (b), $\overline{A} \subset \overline{B}$ is an integral minimal ring extension. Also, by parts (iii) and (iv) of Proposition 4.3 (b), $(\overline{A} : \overline{B}) = (A : B)/Q_0$ and $(A : B)/Q_0 \in \mathrm{Max}(\overline{A}) \cap \mathrm{Max}(\overline{B})$, whence $(\overline{A} : \overline{B}) \in \mathrm{Max}(\overline{B})$. Therefore, by Theorem 3.1, $\Lambda(\overline{B}/\overline{A}) = 1$, the desired contradiction.

(3) $\Rightarrow$ (1): Suppose (3); that is, $\Lambda(B/A) = 1$. By Proposition 4.1, there exists $Q_0 \in \mathrm{Min}(B)$ such that $\Lambda((B/Q_0)/(A/(Q_0 \cap A))) = 1$. Of course, $A/(Q_0 \cap A)$ has been identified with $(A + Q_0)/A$. It cannot be the case that $A + Q_0 = B$ (for otherwise, B/Q_0 would be canonically isomorphic to $A/(Q_0 \cap A)$, so that $\Lambda((B/Q_0)/(A/(Q_0 \cap A))) = 0$, a contradiction). Consequently, as $A \subset B$ are adjacent rings, we get $A + Q_0 = A$; that is, Q_0 is a common (prime) ideal of A and B. As in the preceding paragraph, consider the domains $\overline{B} := B/Q_0$ and $\overline{A} := A/Q_0$. By part (ii) of Proposition 4.3 (b), $\overline{A} \subset \overline{B}$ are adjacent rings (in fact, adjacent domains). Therefore, since $\Lambda(\overline{B}/\overline{A}) = 1$, Theorem 3.1 ensures that $\overline{A} \subset \overline{B}$ is an integral ring extension and $(\overline{A} : \overline{B}) \in \mathrm{Max}(\overline{B})$. Hence, by parts (i), (iii) and (iv) of Proposition 4.3 (b), $A \subset B$ is an integral (minimal) ring extension and $(A : B)/Q_0 \in \mathrm{Max}(\overline{B})$, so that $(A : B) \in \mathrm{Max}(B)$. This establishes (1), thus completing the proof. □

We close the section by recounting some of the developments in the theory of minimal ring extensions that have occurred since [13] was submitted for publication, with an emphasis on the influence of the case analysis in the published "proof" of [13, Theorem 2.5] and some recent work concerning condition (1) in the statement of Theorem 4.4 (that is, concerning the revision of a condition from the statement of [13, Theorem 2.5] which, in its revised form here, permitted the proof of this note's Theorems 3.1 and 4.4).

Remark 4.5. Condition (1) in the statement of Theorem 4.4 is nowadays usually described as "$A \subset B$ is an inert (integral minimal ring) extension." In fact, the (valid) analysis for the integral context in the published "proof" of [13, Theorem 2.5] led (in conjunction with the Ferrand-Olivier classification of the minimal ring extensions of a field [16, Lemme 1.2]) to what has been termed as the inert-decomposed-ramified trichotomy for integral minimal ring extensions (cf. [14, Corollary II.2], [15, Proposition 2.12]). In detail, let $A \subseteq B$ be an integral ring extension, with conductor $M := (A : B)$. A standard homomorphism theorem shows that $A \subset B$ is a minimal ring extension if and only if $A/M \subset B/MB$ $(= B/M)$ is a minimal ring extension. In fact (cf. also [16, Lemme 1.2 and Proposition 4.1], [14, Lemma II.3]), the above-mentioned classification result of Ferrand-Olivier leads to the following trichotomy: $A \subseteq B$ is a (an integral) minimal ring extension if and only if $M \in \mathrm{Max}(A)$ and (exactly) one of the following three conditions holds: $A \subseteq B$ is said to be respectively *inert*, *decomposed*, or *ramified* if B/MB $(= B/M)$ is isomorphic, as an algebra over the field $F := A/M$, to a minimal field extension of F, $F \times F$, or $F[X]/(X^2)$. Notice that in this situation, where the minimal ring extension $A \subset B$ is integral, its conductor is M, which is also the crucial maximal ideal of $A \subset B$. Note also that a minimal ring extension $A \subset B$ is either integrally closed (in the sense that A is integrally closed in B) or integral.

The integrally closed minimal ring extensions were extensively characterized and discussed, with the help of a generalized Kaplansky transform, in [2] (cf. also [15, Section 3]). Of the three kinds of integral minimal ring extensions, the inert ones are perhaps the most enigmatic, in part because both the ramified and the decomposed extensions can be characterized via generator-and-relations [15, Proposition 2.12]. For every nonzero ring A, there exists at least one ramified extension $A \subset B$ [5, Corollary 2.5] and at least one decomposed extension $A \subset B$ [15, page 805]. As for inert extensions, note first that if F is a field, then a ring extension $F \subseteq B$ is inert if and only if (B is a field and) $F \subseteq B$ is a minimal field extension. Thus, it follows from the classical Galois theory of finite fields that if k is a finite field, then there exist denumerably many inert extensions $k \subset L_i$ such that, whenever $i \neq j$, L_i and L_j are not isomorphic as k-algebras. On the other hand, an algebraically closed field has no inert extensions. The same conclusion holds for any SPIR (special principal ideal ring) which is not a field [7, Proposition 8].

More generally, let A be a (nonzero) finite local ring which is not a field. It turns out to be a thorny question to determine the cardinal number, say ν_A, of the collection of A-algebra isomorphism classes that can be represented by rings B such that $A \subset B$ is inert. For any such A, ν_A is finite [9, Theorem 2.3 (a)], in contrast to the case where the base ring is a given finite field. However, as A varies over the class of finite local rings that are not fields, there is no absolute finite upper bound on ν_A [10, Example 4.4]. Much of the recent interest in inert extensions has been due to the discovery of some inert examples whose existence revealed a 45-year-old error in a classic text that had purported to characterize the Galois ring extensions of a finite local ring: for further details, see [8, Theorem 2.5 and Remark 2.6 (a), (d)].

5 Some connections with normal pairs

In Section 2, we recalled the (erroneous) statement of [13, Theorem 2.5] concerning adjacent rings and then provided a couple of families of counterexamples to that statement. Note that all those counterexamples featured quasi-local base rings. It seems natural to ask if a more general family of counterexamples is available. Put differently: does there exist a counterexample to [13, Theorem 2.5] whose base ring is not quasi-local? We begin this final section with Example 5.1, which provides an affirmative answer to the above question by constructing the desired "more general family of counterexamples" to [13, Theorem 2.5], that is, counterexamples each of which has a base ring that is not quasi-local. The development of Example 5.1 will naturally involve the archetypical kind of normal pair, that is, (R, K) where R is a Prüfer domain with quotient field K. Proposition 5.3 identifies an order-theoretic aspect of the prime spectrum of the base ring A in any ring extension $A \subset B$ that satisfies the conclusion of Example 5.1. Not only does that result serve to explain how the authors were led to discover the construction used in the proof of Example 5.1, but it also identifies a commonality between the quasi-local base rings in the counterexamples to [13, Theorem 4.5] that were given in Section 2 and the non-quasi-local base rings appearing in Example 5.1.

Following Proposition 5.3, we will consider a question analogous to the question that is solved in Example 5.1. In the new question, the earlier role that had been played by an integrally closed minimal ring extension is played by a normal pair. This new question is settled in Example 5.4 by using a normal pair whose base ring is a certain conducive Prüfer domain. As the base ring used in Example 5.1 is a Prüfer domain that is not a conducive domain, it is natural to ask if one could have proven Example 5.1 by using a suitable ring extension whose base ring is a conducive Prüfer domain. Remark 5.5 addresses this question, showing that it is not possible for the motivating question about adjacent rings (which is resolved in Example 5.1) to be settled by using a ring extension whose base ring is a two-dimensional conducive Prüfer domain. Then we present a proposition and three corollaries that develop some material on normal pairs in the spirit of some of the work on minimal ring extensions in Section 4. In particular, returning to a theme motivated by the context of [13, Theorem 2.5], Corollaries 5.8-5.10 identify some roles played by Prüfer domains in the theory of normal pairs (A, B) such that $(A : B) \in \text{Max}(B)$.

Example 5.1. For any integer $n \geq 2$, there exist a Prüfer domain R and an overring T of R such that R is not quasi-local (that is, R is not a valuation domain), $\dim(R) = n$, $R \subset T$ is a minimal ring extension, and $(R : T) \in \text{Max}(T)$. It can also be arranged that $|\text{Max}(R)|$ is any preassigned integer $m \geq 2$. Since, in addition, $R \subset T$ is an integrally closed ring extension and $\Lambda(T/R) = 0$, any such data contradicts [13, Theorem 2.5] and shows the need for the integrality condition in the statement of Theorem 3.1.

Proof. By [22, Theorem 3.1], there is a Prüfer domain R with exactly m pairwise distinct maximal ideals, say $M = M_1, M_2, \ldots, M_i, \ldots, M_m$, such that M has height n; for each $i \geq 2$, M_i has height at most n; and whenever $1 \leq i < j \leq m$, 0 is the only prime ideal of R that is contained in both M_i and M_j. (Rather than citing [22] to create a Prüfer domain R with such a vee-shaped prime spectrum, one could construct such R by intersecting a suitable family of m pairwise incomparable valuation domains that have a common quotient field: cf. [18, Theorem 22.8], [20, Theorem 107].) Denote the (chain of) prime ideals of R that are contained in M by $M \supset P_{n-1} \supseteq \ldots \supseteq P_i \supseteq \ldots \supseteq P_1 \supset 0$. Consider

$$T := R_{P_{n-1}} \cap (\cap_{i=2}^{m} R_{M_i}).$$

Of course, T is an overring of R. Moreover, since $R_{P_{n-1}}, R_{M_2}, \ldots, R_{M_n}$ are pairwise incomparable valuation overrings of R, it follows (cf. [18, Theorem 22.8]) that the maximal ideals of T meet R in $P_{n-1}, M_2, \ldots$, and M_n. In particular, no prime ideal of T lies over M. Consequently, $R \subset T$ and $MT = T$. Hence, $MT \not\subseteq R$; that is, $(R : T) \neq M$.

Since R is a Prüfer domain, $R \subset T$ is an integrally closed ring extension and, by [13, Corollary 2.7], $\Lambda(T/R) = 0$. Moreover, since each overring of R is an intersection of localizations of R (at various prime ideals of R) [18, Theorem 26.1 (2)], the nature of $\text{Spec}(R)$ as a poset under inclusion ensures that $R \subset T$ is a minimal ring extension. (The point is that the only possible ring in $[R, T] \setminus \{T\}$ is $R_M \cap T = R$.) As the minimal ring extension $R \subset T$ is also integrally closed and $MT = T$, it follows from [16, Théorème 2.2 (ii)] that M is the crucial maximal ideal of $R \subset T$.

It will suffice to prove that $(R : T) = P_{n-1}$ and that $P_{n-1} \in \text{Max}(T)$. For convenience, henceforth let $P := P_{n-1}$. We can apply [19, Lemma 3] to the present data, thus showing that $PT = P$. In other words, $P \subseteq (R : T)$. Moreover, since R is a seminormal domain and T is an overring of R, [11, Lemma 2.10 (i)] ensures that $(R : T)$ is a radical ideal of both T and R. As $1 \notin (R : T)$ and the only prime ideal of R that properly contains P is M, the task of proving that $(R : T) = P$ has been reduced to proving that $(R : T) \neq M$. That inequality was established two paragraphs ago. Thus, $(R : T) = P = P_{n-1}$.

Recall that $(R : T) = P$ is a radical ideal of T and, hence, an intersection of some prime ideals of T. If $Q \in \text{Spec}(T)$ satisfies $P \subseteq Q$, then $Q \cap R$ is a prime ideal of R that contains P; that is, $Q \cap R$ must be either P or M. As we have seen that $Q \cap R$ cannot be M, it must be the case that $Q \cap R = P$. As P is a proper ideal of T, there does exist a maximal ideal $\mathfrak{q}$ of T such that $P \subseteq \mathfrak{q}$. Taking $Q = \mathfrak{q}$ now shows that $P = Q \cap R = \mathfrak{q} \cap R \in \text{Spec}(R)$. However, since P is not the crucial maximal ideal of the minimal ring extension $R \subset T$, we have $R_P = T_P$ canonically, and it follows that there is at most one prime ideal of T that meets R in P. Consequently, there is exactly one $Q \in \text{Spec}(T)$ which contains P, that Q must be in $\text{Max}(T)$, and P coincides with that Q. In short, $P \in \text{Max}(T)$. The proof is complete. □

Remark 5.2. (a) In the proof of Example 5.1, it was only to simplify the notation that we arranged that the crucial maximal ideal of $R \subset T$ was the maximal ideal of R having maximal height. If $n \geq 3$ then, by taking the maximal height of at least one of $M_2, \ldots, M_m$ to be n, one can adapt the above reasoning and obtain slightly different counterexamples to [13, Theorem 2.5], where the crucial maximal ideal M_1 has height at most $n - 1$.

(b) In view of the proof of Example 5.1, one may ask if there is a valid analogue of Example 5.1 when $n = 1$. More precisely, does there exists a one-dimensional Prüfer domain R, with exactly two distinct maximal ideals, say M and N, such that $R \subset R_N$ is a(n integrally closed) minimal ring extension such that $(R : R_N)$ is a (the) maximal ideal of R_N? The answer is in the negative. If it were otherwise, then $N = NR_N$ and any $r \in N \setminus M$ and $s \in M \setminus N$ would satisfy $r = (rs^{-1})s \in (NR_N)M = NM \subseteq M$, a contradiction.

(c) A significant amount of the first paragraph of the proof of Example 5.1 was devoted to showing that $MT = T$; that is, that $1 \in MT$. There are a number of other interesting ways to establish this fact, and we next indicate one of those methods. Use the Prime Avoidance Lemma [20, Theorem 81] to produce an element $r \in M$ such that $r \notin P$ and $r \notin \cup_{i=2}^{m} M_i$. Then $1 = rr^{-1} \in MT$, as desired. This completes the remark.

It is natural to ask if a qualitatively different kind of construction could have been used to prove Example 5.1. In a sense, the answer is in the negative, as the next result identifies two order-theoretic properties that must be satisfied by the prime spectrum of any base ring A for which the ring extension $A \subset B$ satisfies the conclusion of Example 5.1.

Proposition 5.3. *Let $A \subset B$ be an integrally closed minimal ring extension such that $(A : B) \in \text{Max}(B)$. Then $(A : B)$ is a nonmaximal prime ideal of A, $(A : B)$ is contained in only one maximal ideal M of A, $(A : B)$ is not properly contained in any nonmaximal prime ideal of A, and M is the crucial maximal ideal of $A \subset B$.*

Proof. It will be convenient to let $P := (A : B)$. As $P \in \text{Spec}(B)$, we have $P = P \cap A \in \text{Spec}(A)$. Also, since $1 \notin (A : B)$, there exists $M \in \text{Max}(A)$ such that $P \subseteq M$.

Consider the domain $\overline{A} := A/(A : B) = A/P$ and the field $\overline{B} := B/(A : B) = B/P$. Then, by parts (i) and (ii) of Proposition 4.3 (b), $\overline{A} \subset \overline{B}$ inherits the property of being an integrally closed minimal ring extension from $A \subset B$. As every minimal ring extension of a field is an integral extension (cf. [16, Lemme 1.2]), it follows that $\overline{A}$ is not a field. Hence, $(A : B)$ is not a maximal ideal of A. Recall from [24] that if $D \subset E$ is a minimal ring extension of domains and D is not a field, then E is (D-algebra isomorphic to) an overring of D. Consequently, $\overline{B}$ must be the quotient field of $\overline{A}$. Therefore, the minimality of $\overline{A} \subset \overline{B}$ ensures that $\overline{A}$ is a one-dimensional valuation domain (cf. [20, Exercise 29, page 43]). Thus, $\overline{A}$ is quasi-local, and so M is the only maximal ideal of A that contains P. Furthermore, there cannot exist a nonmaximal prime ideal Q of A such that $P \subset Q$ (for otherwise, $P \subset Q \subset M$, contradicting the fact that $\dim(\overline{A}) = 1$).

It remains to prove that M is the crucial maximal ideal of $A \subset B$. It suffices to show that if N is any maximal ideal of A other than M, then $A_N = B_N$. As $N \neq M$, we have $P \nsubseteq N$; that is, $(A : B) \nsubseteq N$. Then a simple calculation (as in the hints for [20, Exercise 41, page 46]) shows that $A_N = B_N$, as desired. □

The following background will be useful and it will also help to motivate the transition (later in this section) from a context involving minimal ring extensions to a context involving normal pairs. If $A \subseteq B$ are rings, then (A, B) is said to be a *normal pair* if each ring in $[A, B]$ is integrally closed in B. If $A \subset B$ are adjacent rings (that is, if $A \subset B$ is a minimal ring extension), then the ring extension $A \subset B$ is integrally closed if and only if (A, B) is a normal pair. Many characterizations of normal pairs are known: see [21, Theorem 5.2, pages 47-48]. Also, according to a celebrated result of Davis (cf. [18, Theorem 26.2]), if R is a domain with quotient field K, then (R, K) is a normal pair (if and) only if R is a Prüfer domain. As a number of authors have noted (cf. [23]), if $D \subseteq E$ are domains such that (D, E) is a normal pair, then E is (D-algebra isomorphic to) an overring of D (inside the quotient field of D).

The (counter)examples in Section 2, along with Example 5.1, lead naturally to the following question. Does there exist a normal pair (A, B) with $(A : B) \in \text{Max}(B)$ such that $A \subset B$ is not a minimal ring extension? (To avoid certain trivialities, we will tacitly also require that B is not a total quotient ring.) This question is easy to answer. If R is a valuation domain in which $P \subset Q$ are (two of the) nonzero prime nonmaximal ideals, then (R, R_P) is a normal pair and $(R : R_P) = P = PR_P \in \text{Max}(R_P)$, but $R \subset R_P$ is not a minimal ring extension since $R \subset R_Q \subset R_P$.

It now seems natural to raise the following variant of the above question. Does there exist a normal pair (A, B) with $(A : B) \in \text{Max}(B)$ such that $A \subset B$ is not a minimal ring extension and A is not quasi-local? Example 5.4 will answer this question in the affirmative. First, we devote the following paragraph to some background material.

Recall from [11] that if R is a domain with quotient field K, then R is said to be a *conducive domain* if $(R : V) \neq 0$ for each valuation ring V of R other than K. In fact, R satisfies the "conducive" property if a single such V can be found. (Although this result was given in [11, Theorem 3.2], it was obtained much earlier in [1]. As noted belatedly in a note added in galley proof to [11], conducive domains were introduced and studied, without the "conducive" terminology, in [1].) Several ideal-theoretic characterizations of conducive Prüfer domain are known [11, Corollary 3.4], as is a pullback-theoretic characterization [11, Proposition 2.12] of seminormal conducive domains. Examples of conducive domains include any ring obtained via the classical $D + M$ construction [11, Proposition 2.2].

We next answer the above question.

Example 5.4. For any integer $n \geq 2$, there exist a Prüfer domain R and an overring T of R such that R is not quasi-local (that is, R is not a valuation domain), $\dim(R) = n$, $(R : T) \in \text{Max}(T)$, and $R \subset T$ is not a minimal ring extension. It can also be arranged, for any preassigned integer $m \geq 2$ and for any integer ν such that $n + m \leq \nu \leq (n-1)m + 2$, that $|\text{Max}(R)| = m$ and $|\text{Spec}(R)| = \nu$. Necessarily, (R, T) is a normal pair.

Proof. By [22, Theorem 3.1], there exists a Prüfer domain R with exactly m pairwise distinct maximal ideals, say $M_1, M_2, \ldots, M_i, \ldots, M_m$, such that M_1 has height n; R has a unique height 1 prime ideal, P; P is the only nonzero prime ideal of R that is contained in more than one of the M_j; if $2 \leq i \leq m$, then the height of M_i is an integer h_i such that $2 \leq h_i \leq n$; and (as would be necessary for any Prüfer domain) the set of prime ideals of R that are contained in any specific M_j forms a chain. This prime spectrum is an example of what has been called a "generalized Y-shaped prime spectrum," but one should notice the extra feature here that the "vertex" of the "Y" is at the height 1 prime ideal.

We next show that it is possible to choose $h_2, \ldots, , h_m$ so that $|\text{Spec}(R)| = \nu$. We have

$$|\text{Spec}(R)| = (n+1) + \sum_{i=2}^{m}(h_i - 1) = n - m + 2 + \sum_{i=2}^{m} h_i.$$

Requiring the displayed value to be ν satisfying the stipulated inequalities is equivalent to requiring that $\sum_{i=2}^{m} h_i = \nu - n + m - 2$. It is easy to see that the h_i can be chosen so that this equation holds.

Put $T := R_P$. Since R is a Prüfer domain and T is an overring of R, (R, T) is a normal pair. Moreover, $R \subset T$ is not a minimal ring extension, since $R \subset R_{M_1} \subset R_P = T$. As $|\text{Max}(R)|$ and $|\text{Spec}(R)|$ have the asserted values, it remains only to show that $(R : T) \in \text{Max}(T)$, that is, that $(R : R_P) = PR_P$.

Since R is a Prüfer domain, but not a field, such that $\text{Spec}(R)$ is pinched at a nonzero prime ideal (namely, at P), it follows from [11, Corollary 3.4] that R is a conducive Prüfer domain. Therefore, by [11, Corollary 3.4], there exists at least one nonzero prime ideal Q of R such that $Q = QR_Q$. Pick one such Q. We claim that $Q = P$.

Suppose that the claim fails. We have that $P \subset Q$ by the "pinched" condition, and so there exist at least $m - 1$ maximal ideals M_j of R such that $Q \not\subseteq M_j$. Pick one such M_j. In particular, $M_j \not\subseteq Q$. We can now choose nonzero elements $r \in Q \setminus M_j$ and $s \in M_j \setminus Q$. Hence $rs^{-1} \in QR_Q = Q \subseteq R$, and so $r = (rs^{-1})s \in RM_j = M_j$, a contradiction. This proves the above claim.

As we have proven that $P = Q$, we have $P = PR_P$. Hence $PR_P \subseteq (R : R_P)$. Since $R \neq R_P$ and PR_P is a (the) maximal ideal of R_P, it follows that $(R : R_P) = PR_P$. The proof is complete. □

Remark 5.5. The Prüfer domain R in Example 5.4 is a conducive domain of finite Krull dimension $n \geq 2$. That fact was central in proving that the data in Example 5.4 satisfies $(R : R_P) = P = PR_P \in \text{Max}(R_P)$. By way of contrast, the proof of Example 5.1 could not have proceeded in a similar fashion. Indeed, the base ring in Example 5.1, while being a Prüfer domain, is definitely not a conducive domain. In fact, if D is any finite-dimensional non-quasi-local domain, but not a field, with a vee-shaped prime spectrum, then D is not a conducive domain. The underlying reason is [11, Theorem 2.4 (ii), (iii)]: if E is a conducive domain and Q is a (necessarily, the unique) height 1 prime ideal of E, then $\text{Spec}(E)$ is pinched at Q. On the other hand, recall that the proof of Example 5.4 used the fact [11, Corollary 3.4] that if E is a Prüfer domain but not a field, then E is a conducive domain if and only if there exists a nonzero prime ideal Q of R such that $QE_Q = Q$ (and, necessarily, $\text{Spec}(E)$ is pinched at Q).

The above facts explain why it was natural to use the "conducive Prüfer domain" condition in constructing a suitable non-quasi-local base ring in Example 5.4. However, these facts also serve to explain why it would have been impossible to use something as prosaic and concrete as a two-dimensional conducive non-quasi-local Prüfer domain D as a suitable base ring in proving Example 5.1. (Recall that the construction in the proof of Example 5.1 did allow for the base ring there to be two-dimensional.) More precisely, we will obtain a contradiction from the assumption that there exist domains $D \subset E$ such that D is a non-quasi-local two-dimensional conducive Prüfer domain, $D \subset E$ is a(n integrally closed) minimal ring extension, and $(D : E) \in \text{Max}(E)$. As noted above, the "conducive Prüfer domain" property ensures that there exists a nonzero prime ideal Q of D such that $QD_Q = Q$. Necessarily, $\text{Spec}(D)$ is pinched at Q. By Proposition 5.3, $\mathfrak{P} := (D : E)$ is a nonmaximal prime ideal of A which is contained in only one maximal ideal of R (namely, the crucial maximal ideal M of $D \subset E$). As R is not quasi-local, $\mathfrak{P} \neq 0$. Hence, since $\dim(D) = 2$ and the prime ideals of D that are contained in M are linearly ordered by inclusion, a process of elimination shows that Q must be $\mathfrak{P}$. This is the desired contradiction (since $\mathfrak{P}$ is not comparable with any of the maximal ideals of D other than M). This completes the remark.

Just as the (counter)examples in Section 2, along with Example 5.1, led naturally to the questions that were answered in Examples 5.1 and 5.4, the nature of those answers lead us to ask about possible roles for Prüfer domains in the theory of arbitrary normal pairs (A, B) such that $(A : B) \in \text{Max}(B)$. The rest of this note is devoted to identifying such roles. To do so, we will need to develop some material concerning normal pairs. The choice of material to be considered is informed by the following observation. The approach to Theorem 4.4 was based on results on minimal ring extensions involving a passage from an extension $A \subset B$ to the induced extension $A/(A : B) \subset B/(A : B)$. The proof of Proposition 5.3 involved the same kind of passage. Accordingly, we will study the behavior of normal pairs and some associated concepts for that kind of passage. While some of the material developed below will be in the spirit of some of the work involving integrally closed minimal ring extensions in Section 4, some of the technical details for normal pairs will be different, owing to the possible absence of crucial maximal ideals. We begin by collecting some rather easy facts about normal pairs and some related matters in the spirit of Proposition 4.3.

Proposition 5.6. *Let $A \subset B$ be rings, with J a common ideal of A and B. Put $\overline{A} := A/J$ and $\overline{B} := B/J$. Then:*

(*a*) (*Rhodes [23, Proposition 3.1.1]*) *(A, B) is a normal pair if and only if $(\overline{A}, \overline{B})$ is a normal pair.*
(*b*) *$(A : B) \in \text{Max}(B)$ if and only if $(\overline{A} : \overline{B}) \in \text{Max}(\overline{B})$.*
(*c*) *If $\Lambda(B/A) = 0$, then $\Lambda(\overline{B}/\overline{A}) = 0$.*

Proof. (b) The hypotheses ensure that $J \subseteq (A : B)$. Hence, by Proposition 4.3 (b) (iv), $(A : B)/J \in \text{Max}(\overline{B})$ if and only if $(A : B) \in \text{Max}(B)$. As $(\overline{A} : \overline{B}) = (A : B)/J$ by Proposition 4.3 (b) (iii), the assertion follows.

(c) By the definition of the Λ invariant, it suffices to show that if $\overline{Q} \in \text{Spec}(\overline{B})$ and $\overline{P} := \overline{Q} \cap \overline{A}$, then $\lambda(\mathbf{k}_{\overline{B}}(\overline{Q})/\mathbf{k}_{\overline{A}}(\overline{P})) = 0$. Let Q denote the unique prime ideal of B such that $J \subseteq Q$ and $Q/J = \overline{Q}$; and let P denote the unique prime ideal of A such that $J \subseteq P$ and $P/J = \overline{P}$. Then $(Q/J) \cap (A/J) = P/J$; that is, $Q \cap A = P$. As $\Lambda(B/A) = 0$ by hypothesis, we have $\lambda(\mathbf{k}_B(Q)/\mathbf{k}_A(P)) = 0$. Therefore, it suffices to show that

$$\lambda(\mathbf{k}_{\overline{B}}(\overline{Q})/\mathbf{k}_{\overline{A}}(\overline{P})) = \lambda(\mathbf{k}_B(Q)/\mathbf{k}_A(P)).$$

This, in turn, holds since the canonical isomorphisms $\overline{B}/\overline{Q} \cong B/Q$ and $\overline{A}/\overline{P} \cong A/P$ induce identifications $\mathbf{k}_{\overline{B}}(\overline{Q}) = \mathbf{k}_B(Q)$ and $\mathbf{k}_{\overline{A}}(\overline{P}) = \mathbf{k}_A(P)$. The proof is complete. □

The change-of-ring considerations in the proof of Proposition 5.6 (c) (cf. also the proof of [13, Proposition 2.1]) suggest the following result about the Λ-invariant which seems to have not been noticed in [13].

Proposition 5.7. *Let $A \subset B$ be rings, with $Q \in \text{Spec}(B)$ and $P := Q \cap A$ ($\in \text{Spec}(A)$). Put $\overline{A} := A/P$ and $\overline{B} := B/Q$, and view $\overline{A} \subseteq \overline{B}$ in the usual way. Then $\Lambda(\overline{B}/\overline{A}) \leq \Lambda(B/A)$. If, in addition, B has a unique minimal prime ideal (that is, if the associated reduced ring of B is a domain), then $\Lambda(\overline{B}/\overline{A}) = \Lambda(B/A)$.*

Proof. For prime ideals $\mathfrak{Q} \supseteq Q$ of B and prime ideals $\mathfrak{P} \supseteq P$ of A, with $\overline{\mathfrak{Q}} := \mathfrak{Q}/Q$ and $\overline{\mathfrak{P}} := \mathfrak{P}/P$, it is straightforward to check that $\overline{\mathfrak{Q}} \cap \overline{A} = \overline{\mathfrak{P}}$ if and only if $\mathfrak{Q} \cap A = \mathfrak{P}$. Therefore, the asserted inequality follows from the observation that the field extension $\mathbf{k}_{\overline{A}}(\overline{\mathfrak{P}}) \subseteq \mathbf{k}_{\overline{B}}(\overline{\mathfrak{Q}})$ can be identified with $\mathbf{k}_A(\mathfrak{P}) \subseteq \mathbf{k}_B(\mathfrak{Q})$. Similar reasoning (cf. also the proof of [13, Proposition 2.1]) shows that the reverse inequality holds if there exists $Q_0 \in \text{Min}(B)$ such that Q_0 is contained in each prime ideal of B, that is, if $\text{Min}(B)$ is a singleton set. The proof is complete. □

We close with three corollaries, each of which shows that Prüfer domains play a role in studying the ambient property involving normal pairs. Note that Corollary 5.8 characterizes the property that was central to Example 5.4, while the "Prüfer domain" condition is part of the conclusions in Corollaries 5.9 and 5.10.

Corollary 5.8. *Let $A \subset B$ be rings. Put $\overline{A} := A/(A : B)$ and $\overline{B} := B/(A : B)$. Then the following conditions are equivalent:*

(1) (A, B) *is a normal pair and* $(A : B) \in \text{Max}(B)$*;*

(2) $(\overline{A}, \overline{B})$ *is a normal pair and* $(A : B) \in \text{Max}(B)$*;*

(3) $(\overline{A}, \overline{B})$ *is a normal pair and* $(\overline{A} : \overline{B}) \in \text{Max}(\overline{B})$ *(and, necessarily,* $\overline{A}$ *is a domain, but not a field, whose quotient field is* $\overline{B}$*);*

(4) $\overline{A}$ *is a Prüfer domain, but not a field, whose quotient field is* $\overline{B}$*.*

Proof. We first address the parenthetical assertions in the statement of (3). Assume that $(\overline{A}, \overline{B})$ is a normal pair and $(\overline{A} : \overline{B}) \in \text{Max}(\overline{B})$. Then, by Proposition 5.6 (b), $(A : B) \in \text{Max}(B)$. It follows that $\overline{B}$ is a field, and so $\overline{A}$ is a domain. Since $(\overline{A}, \overline{B})$ is a normal pair and $\overline{B}$ is a domain, it follows from the above comments that (the field) $\overline{B}$ is the quotient field of $\overline{A}$. However, it follows from Proposition 5.6 (b) that $\overline{A}$ is not a (that is, $\overline{A}$ is not that) field, since $(\overline{A} : \overline{B})$ is a proper ideal of $\overline{B}$.

(1) ⇔ (2): Apply Proposition 5.6 (a), the above-mentioned result of Rhodes.

(2) ⇒ (3): Assume (2). In view of the first paragraph of this proof, it remains only to prove that $(\overline{A} : \overline{B}) \in \text{Max}(\overline{B})$. Thus, in turn, follows from Proposition 5.6 (b).

(3) ⇒ (4): This implication follows from the above-mentioned result of Davis.

(4) ⇒ (2): Assume (4). As $B/(A : B) = \overline{B}$ is a field, $(A : B) \in \text{Max}(B)$. Also, the conclusion that $(\overline{A}, \overline{B})$ is a normal pair follows because $\overline{A}$ is a Prüfer domain whose quotient field is $\overline{B}$. The proof is complete. □

Corollary 5.9. *Let* $A \subset B$ *be rings. Put* $\overline{A} := A/(A : B)$ *and* $\overline{B} := B/(A : B)$*. Suppose also that* (A, B) *is a normal pair and* $(A : B) \in \text{Max}(B)$*. Then* $\overline{A}$ *is a Prüfer domain, but not a field, whose quotient field is* $\overline{B}$ *and* $\Lambda(\overline{B}/\overline{A}) = 0$*.*

Proof. By Corollary 5.8, $\overline{A}$ is a Prüfer domain, but not a field, whose quotient field is $\overline{B}$. Then $\Lambda(\overline{B}/\overline{A}) = 0$ since $\Lambda(E/D) = 0$ whenever E is an overring of a Prüfer domain D [13, Corollary 2.7]. □

We close by characterizing a kind of ring extension that figured in Examples 5.1 and 5.4.

Corollary 5.10. *Let* $A \subset B$ *be rings. Put* $\overline{A} := A/(A : B)$ *and* $\overline{B} := B/(A : B)$*. Suppose also that* (A, B) *is a normal pair,* $(A : B) \in \text{Max}(B)$*, and* $(A : B)$ *is contained in at least two distinct maximal ideals of* A*. Then:*

(a) $\overline{A}$ *is a Prüfer domain whose quotient field is* $\overline{B}$ *and* $\overline{A}$ *is not a field.*

(b) $\overline{A}$ *is not quasi-local (that is,* $\overline{A}$ *is not a valuation domain) and* $(\overline{A} : \overline{B}) = 0 \in \text{Max}(\overline{B})$*.*

Proof. The assertions in (a) follow from Corollary 5.9; (b) is then clear. □

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