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A classification up to algebra isomorphism of the ramified minimal ring extensions of a principal ideal ring

David E. Dobbs

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Abstract. For any (unital commutative) principal ideal ring A, we produce a set of A-algebra isomorphism class representatives of the ramified (unital minimal) ring extensions of A. Special attention is paid to the case of a finite principal ideal ring. The first step in this process involves sharpening some of our earlier results from the case where A is a finite special principal ideal ring. One consequence of this work is the completion of the classification, up to ring isomorphism, of the commutative (unital) rings having a unique proper (unital) subring.

1 Introduction

The titular classification is given in Proposition 2.5, with its application to the case of finite base rings being isolated in Corollary 2.6. One offshot is the completion, in Remark 2.7 (b), of the probabilistic studies that were begun in [3, Corollary 3.6]. In short, this paper completely solves all the fundamental questions about ramified (minimal ring) extensions of a special principal ideal ring (SPIR) or, more generally, of a principal ideal ring (PIR), that underlay [3, Section 3]. The results for SPIRs are largely accomplished with the aid of Theorem 2.1 and its application to the case of finite base rings in Corollary 2.2. These two results serve to generalize and sharpen [3, Theorem 3.4]. Indeed, Theorem 2.1 (g) establishes that the specific examples of ramified extensions of any given finite SPIR, A, which were found in [3, Theorem 3.4] are, up to A-algebra isomorphism, the only possible ramified extensions of A and that the possible finiteness of A is irrelevant in this regard. Moreover, in case the given SPIR, A, *is* finite, the "at least" assertion in [3, Theorem 3.4 (f)] (estimating the number of A-algebra ismorphism classes represented by ramified extensions of A) is sharpened and generalized in Theorem 2.1 (h) and Corollary 2.2 by revisiting more deeply some group-theoretic and some number-theoretic aspects of the approach in [3]. One should perhaps also alert the reader that Theorem 2.1 (g) features what is arguably the hardest proof in this paper.

To carry out the task of extending the classification (from Theorem 2.1) of the relevant algebra isomorphism classes from the context of base rings that are SPIRs to the context where the base rings are arbitrary PIRs (in Proposition 2.5), we apply the fact that ramified (minimal ring) extensions behave well with respect to finite direct products of base rings (as in [3, Lemma 2.2 (b)]). By using this methodology from [3, Lemma 2.2 (b)] and combining our present results with some other earlier work, Corollary 2.8 completes the project of classifying, up to ring isomorphism, the commutative (unital) rings having a unique proper (unital) subring. This project had been begun in [1], it was completed for rings of characteristic 0 in [1, Theorem 5], and it was settled for rings of positive characteristic for all but one case in [1, Theorem 13]. The context for that one open case involves classifying (up to isomorphism) the ramified (minimal ring) extensions of rings of the form $\mathbb{Z}/p^n\mathbb{Z}$ (with p prime and $n \geq 2$), thus explaining in part our recent interest in the ramified extensions of SPIRs.

This paragraph collects some notational conventions and the next paragraph collects some background on minimal ring extensions that will be useful below. All rings and algebras considered here are commutative and unital, typically nonzero; all inclusions of rings, ring extensions and algebra/ring homomorphisms are unital. If A is a ring, then $\mathrm{Max}(A)$ denotes the set of all maximal ideals of A. As usual, $|\mathcal{U}|$ denotes the cardinal number of a set $\mathcal{U}$; $\subset$ denotes proper

inclusion; and X and Y denote commuting, algebraically independent indeterminates over the ambient ring(s). Also, it will be convenient to modify the SPIR concept as defined in [18, page 245], by adding the restriction that no field will be considered to be an SPIR here (although all fields are, of course, PIRs). Finally, we would mention [12] as a suitable reference for the basics about the idealization construction.

If $A \subset B$ are (distinct) rings then we say, following [10], that $A \subset B$ is a *minimal ring extension* if there is no ring C such that $A \subset C \subset B$. A minimal ring extension $A \subset B$ is either integrally closed (in the sense that A is integrally closed in B) or integral. We can mostly ignore integrally closed minimal ring extensions here since, for any minimal ring extension $A \subset B$ such that A is a finite ring, B is also finite [1, Proposition 7] and any ring extension whose "top" ring is finite must be integral (cf. [16, Theorem XIII.1], [4, Lemma 2.1 (c)]). (By using a deep noncommutative result on rng extensions, due independently to Klein [14] and Laffey [15], it was recently shown in [6, Lemma 2.1 (c)] that if $A \subset B$ is a minimal ring extension of possibly noncommutative rings, in the obvious sense, and A is finite, then B is also finite.) The first classification result on minimal ring extensions was due to Ferrand-Olivier [10, Lemme 1.2]: if k is a field, then a nonzero k-algebra B is a minimal ring extension of k (when we view $k \subseteq B$ via the injective structural map $k \to B$) if and only if B is k-algebra isomorphic to (exactly one of) a minimal field extension of k, $k \times k$ or $k[X]/(X^2)$. Now let $A \subset B$ be an integral ring extension, with its conductor $M := (A : B)$ $(:= \{b \in B \mid bB \subseteq A\})$. A standard homomorphism theorem shows that $A \subset B$ is a minimal ring extension if and only if $A/M \subset B/MB$ $(= B/M)$ is a minimal ring extension. In fact (cf. also [10, Lemme 1.2 and Proposition 4.1], [7, Lemma II.3]), the above-mentioned classification result of Ferrand-Olivier leads to the following trichotomy: $A \subset B$ is a (an integral) minimal ring extension if and only if $M \in \mathrm{Max}(A)$ and (exactly) one of the following three conditions holds: $A \subset B$ is said to be respectively *ramified*, *decomposed*, or *inert* if B/MB $(= B/M)$ is isomorphic, as an algebra over the field $k := A/M$, to $k[X]/(X^2)$, $k \times k$, or a minimal field extension of k. It is known that if A is any finite ring that is not a field, then the (cardinal) number of A-algebras that can be represented by a ramified (resp., decomposed) extension of A is finite. To see this, combine the (essentially unique) expression of A as a direct product of finitely many local rings (cf. [18, Theorem 3, page 205]) and an above-mentioned result about finite direct products [3, Lemma 2.2 (b)] with [4, Corollary 2.4 (a)]. The answer to the analogous question about inert extensions is more delicate (cf. [3, Theorem 2.5] and [4, Theorem 2.3]). Fortunately for the present work, an SPIR (which is not a field) has no inert extensions [1, Proposition 8] and an SPIR (A, M) admits a unique class of A-algebras represented by decomposed extensions of A, with representative $A \times A/M$ [1, Proposition 10].

Any unexplained material is standard, as in [11], [13].

2 Results

We begin with two results that significantly sharpen the statement of [3, Theorem 3.4]. Theorem 2.1 deletes the riding assumption from [3, Theorem 3.4] that the given SPIR (A, M) is finite. In our opinion, the new foundational result in this paper is Theorem 2.1 (h), as it classifies, up to algebra isomorphism, the ramified (minimal ring extensions) of any SPIR (that is not a field). This classification result is made possible by Theorem 2.1 (g), whose proof is arguably the most technically demanding in this paper. Corollary 2.2 gives a complete treatment for the special case of Theorem 2.1 where the given SPIR is finite. That includes replacing the earlier estimates from [3, Theorem 3.4 (f)] with equalities. We should mention that part (i) of Theorem 2.1 is being included not only for the sake of completeness but also for readers interested in pursuing probabilistic studies in the spirit of [3, Corollary 3.6] (bearing in mind that the earlier estimates in those probabilistic studies can be sharpened to equalities by combining Theorem 2.1 (i) with the formulas in Corollary 2.2).

Theorem 2.1. *Let (A, M) be an* SPIR *(but not a field). Let α (≥ 2) denote the index of nilpotence of M (so that $M^\alpha = 0 \neq M^{\alpha-1}$). Fix $\pi \in M$ such that $A\pi = M$. For each $u \in U(A)$, let $B_u := A[X]/(\pi X, X^2 - u\pi^{\alpha-1})$. Then:*

(a) If $u \in U(A)$, then the canonical A-algebra homomorphism $A \to B_u$ is an injection, and we use that fact to view $A \subseteq B_u$.

(*b*) *If $u \in U(A)$, then $A \neq B_u$ and $A \subset B_u$ is a ramified (minimal ring) extension.*

(*c*) *$A \subset A(+)A/M$ is a ramified (minimal ring) extension (when the idealization $A(+)A/M$ is viewed as a ring extension of A via the canonical injective A-algebra homomorphism $A \to A(+)A/M$ given by $a \mapsto (a, a + M)$).*

(*d*) *If $u \in \mathrm{U}(A)$, then B_u is not A-algebra isomorphic to $A(+)A/M$.*

(*e*) *Let $u, v \in \mathrm{U}(A)$. Then the following three conditions are equivalent:*

(*1*) *B_u and B_v are A-algebra isomorphic;*

(*2*) *There exists $\sigma \in \mathcal{S}^*$ such that $u - \sigma^2 v \in M$;*

(*3*) *uv^{-1} is a quadratic residue modulo M (in the sense that there exists $w \in \mathrm{U}(A)$ such that $u - w^2 v \in M$).*

In the following, let $K := A/M$; let $K^* := K \setminus \{0\}$, the multiplicative abelian group consisting of the nonzero elements of K; and let K^{*2} be the subgroup of K^* consisting of the squares of the elements of K^*.

(*f*) *Let $u, v \in U(A)$ $(= A \setminus M)$. Then B_u and B_v are A-algebra isomorphic if and only if $u + M$ and $v + M$ represent the same coset of K^{*2} in K^*.*

(*g*) *Let $A \subset B$ be a ramified (minimal ring) extension. Then B is A-algebra isomorphic to either $A(+)A/M$ or B_u for some $u \in U(A)$.*

(*h*) *Let $\mathcal{R}$ denote the collection of A-algebra isomorphism classes that can be represented by rings B such that $A \subset B$ is a ramified (minimal ring) extension. Write K^* as the disjoint union $\cup_{i \in I} K^{*2}(u_i + M)$; that is, for certain elements $u_i \in A \setminus M$, $\{u_i + M \mid i \in I\}$ is a set of coset representatives for the cosets of K^{*2} in K^*. Then one set of (A-algebra isomorphism class) representatives for $\mathcal{R}$ is found by taking the (disjoint) union of the singleton set $\{A(+)A/M\}$ with (the disjoint union) $\cup_{i \in I} B_{u_i}$. Thus, $\mathcal{R}$ is a set and its cardinal number is*

$$|\mathcal{R}| = 1 + |I| = 1 + [K^* : K^{*2}],$$

where the above addition is that of cardinal numbers.

(*i*) *The decomposed (minimal ring) extensions of A constitute a unique A-algebra isomorphism class, with $A \times A/M$ as a representative of it. There do not exist any inert or integrally closed minimal ring extensions of A.*

Proof. The proofs for (a)-(e) that were given in [3, Theorem 3.4] apply *verbatim* here, as those proofs did not use the hypothesis that A is finite.

(f) We have that $u + M$ and $v + M$ represent the same coset of K^{*2} in K^* if and only if there exists $w \in A \setminus M$ such that $u + M = (w + M)^2(v + M)$; that is, if and only if there exists $w \in U(A)$ such that $u - w^2 v \in M$; that is, by (e), if and only if $B_u \cong B_v$ as A-algebras.

(g) Suppose that B is not A-algebra isomorphic to $A(+)A/M$. It will suffice to prove that B is A-algebra isomorphic to B_u for some $u \in U(A)$. As $A \subset B$ is ramified, B is quasi-local and its maximal ideal, say N, satisfies $N^2 \subseteq M \subset N$. Moreover, the generator-and-relations characterization of ramified extensions (cf. [9, Proposition 2.12], [2, Lemma 2.1]) supplies $y \in N \setminus A$ such that $B = A[y], yM \subseteq M$ and $y^2 \in M$. Note that the literature's condition that $y^3 \in M$ is implied by the other conditions here, since $y^3 = yy^2 \in yM \subseteq M$. Also, since $M = A\pi$, the condition that $yM \subseteq M$ is equivalent to $\pi y \in M$. Since B is not A-algebra isomorphic to $A(+)A/M$, it follows from a characterization of idealizations [5, Lemma 3.1 (b)] that either $y^2 \neq 0$ or $\pi y \neq 0$ (or both).

Suppose first that $y^2 \neq 0$ *and* $\pi y \neq 0$. We claim that $\pi y = \pi^2 w$ for some $w \in A$. (It follows from this claim that $\alpha \geq 3$, as $\pi y \neq 0$ by hypothesis.) This claim can be established by adapting the reasoning in the second paragraph of the proof of [1, Proposition 12 (a)]. For the sake of completeness, we next provide the details. Suppose, on the contrary, that no such w exists. Then, as A is an SPIR, the factorization theory for SPIRs (as in [18, page 245]) supplies $\lambda \in U(A)$ such that $\pi y = \pi\lambda$, whence $(y - \lambda)\pi = 0$. As $y - \lambda \in N + U(B) \subseteq U(B)$, we have

$$\pi = 1 \cdot \pi = ((y - \lambda)^{-1}(y - \lambda))\pi = (y - \lambda)^{-1}((y - \lambda)\pi) = (y - \lambda)^{-1} \cdot 0 = 0,$$

which is a contradiction to the factorization theory for SPIRs since $\alpha \geq 2$. This proves the claim that $\pi y = \pi^2 w$ for some $w \in A$. With suitable w in hand, consider $z := y - \pi w$. As $\pi w \in M \subseteq A$, we have $z \in N \setminus A$. Since $A \subset B$ is a minimal ring extension, it follows that $A[z] = B$. Also, $zM \subseteq yM + \pi wM \subseteq M + M = M$. Moreover, since we know that $\pi y = \pi^2 w$

with $y^2 \in M$, we get

$$z^2 = (y - \pi w)^2 = y^2 - 2w\pi y + \pi^2 w^2 = y^2 - 2w\pi^2 w + \pi^2 w^2 = y^2 - \pi^2 w^2 \in M.$$

Consequently, by replacing y with z, we can assume (for the subcase where $y^2 \neq 0$), by *abus de langage*, that $\pi y = 0$ (and also that $y^2 \neq 0$ and, in addition, $y^3 = 0$ since $y^3 = yy^2 \in yM = y\pi A = 0$).

By adapting the reasoning in the proof of the next-to-last assertion in the statement of [1, Proposition 12 (a)], we get $u, v \in U(A)$ and uniquely determined integers i, j such that $1 \leq i, j \leq \alpha$, $y^2 = u\pi^i$ and $\pi y = v\pi^j$, along with the fact that if $i < \alpha - 2$, then $j < \alpha/2$. However, since $\pi y = 0$ forces $j = \alpha$ and $\alpha > \alpha/2$, we can conclude that $i \geq \alpha - 2$. If $i = \alpha - 1$, we get that $B \cong B_u$. (The preceding assertion is rather subtle. By the universal mapping property of factor algebras, there is a surjective A-algebra homomorphism, say φ, from B_u to B that sends $\xi := X + (\pi X, X^2 - u\pi^{\alpha-1})$ to y. It remains to check that the kernel of φ is 0. Recall (cf. the reasoning in [2, pages 3463-3464]) that the "ramified" property of $A \subset B_u = A + \mathcal{S}\xi$ ensures that if $a_1, a_2 \in A$ and $\sigma_1, \sigma_2 \in \mathcal{S}$ are such that $a_1 + \sigma_1\xi = a_2 + \sigma_2\xi$, then $a_1 = a_2$ and $\sigma_1 = \sigma_2$. A similar comment holds for the addition in $B = A + \mathcal{S}y$. Since $\varphi(a + \sigma\xi) = a + \sigma y$ for all $a \in A$ and all $\sigma \in \mathcal{S}$, it is now easy to check that $\ker(\varphi) = 0$.) Therefore, without loss of generality, $i = \alpha - 2$; that is, $y^2 = u\pi^{\alpha-2}$. Next, note that the factorization theory for SPIRs ensures that $u\pi^{\alpha-1} \neq 0$. However,

$$u\pi^{\alpha-1} = \pi(u\pi^{\alpha-2}) = \pi y^2 = (\pi y)y = 0y = 0,$$

a contradiction.

It remains only to consider the case where $y^2 = 0$ and $\pi y \neq 0$. Recall that $A \subset B = A[y]$ is ramified and $\pi y \in M$. We have $\pi y = v\pi^j$ for some $v \in U(A)$ and a uniquely determined integer j such that $1 \leq j \leq \alpha - 1$. We claim that if $\alpha = 2$, then no such data can arise. If the claim fails, then $j = 1$, whence $\pi y = v\pi$, and so $(y - v)\pi = 0$. As explained above, this leads to the contradiction that $\pi = 0$ (by using the fact that $y - v \in U(B)$). This proves the claim, and so we can henceforth assume, without loss of generality, that $\alpha \geq 3$. We will complete the proof of (g) by reducing to the case that was settled earlier. In other words, we will show that if $\alpha \geq 3$ and $A \subset B = A[y]$ is ramified with $y^2 = 0$ and $0 \neq \pi y \in M$, then there exists $z \in N$ such that $B = A[z]$, $\pi z = 0$ and $z^2 \in M$.

As πy is a nonzero element of M, the factorization theory for SPIRs provides an element $w \in A$ such that $\pi y = w\pi^2$. (Note that the preceding step required that $\alpha \geq 3$.) We will show that $z := y - \pi w$ has the desired properties that were listed above. Of course, $A[z] = B$ since $z \in N \setminus A$. Also, it is clear that $\pi z = 0$. Finally, since $y^2 = 0$, we get

$$z^2 = -2(\pi y)w + \pi^2 w^2 \in M^2 A + M^2 A \subset M.$$

This completes the proof of (g).

(h) In view of the standard definition of the sum of two cardinal numbers, it suffices to combine parts (g), (c), (b), (d) and (f) of the present result.

(i) These assertions were established in Propositions 10, 8 and 7, respectively, of [1]. □

Corollary 2.2. *Let (A, M) be a finite* SPIR. *Then $K := A/M$ can be identified with $\mathbb{F}_q$ for some $q = p^n$, with p a prime number and $n \geq 2$. Let α (≥ 2) denote the index of niplotence of M and choose $\pi \in M$ such that $M = A\pi$. For each $u \in U(A)$, let $B_u := A[X]/(\pi X, X^2 - u\pi^{\alpha-1})$. Let $\mathcal{R}$ denote the collection of A-algebra isomorphism classes that can be represented by rings B such that $A \subset B$ is a ramified (minimal ring) extension. Then:*

(a) Suppose that $2 \in M$ (equivalently, the characteristic of A is even; equivalently, $p = 2$; equivalently, $|K|$ is even). Then one set, call it $\mathcal{C}$, of (A-algebra isomorphism class) representatives for $\mathcal{R}$ consists of $A(+)A/M$ and B_1, and so $|\mathcal{R}| = 2$.

(b) Suppose that $2 \notin M$ (equivalently, the characteristic of A is odd; equivalently, $p \neq 2$; equivalently, $|K|$ is odd). Pick $v \in U(A)$ such that v is a quadratic nonresidue modulo M (in the sense that there does not exist $w \in \mathrm{U}(A)$ such that $v - w^2 \in M$). Then one set, call it $\mathcal{D}$, of (A-algebra isomorphism class) representatives for $\mathcal{R}$ consists of $A(+)A/M$, B_1 and B_v, and so $|\mathcal{R}| = 3$.

Proof. Since A is a finite (nonzero) local ring but not a field, $|A| = p^d$ for some prime number p and some integer $d \geq 2$. By Lagrange's Theorem and the Fundamental Theorem of Arithmetic, $|M| = p^e$ for some positive integer $e < d$; and $|K| = |A|/|M| = p^n$ with $n := d - e$, whence K can be identified with $\mathbb{F}_q$ where $q := p^n$. Also, the characteristic of A is p^s for some positive integer s (cf. [16, pages 1-3]). Hence $p \cdot 1 \in M$ (as M is a prime ideal of A). By expressing the integral greatest common divisor of 2 and p as an integral linear combination and arguing as in the proof of [3, Theorem 3.4 (f)], we see that: $2 \in M \Leftrightarrow p = 2 \Leftrightarrow$ the characteristic of A (namely, p^s) is even $\Leftrightarrow |K|$ (namely, $q = p^n$) is even. Similarly, $2 \notin M \Leftrightarrow p \neq 2 \Leftrightarrow$ the characteristic of A is odd $\Leftrightarrow |K|$ is odd. This completes the proofs of the initial assertions in both (a) and (b). In view of parts (e), (f), (g) and (h) of Theorem 2.1, it remains only to establish the following facts: if $p = 2$, then $[K^* : K^{*2}] = 1$; and if $p \neq 2$, then the asserted element v exists and $[K^* : K^{*2}] = 2$.

Suppose first that $p = 2$. As $K \cong \mathbb{F}_{2^n}$, each $\xi \in K^*$ satisfies $\xi^{2^n} = \xi$, whence $\xi = (\xi^{2^{n-1}})^2 \in K^{*2}$. Hence $[K^* : K^{*2}] = 1$ in this case, thus completing the proof of (a).

In the remaining case, $p \neq 2$. Given $\lambda, \mu \in K^*$, we have that $\mu^2 = \lambda^2$ if and only if $\mu \in \{\lambda, -\lambda\}$. As $p \neq 2$, each $\lambda \in K^*$ satisfies $\lambda \neq -\lambda$. Consequently $|K^{*2}| = |K^*|/2$, whence $[K^* : K^{*2}] = |K^*|/|K^{*2}| = 2$. It follows from Theorem 2.1 (f) that, since $|K^{*2}| = |K^*|/2 < |K^*|$, we can choose $v \in A \setminus M$ $(= U(A))$ such that $v + M \in K^* \setminus K^{*2}$ (and so B_v is not A-algebra isomorphic to B_1). This completes the proof of (b). □

Before enlarging the context to base rings that are not necessarily quasi-local, we next collect some comments that pertain to the above context where the base rings were SPIRs.

Remark 2.3. (a) In the interest of full disclosure, we next identify a minor error in the proof of the "at least" assertion in [3, Theorem 3.4 (f)]. The context of that result involved a finite SPIR (A, M), where $|A| = p^m$ and $|M| = p^t$ for some odd prime number p and some positive integers $t < m$. Of course, by arguing as in the second sentence of the above proof of Corollary 2.2, we see that the number of nonzero elements of A/M is then $p^{m-t} - 1$. Unfortunately, the proof of [3, Theorem 3.4 (f)] misidentified that number as being $p^m - p^t - 1$. Fortunately, the statement of all of [3, Theorem 3.4] is correct. In addition, the "at least" assertion in the statement of [3, Theorem 3.4 (f)] (whose proof was actually not seriously affected by the above error) has been improved to a complete answer to the underlying question in Corollary 2.2 (b). That improvement permits us to sharpen the assertions of [3, Corollaries 3.6 and 3.7]. We will say more about such sharpenings while working in a more general context in Remark 2.7 (b).

(b) The "complete answer to the underlying question" that was mentioned in (a) was made possible by applying Theorem 2.1 (h) and calculating the group-theoretic index $[K^* : K^{*2}]$ for finite fields K, where this index turns out to be either 1 or 2, depending on whether the integer $|K|$ is even or odd. One should note that this index can have qualitatively different values when K is an infinite field. For instance, $[\mathbb{Q}^* : \mathbb{Q}^{*2}] = \aleph_0$. (A consideration of the infinite field $\mathbb{Q}$ is not out of place when considering SPIRs, as it is, up to isomorphism, the residue field of the SPIR $\mathbb{Q}[X]/(X^2)$.) To prove this, it follows (from the infinitude of the set of prime numbers) that it suffices to show that if p_1 and p_2 are distinct prime numbers, then $p_1 \notin \mathbb{Q}^{*2}p_2$. This, in turn, is an easy consequence of the Fundamental Theorem of Arithmetic.

(c) Calculating the above group-theoretic index is not simply a matter of noticing the characteristic of an ambient ring. For instance, consider $A := \mathbb{F}_q(Y)[X]/(X^2)$ for any prime-power q. Up to isomorphism, the residue ring of this SPIR is $\mathbb{F}_q(Y)$. As an easy degree argument shows that $Y \notin F(Y)^{*2}$ for any field F, we get that $[F(Y)^* : F(Y)^{*2}] > 1$. In particular, $[\mathbb{F}_q(Y)^* : \mathbb{F}_q(Y)^{*2}] > 1$ for any prime-power q. If such q is even, we saw in the proof of Corollary 2.2 that $[\mathbb{F}_q^* : \mathbb{F}_q^{*2}]$ equals 1 and, hence, is unequal to $[\mathbb{F}_q(Y)^* : \mathbb{F}_q(Y)^{*2}]$ (> 1). One could say more about the value of $[F^* : F^{*2}]$ for various historically important kinds of fields F. But our main purpose in presenting the variety of behavior in (b) and (c) was to alert potential users of Theorem 2.1 (h) and Corollary 2.2 that one needs to be careful in addressing the group-theoretic specifics of any application. This completes the remark.

To prepare for the generalization of Theorem 2.1 from the context where the base ring A is an SPIR to the context where A is an arbitrary PIR, it will be convenient to next state a result that was mentioned above concerning the good behavior of minimal ring extensions with respect to finite direct products.

Lemma 2.4. ([3, Lemma 2.2]) *(a) Let $A \subset B_1$ be a minimal ring extension and let B_2 be an A-algebra such that $B_1 \cong B_2$ as A-algebras. Then $A \subset B_2$ is a minimal ring extension. Moreover, $A \subset B_1$ is ramified (resp., decomposed; resp., inert) if and only if $A \subset B_2$ is ramified (resp., decomposed; resp., inert).*

(b) Let $A = A_1 \times \cdots \times A_n$ be a finite direct product of nonzero rings (where possibly $A_i \cong A_j$ for some $i \neq j$). Let $A \subseteq B$ be a ring extension. Then B is A-algebra isomorphic to $B_1 \times \cdots \times B_n$ where, for each j, $A_j \subseteq B_j$ is a ring extension and B_j is uniquely determined up to A_j-algebra isomorphism. Pick/fix such $B_1, \ldots, B_n$. Then $A \subset B$ is a minimal ring extension if and only if there exists a (necessarily unique) index i such that $A_i \subset B_i$ is a minimal ring extension and $A_j = B_j$ for all $j \neq i$. Moreover, when these conditions hold, $A \subset B$ is the same kind of minimal ring extension (this is, integrally closed, ramified, decomposed, or inert) as $A_i \subset B_i$.

Proposition 2.5 next gives the promised generalization of Theorem 2.1. For any PIR, A, Proposition 2.5 classifies (up to A-algebra isomorphism) the rings B such that $A \subset B$ is a ramified (minimal ring) extension.

Proposition 2.5. *Let A be a* PIR. *It is known (cf. [18, Theorem 33, pages 245 – 246]) that A is ring isomorphic to a finite direct product $\prod_{i=1}^{n} A_i$ where each A_i is either a field, a principal ideal domain* (PID) *which is not a field, or an* SPIR (*which is not a field*). *Fix one such direct product. Let $\mathcal{S}_1$ (resp., $\mathcal{S}_2$; resp., $\mathcal{S}_3$) be the set of indexes i such that A_i is a field (resp., a* PID *which is not a field; resp., an* SPIR (*which is not a field*)). *Put $n_1 := |\mathcal{S}_1|$, $n_2 := |\mathcal{S}_2|$ and $n_3 := |\mathcal{S}_3|$; of course, $n_1 + n_2 + n_3 = n$. For each $i \in \mathcal{S}_3$, let M_i denote the maximal ideal of A_i and put $K_i := A_i/M_i$. Let $\mathcal{R}$ denote the collection of A-algebra isomorphism classes that can be represented by rings B such that $A \subset B$ is a ramified (minimal ring) extension. Then one set, let us call it $\mathcal{C}$, of (A-algebra isomorphism class) representatives for $\mathcal{R}$ can be found by having each $i = 1, \ldots, n$ contribute at least one member to $\mathcal{C}$, as follows. Each contribution that is due to such an index i is an alteration of the above direct product description of A in which the occurrence of A_i is replaced by a ring that properly contains A_i. Each $i \in \mathcal{S}_1$ contributes one member of $\mathcal{C}$ determined by replacing A_i with $A_i[X]/(X^2)$ (which is A_i-algebra isomorphic to $A_i(+)A_i$ because this kind of A_i is a field). Each $i \in \mathcal{S}_2$ contributes $|\mathrm{Max}(A_i)|$ members of $\mathcal{C}$ determined by the replacement of A_i with the elements of the form $A_i(+)A_i/M$ as M varies over $\mathrm{Max}(A_i)$. If $i \in \mathcal{S}_3$ and $\{u_{ij} \mid j \in I_j\}$ is a chosen subset of $A_i \setminus M_i$ such that $\{u_{ij} + M \mid i \in I_j\}$ is a set of coset representatives for the cosets of K_i^{*2} in K_i^*, then i contributes $1 + [K_i^* : K_i^{*2}]$ members of $\mathcal{C}$ determined by the replacement of A_i with $A_i(+)A_i/M_i$ or with $B_{u_{ij}}$ (adapting notation from Theorem 2.1 for use here with the base ring A_i) as j varies over I_j. Thus, $\mathcal{R}$ is a set and its cardinal number is*

$$|\mathcal{R}| = n_1 + \sum_{i \in \mathcal{S}_2} |\mathrm{Max}(A_i)| + n_3 + \sum_{i \in \mathcal{S}_3} [K_i^* : K_i^{*2}],$$

where the above addition is that of cardinal numbers.

Proof. In view of Lemma 2.4, it suffices to establish the following three facts. If $i \in \mathcal{S}_1$ (resp., $i \in \mathcal{S}_2$; resp., $i \in \mathcal{S}_3$) then the collection of A_i-algebra isomorphism classes represented by ramified extensions of A_i has cardinality 1, with representative $A_i[X]/(X^2)$ (resp., is in one-to-one correspondence with $\mathrm{Max}(A_i)$ where the typical $M \in \mathrm{Max}(A_i)$ corresponds to the A_i-algebra isomorphism class represented by $A_i(+)A_i/M$; resp., has cardinality $1 + [K_i^* : K_i^{*2}]$, with representatives $A_i(+)A_i/M_i$ and the $B_{u_{ij}}$ as j varies over I_j). For the first of these facts, apply the Ferrand-Olivier classification of the minimal ring extensions of a field [10, Lemme 1.2]. For the second of these facts, apply the classification, due to Shapiro and the author [9, Corollary 2.5 and Proposition 2.12] (cf. also [8, Theorem 2.7 and Remark 2.8 (a)] and [17]), of the minimal ring extensions of an integral domain which is not a field. For the third of these facts, apply Theorem 2.1 (h). The proof is complete. □

In view of the length of the statement of Proposition 2.5, we will present only one corollary of it, leaving interested readers to formulate additional corollaries that may be more suited to their needs. Corollary 2.6 presents the application of Proposition 2.5 to finite PIRs. Its specification of A-isomorphism class representatives is relatively concise and has a number-theoretic flavor

that neatly generalizes the results on finite SPIR base rings in Corollary 2.2 (which, in turn, generalized results on base rings of the form $\mathbb{Z}/p^2\mathbb{Z}$ with which our work in this direction had begun in [1, Proposition 12 (e), (f)]).

Corollary 2.6. *Let A be a finite* PIR. *Fix a finite direct product description of A as $\prod_{i=1}^{n} A_i$ where each A_i is either a (finite) field or a (finite)* SPIR *(which is not a field). Let $\mathcal{T}_1$ be the set of indexes i such that A_i is a field. Let $\mathcal{T}_2$ be the set of indexes i such that A_i is an SPIR (but not a field) and the characteristic of A_i is even. Let $\mathcal{T}_3$ be the set of indexes i such that A_i is an SPIR (but not a field) and the characteristic of A_i is odd. Put $\nu_1 := |\mathcal{T}_1|$, $\nu_2 := |\mathcal{T}_2|$ and $\nu_3 := |\mathcal{T}_3|$; of course, $\nu_1 + \nu_2 + \nu_3 = n$. For each $i \in \mathcal{T}_2 \cup \mathcal{T}_3$, let $M_i = A_i\pi_i$ denote the maximal ideal of A_i, put $K_i := A_i/M_i$, and let α_i (≥ 2) denote the index of nilpotence of M_i. Let $\mathcal{R}$ denote the collection of A-algebra isomorphism classes that can be represented by rings B such that $A \subset B$ is a ramified (minimal ring) extension. Then one set, let us call it $\mathcal{C}$, of (A-algebra isomorphism class) representatives for $\mathcal{R}$ can be found by having each $i = 1, \ldots, , n$ contribute at least one member to $\mathcal{C}$, as follows. Each contribution that is due to such an index i is an alteration of the above direct product description of A in which the occurrence of A_i is replaced by a ring that properly contains A_i. Each $i \in \mathcal{T}_1$ contributes one member of $\mathcal{C}$ determined by replacing A_i with $A_i[X]/(X^2)$. Each $i \in \mathcal{T}_2$ contributes two distinct members of $\mathcal{C}$ determined by replacing A_i with $A_i(+)A_i/M_i$ or $A_i[X]/(\pi_i X, X^2 - \pi_i^{\alpha_i - 1})$. Each $i \in \mathcal{T}_3$ contributes three pairwise distinct members of $\mathcal{C}$ determined by replacing A_i with $A_i(+)A_i/M_i$, $A_i[X]/(\pi_i X, X^2 - \pi_i^{\alpha_i - 1})$ or $A_i[X]/(\pi_i X, X^2 - v_i\pi_i^{\alpha_i - 1})$, where v_i is a preassigned element of $U(A_i)$ which is a quadratic nonresidue modulo M_i (in the sense that there does not exist $w \in \mathrm{U}(A_i)$ such that $v_i - w^2 \in M_i$). Then the number of A-algebra isomorphism classes represented by ramified (minimal ring) extensions of A is*

$$|\mathcal{R}| = n + \nu_2 + 2\nu_3 = \nu_1 + 2\nu_2 + 3\nu_3,$$

where the above addition is that of cardinal numbers.

Proof. We apply Proposition 2.5. Using some notation from that result, we have that $\mathcal{T}_1 = \mathcal{S}_1$ (and so $\nu_1 = n_1$); $\mathcal{S}_2$ is empty since every finite integral domain is a field (and so $n_2 = 0$); and $\mathcal{S}_3$ is the disjoint union $\mathcal{T}_2 \cup \mathcal{T}_3$ (and so $n_3 = \nu_2 + \nu_3$). By combining parts (e)-(h) of Theorem 2.1 with Corollary 2.2 (b), we obtain the asserted contributions to $\mathcal{C}$ that are attributable to each index $i \in \mathcal{T}_2$ (resp., $i \in \mathcal{T}_3$) and the fact that the number of such contributions (including $A_i(+)A_i/M_i$) from each such i is 2 (resp., 3). Hence, by Proposition 2.5,

$$|\mathcal{R}| = n_1 + \sum_{i \in \mathcal{S}_2} |\mathrm{Max}(A_i)| + n_3 + \sum_{i \in \mathcal{S}_3} [K_i^* : K_i^{*2}] = \nu_1 + 0 + \nu_2 + \nu_3 + \sum_{i \in \mathcal{T}_2 \cup \mathcal{T}_3} [K_i^* : K_i^{*2}] =$$

$$\nu_1 + \nu_2 + \nu_3 + \sum_{i \in \mathcal{T}_2} 1 + \sum_{i \in \mathcal{T}_3} 2 = n + 1 \cdot |\mathcal{T}_2| + 2 \cdot |\mathcal{T}_3| = n + \nu_2 + 2\nu_3 = \nu_1 + 2\nu_2 + 3\nu_3.$$

The proof is complete. □

Remark 2.7. (a) One may ask how "effectively" one can apply Corollary 2.6. In that regard, note that the proof of Corollary 2.6 made heavy use of Theorem 2.5 and that the proof of Theorem 2.5 made significant use of [18, Theorem 33, pages 245-246]. The proof of the latter result used a primary decomposition of 0 in a given PIR (hence Noetherian ring) R. In our opinion, determining whether a use of Corollary 2.6 is "effective" may often depend on how "effectively" one can carry out the above-mentioned primary decomposition for the data at hand. We will leave further analysis along these lines to specialists in computer algebra and related packages, as that area has shown extensive activity in recent decades in developing primary decompositions for certain appropriate Noetherian rings.

(b) The above work allows us to pursue a probabilistic study of the minimal ring extensions of a finite ring more generally and more sharply than in [3]. Let A be a finite ring. Fix an expression of A as a finite direct product $\prod_{j=1}^{n} A_j$ of (necessarily finite) local rings A_j. (As is well known, this can be done in essentially only one way, apart from reordering the direct factors A_j or replacing them with isomorphic copies.) Further restrict A so that none of the A_j is a field; this restriction is equivalent to requiring that no localization of A (at a prime ideal of A) is a field. By

combining Lemma 2.2 with some of the results from [4] that were mentioned in the Introduction, we see that the collection $\mathcal{E}$ of A-algebra isomorphism classes that can be represented by a ring B such that $A \subset B$ is a minimal ring extension is a finite set. Impose the uniform distribution on $\mathcal{E}$. For each j, let ρ_j (resp., δ_j; resp., θ_j) denote the (finite) number of A_j-algebra isomorphism classes that can be represented by a ramified (resp., decomposed; resp., inert) minimal ring extension of A_j. Then, by Lemma 2.2, the (finite) number of A-algebra isomorphism classes that can be represented by a ramified (resp., decomposed; resp., inert) extension of A is $\rho := \sum_{j=1}^{n} \rho_j$ (resp., $\delta := \sum_{j=1}^{n} \delta_j$; resp., $\theta := \sum_{j=1}^{n} \theta_j$). Then $|\mathcal{E}| = \rho + \delta + \theta$. Furthermore, the probability that a random A-algebra isomorphism class that can be represented by a minimal ring extension of A is actually represented by a ramified (resp., decomposed; resp., inert) extension of A is $\rho/(\rho + \delta + \theta)$; resp., $\delta/(\rho + \delta + \theta)$; resp., $\theta/(\rho + \delta + \theta)$. Implementing the above formulas precisely ultimately requires one to know the values of ρ_j, δ_j and θ_j for each j. While δ_j can be determined by using [5, Theorem 2.6], the above general context admits only estimates for ρ_j and θ_j at this time. For that reason, we next focus on the subcontext that addresses PIRs, where our work will provide exact calculations of ρ, δ and θ.

Let us focus on the case where the above ring A is a PIR. In other words, we are studying a finite PIR, A, none of whose localizations (at a prime ideal of A) is a field. We have identified A with a fixed finite direct product $\prod_{j=1}^{n} A_j$, where each A_j is a finite SPIR (and no A_j is a field). The symbols $\mathcal{E}$, ρ_j, δ_j, θ_j, ρ, δ and θ will retain their respective meanings from the preceding paragraph. Let e denote the number of indexes j such that the characteristic of A_j is even; of course, the characteristic of A_j is odd for each of the other $n - e$ indexes j . Therefore by Corollary 2.2, $\rho_j = 2$ for e values of j and $\rho_j = 3$ for the other $n - e$ values of j. It follows that $\rho = 2e + 3(n - e) = 3n - e$. Moreover, for each j, we have that $\delta_j = 1$ [1, Proposition 10] and $\theta_j = 0$ [1, Proposition 8] . Consequently, $\delta = n$ and $\theta = 0$, whence $|\mathcal{E}| = \rho+\delta+\theta = 4n-e$. We now have the following formulas for the probability that a random A-algebra isomorphism class that can be represented by a minimal ring extension of A is actually represented by a ramified (resp., decomposed; resp., inert) extension of A:

$$\rho = \frac{3n-e}{4n-e}, \quad \delta = \frac{n}{4n-e} \quad \text{and} \quad \theta = 0.$$

The above displayed formula for ρ generalizes and sharpens the "at least" assertions in [3, Corollary 3.6] (where A had been taken to be a finite SPIR), while also sharpening the assertion in [3, Corollary 3.7] that $\rho \geq 2/3$ (for A as above). One may get the impression from [3, items 3.6-3.8] that $2/3 \leq \rho \leq 3/4$ in general (that is, for any finite PIR, A, none of whose localizations at a prime ideal is a field). This impression is actually a theorem. (Proof: Since $0 \leq e \leq n$, it follows easily that $2/3 \leq (3n - e)/(4n - e) \leq 3/4$.) The above formula for ρ can also be used to prove the following realization result (whose proof is left to the reader).

Theorem. Let $r \in \mathbb{Q}$ such that $2/3 \leq r \leq 3/4$. Then there exists a finite PIR, A, none of whose localizations (at a prime ideal of A) is a field, such that r is the probability that a random A-algebra isomorphism class which can be represented by a minimal ring extension of A is actually represented by a ramified extension of A. Indeed, if $r \in \mathbb{Q}$ such that $2/3 \leq r \leq 3/4$ and if A^* is a ring that is (isomorphic to) a finite direct product $\prod_{j=1}^{n^*} A_j^*$ of finite SPIRs A_j^*, where exactly e^* of the indexes j are such that A_j^* has even characteristic (and so each of the other n^*-e^* indexes j is such that A_j^* has odd characteristic) and where the n^* SPIR direct factors may be listed with repetition, then a necessary and sufficient condition for r to be the probability that a random A^*-algebra isomorphism class which can be represented by a minimal ring extension of A^* is actually represented by a ramified extension of A^* is that $(3 - 4r)/(1 - r) = e^*/n^*$. It follows that each r as above can be obtained as the associated probability ρ for denumerably many (pairwise nonisomorphic) finite PIRs.

(c) Much of the above is considered good news by the author. For instance, it was explained in (b) how the results in this paper generalize and sharpen [3, items 3.6-3.8]. In addition, Remark 2.3 (a) documented that a minor error in the proof of [3, Theorem 3.4 (f)] did not deter us from generalizing and sharpening [3, Theorem 3.4] (in Theorem 2.1 and Corollary 2.2). Moreover, the completeness of the information obtained in those results allowed us to prove the titular classification (in Proposition 2.5) and its application to finite rings (in Corollary 2.6). No doubt, there is serendipity in the fact that the above-mentioned minor error did not prevent this paper from solving the more general problems that underlay [3, Section 3]. However, not all errors are

minor and not all errors are followed by only good news. Often, serendipity is not to be expected when one finds an error in earlier work. We next give some facts that illustrate the points made in the preceding two sentences.

If $A = \mathbb{F}_p[X]$ for some prime number p, it follows from Corollary 2.2 that the number of A-algebra isomorphism classes that can be represented by a ramified extension of A is 2 (resp., 3) if $p = 2$ (resp., $p \neq 2$). This conclusion was also made recently in the first paragraph of [5, Remark 3.10 (d)]. However, it contradicts the statement of [1, Proposition 11]. In fact, [1, Proposition 11] is incorrect, and the second paragraph of [5, Remark 3.10 (d)] pinpoints the error in the published "proof" of [1, Proposition 11] (namely, the author's failure to recognize that when $p \neq 2$, certain ramified extensions of A that are isomorphic as rings could fail to be isomorphic as A-algebras). Thus, contrary to the (erroneous) statement of [1, Proposition 11] (but in keeping with the statement of Corollary 2.2), the number of A-algebra isomorphism classes that can be represented by a ramified extension of A depends on the parity of p. In this regard, we must also report that our use of the (erroneous) statement of [1, Proposition 11] caused the statement of [3, Example 3.9] to be incorrect. Accordingly, one should delete [3, Example 3.9] and the sentence that immediately precedes the statement of that (erroneous) result. This completes the remark.

In closing, we use the above results to finish a classification project that was begun in [1].

Corollary 2.8. (*1*) ([*1*, Theorem 5]) *Up to isomorphism, the rings R of characteristic zero that have exactly one proper subring can be classified as the rings satisfying (exactly) one of the following conditions:*

(*i*) $R = \mathbb{Z}[1/p]$, *where p is a prime number (which is uniquely determined by R);*

(*ii*) $R = \mathbb{Z} \times \mathbb{F}_p$, *where p is a prime number (which is uniquely determined by R);*

(*iii*) $R = \mathbb{Z}(+)\mathbb{F}_p$, *where p is a prime number (which is uniquely determined by R).*

(*2*) (*cf.* ([*1*, Theorem 15]) *One set $\mathcal{C}$ of isomorphism class representatives of the rings R of positive characteristic that have exactly one proper subring can be constructed as follows. The prime ring of R is isomorphic to the direct product $\prod_{i=1}^{k} \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}$, where $p_1, p_2, \ldots, p_k$ are pairwise distinct prime numbers for some positive integer k and $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers (and possibly $\alpha_{i_1} = \alpha_{i_2}$ for some $i_1 \neq i_2$). Fix one such direct product that is isomorphic to the prime ring of R. Then a list of the elements of $\mathcal{C}$ (without any repetitions up to isomorphism) can be obtained. For each index i such that $\alpha_i = 1$ (resp., $\alpha_i \geq 2$ and $p_i = 2$; resp., $\alpha_i \geq 2$ and $p_i \neq 2$), there are exactly $\aleph_0$ (resp., 2; resp., 3) members of $\mathcal{C}$ in accordance with the following specifications. The elements of $\mathcal{C}$ are the rings obtained in the form of the direct product*

$$\prod_{j=1}^{i-1} \mathbb{Z}/p_j^{\alpha_j}\mathbb{Z} \times E \times \prod_{j=i+1}^{k} \mathbb{Z}/p_j^{\alpha_j}\mathbb{Z},$$

as i successively takes on the values of $1, \ldots, k$ (and, as usual, empty direct products should be ignored), where the ring E successively satisfies those of the following nine conditions that are pertinent for the index i (noting also that if $\alpha_i = 1$, then denumerably many E are generated in using condition (a) as, in that situation, q can be any prime number):

(a) ($\alpha_i = 1$ *and*) $E = \mathbb{F}_{p_i^q}$, *where q is a prime number;*

(b) ($\alpha_i = 1$ *and*) $E = \mathbb{F}_{p_i} \times \mathbb{F}_{p_i}$;

(c) ($\alpha_i = 1$ *and*) $E = \mathbb{F}_{p_i}[X]/(X^2)$;

(d) $\alpha_i \geq 2$ *and* $E = \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z} \times \mathbb{F}_{p_i}$;

(e) $\alpha_i \geq 2$, $p_i = 2$ *and* $E = \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}(+)\mathbb{F}_{p_i}$;

(f) $\alpha_i \geq 2$, $p_i = 2$ *and* $E = B_1 = (\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})[X]/(p_iX, X^2 - p_i^{\alpha_i - 1})$;

(g) $\alpha_i \geq 2$, $p_i \neq 2$ *and* $E = \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}(+)\mathbb{F}_{p_i}$;

(h) $\alpha_i \geq 2$, $p_i \neq 2$ *and* $E = B_1 = (\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})[X]/(p_iX, X^2 - p_i^{\alpha_i - 1})$;

(i) $\alpha_i \geq 2$, $p_i \neq 2$ *and* $E = B_v = (\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})[X]/(p_iX, X^2 - p_i^{\alpha_i - 1}v)$, *where $v \in U(A)$ is a preassigned element such that $v + M \in \mathbb{F}_{p_i}^* \setminus \mathbb{F}_{p_i}^{*2}$.*

Moreover, in (a)-(i), the parameters $p_1, p_2, \ldots, p_k, \alpha_1, \alpha_2, \ldots, \alpha_k$ are determined by R.

Proof. The statement of (1) is essentially that of [1, Theorem 2.5]. The formulation of the wording at the outset of the statement of (2) differs slightly from the wording that was used in the statement of [1, Theorem 15]. This change in wording has been made here, in keeping with

Lemma 2.4, to make clear that each relevant isomorphism class has exactly one representative generated when one constructs $\mathcal{C}$ by following the above-stated procedure. Apart from this (which we see as a) minor change, conditions (a)-(d) in the statement of (2) are unchanged from the statement of [1, Theorem 2.5]. On the other hand, the above five conditions (e)-(i) in the statement of (2) are the required sharpening of the remaining condition from the statement of [1, Theorem 15], namely, the condition stipulating that $\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z} \subset E$ is a ramified extension. To prove that this sharpening has been achieved, it suffices to combine Lemma 2.4 with Corollary 2.2. (Note that in applying Corollary 2.2 to obtain conditions (f), (h) and (i), we have taken π_i to be the coset represented by p_i.) Also, the final assertion was taken directly from [1, Theorem 15]. This completes the proof. □

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Author information

David E. Dobbs, Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1320, U.S.A..
E-mail: ddobbs1@utk.edu

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