Palestine Journal of Mathematics

Vol. 10(2)(2021) , 389–393

ADDITIVITY OF n-MULTIPLICATIVE (α, β)-DERIVATIONS ON ASSOCIATIVE RINGS

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Communicated by Mohammad Ashraf

MSC 2010 Classifications: Primary 16N60; Secondary 16W99.

Keywords and phrases: Associative prime rings, n-Multiplicative (α, β)-derivations, additivity.

The authors grateful thank the anonymous reviewer for his/her valuable suggestions and comments.

Abstract. In the present paper, we discuss the additivity of n-multiplicative (α, β)-derivations, for the class of associative rings satisfying Martindale's conditions.

1 Introduction

Let $\mathfrak{R}$ be an associative ring, $\alpha, \beta : \mathfrak{R} \to \mathfrak{R}$ automorphisms and n be a positive integer ≥ 2. A mapping $\delta : \mathfrak{R} \to \mathfrak{R}$ is called a *n-multiplicative (α, β)-derivation of* $\mathfrak{R}$ if

$$\delta(a_1 a_2 \cdots a_n) = \sum_{i=1}^{n} \beta(a_1) \cdots \beta(a_{i-1}) \delta(a_i) \alpha(a_{i+1}) \cdots \alpha(a_n),$$

for arbitrary elements $a_1, \cdots, a_n \in \mathfrak{R}$. If $\delta(a_1 a_2) = \delta(a_1)\alpha(a_2) + \beta(a_1)\delta(a_2)$, for arbitrary elements $a_1, a_2 \in \mathfrak{R}$, we just say that δ is a *multiplicative (α, β)-derivation of* $\mathfrak{R}$. A mapping $\delta : \mathfrak{R} \to \mathfrak{R}$ is called a *n-multiplicative derivation of* $\mathfrak{R}$ if

$$\delta(a_1 a_2 \cdots a_n) = \sum_{i=1}^{n} a_1 \cdots \delta(a_i) \cdots a_n,$$

for arbitrary elements $a_1, \cdots, a_n \in \mathfrak{R}$. If $\delta(a_1 a_2) = \delta(a_1)a_2 + a_1\delta(a_2)$, for arbitrary elements $a_1, a_2 \in \mathfrak{R}$, we just say that δ is a *multiplicative derivation of* $\mathfrak{R}$.

The study of the question of when a n-multiplicative derivation is additive has become an active research area in associative rings theory. The first result in this direction is due to Daif [3] who obtained a pioneer result in 1991, which in his condition requires that the ring possess idempotents. It is worth noting that this question is not limited only to the scope of n-multiplicative derivations. Over the last two decades, several papers have been published on the additivity of various mappings on rings. For instance, in the papers [1, 2, 4, 5, 6, 7, 8, 9] we can find important investigations involving studies on the additivity of Jordan (triple) derivations, of Jordan (triple) higher derivable mappings, of Jordan (triple) multiplicative maps and Jordan elementary maps. Within the scope of n-multiplicative derivations Wang [10] considered this question, presenting a unified technique for the discussion of the additivity of n-multiplicative maps on associative rings with idempotents, satisfying Martindale's conditions [9]. To this end, he proved the following main theorem:

Theorem 1.1. *[10, Theorem 1.2.] Let $\mathfrak{R}$ be an associative ring containing a family $\{e_\alpha\}_{\alpha \in \Lambda}$ of non-trivial idempotents which satisfies as follows:*

(i) If $x \in \mathfrak{R}$ is such that $x\mathfrak{R} = 0$, then $x = 0$;

(ii) If $x \in \mathfrak{R}$ is such that $e_\alpha \mathfrak{R} x = 0$ for all $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{R}x = 0$ implies $x = 0$);

(iii) For each $\alpha \in \Lambda$ and $x \in \mathfrak{R}$ if $e_\alpha x e_\alpha \mathfrak{R}(1 - e_\alpha) = 0$, then $e_\alpha x e_\alpha = 0$.

Suppose that $f : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ *is a mapping and* k *a positive integer satisfying:*

(iv) $f(x, 0) = f(0, x) = 0$;

(v) $u_1 \cdots u_k f(x, y) = f(u_1 \cdots u_k x, u_1 u_2 \cdots u_k y)$;

(vi) $f(x, y) u_1 u_2 \cdots u_k = f(x u_1 u_2 \cdots u_k, y u_1 u_2 \cdots u_k)$;

for all elements $x, y, u_1, u_2, \cdots, u_k \in \mathfrak{R}$.
Then $f(x, y) = 0$, *for all elements* $x, y \in \mathfrak{R}$.

As a consequence of this result, he obtained the following corollary:

Corollary 1.2. *[10, Corollary 3.3.] Let* $\mathfrak{R}$ *be an associative ring containing a family* $\{e_\alpha\}_{\alpha \in \Lambda}$ *of non-trivial idempotents which satisfies as follows:*

(i) If $x \in \mathfrak{R}$ *is such that* $x\mathfrak{R} = 0$, *then* $x = 0$;

(ii) If $x \in \mathfrak{R}$ *is such that* $e_\alpha \mathfrak{R} x = 0$ *for all* $\alpha \in \Lambda$, *then* $x = 0$ *(and hence* $\mathfrak{R}x = 0$ *implies* $x = 0$*);*

(iii) For each $\alpha \in \Lambda$ *and* $x \in \mathfrak{R}$ *if* $e_\alpha x e_\alpha \mathfrak{R}(1 - e_\alpha) = 0$, *then* $e_\alpha x e_\alpha = 0$.

Then any n*-multiplicative derivation* δ *of* $\mathfrak{R}$ *is additive.*

In the present paper, we also investigate this same question and consider the same approach as was taken by Wang. To this end, we generalized the Theorem 1.1 and as a consequence of this fact we discuss the additivity of n-multiplicative (α, β)-derivations, for the class of associative rings satisfying Martindale's conditions.

Let $\mathfrak{R}$ be an associative ring and $\delta : \mathfrak{R} \to \mathfrak{R}$ a mapping. According to Wang, let us set $f(x, y) = \delta(x + y) - \delta(x) - \delta(y)$, for all elements $x, y \in \mathfrak{R}$. Then, we see that $f(x, y) = 0$, if and only if, δ is additive. This observation also gives us a unified technique in the discussion of the additivity of n-multiplicative (α, β)-derivations on associative rings.

2 The results

Let us state our main theorem.

Theorem 2.1. *Let* $\mathfrak{R}$ *be an associative ring containing a family* $\{e_\gamma\}_{\gamma \in \Gamma}$ *of non-trivial idempotents and* $\alpha, \beta : \mathfrak{R} \to \mathfrak{R}$ *be automorphisms which satisfy as follows:*

(i) If $x \in \mathfrak{R}$ *is such that* $x\mathfrak{R} = 0$, *then* $x = 0$;

(ii) If $x \in \mathfrak{R}$ *is such that* $\beta(e_\gamma)\mathfrak{R}x = 0$ *for all* $\gamma \in \Gamma$, *then* $x = 0$ *(and hence* $\mathfrak{R}x = 0$ *implies* $x = 0$*);*

(iii) For each $\gamma \in \Gamma$ *and* $x \in \mathfrak{R}$ *if* $\beta(e_\gamma) x \alpha(e_\gamma) \alpha\big(\mathfrak{R}(1 - e_\gamma)\big) = 0$, *then* $\beta(e_\gamma) x \alpha(e_\gamma) = 0$.

Suppose that $f : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ *is a mapping and* k *a positive integer satisfying:*

(iv) $f(x, 0) = f(0, x) = 0$;

(v) $\beta(u_1)\beta(u_2) \cdots \beta(u_k) f(x, y) = f(u_1 u_2 \cdots u_k x, u_1 u_2 \cdots u_k y)$;

(vi) $f(x, y)\alpha(u_1)\alpha(u_2) \cdots \alpha(u_k) = f(x u_1 u_2 \cdots u_k, y u_1 u_2 \cdots u_k)$;

for all elements $x, y, u_1, u_2, \cdots, u_k \in \mathfrak{R}$.
Then $f(x, y) = 0$, *for all elements* $x, y \in \mathfrak{R}$.

Following the techniques presented by Wang [10], we organize the proof of Theorem 2.1 in a series of Lemmas which have the same hypotheses. We begin with the following.

Lemma 2.2. $\beta(u) f(x, y) = f(ux, uy)$ *and* $f(x, y)\alpha(u) = f(xu, yu)$ *for all elements* $x, y, u \in \mathfrak{R}$.

Proof. For arbitrary elements $x, y, u, u_1, u_2, \cdots, u_k \in \mathfrak{R}$ we have

$$\beta(u_1)\cdots\beta(u_k)\beta(u)f(x,y) = \beta(u_1)\cdots\beta(u_k u)f(x,y) = f(u_1u_2\cdots u_k ux, u_1u_2\cdots u_k uy) = \beta(u_1)\beta(u_2)\cdots\beta(u_k)f(ux,uy).$$

It follows that $\beta(u_1)\beta(u_2)\cdots\beta(u_k)\big(\beta(u)f(x,y) - f(ux,uy)\big) = 0$. In view of conditions on β and (ii), of the Theorem 2.1, we conclude that $\beta(u)f(x,y) = f(ux,uy)$. Similarly, we prove that $f(x,y)\alpha(u) = f(xu,yu)$. □

Lemma 2.3. $f(x_{ii}, y_{jk}) = 0 = f(y_{jk}, x_{ii})$, *for* $j \neq k$.

Proof. Two cases are considered. First case. $(i = j)$. For an arbitrary element $a_{is} \in \mathfrak{R}_{is}$, we have

$$f(x_{ii}, y_{jk})\alpha(a_{is}) = f(x_{ii}a_{is}, y_{jk}a_{is}) = f(x_{ii}a_{is}, 0) = 0.$$

Also, for an arbitrary element $a_{ks} \in \mathfrak{R}_{ks}$, we have

$$f(x_{ii}, y_{jk})\alpha(a_{ks}) = f(x_{ii}a_{ks}, y_{jk}a_{ks}) = f(0, y_{jk}a_{ks}) = 0.$$

Since α is an epimorphism, then we conclude that $f(x_{ii}, y_{jk})\mathfrak{R} = 0$ which implies that $f(x_{ii}, y_{jk}) = 0$, by condition (i) of the Theorem 2.1. Second case. $(i \neq j)$. For an arbitrary element $a_{si} \in \mathfrak{R}_{si}$, we have

$$\beta(a_{si})f(x_{ii}, y_{jk}) = f(a_{si}x_{ii}, a_{si}y_{jk}) = f(a_{si}x_{ii}, 0) = 0.$$

Also, for an arbitrary element $a_{kj} \in \mathfrak{R}_{kj}$, we have

$$\beta(a_{kj})f(x_{ii}, y_{jk}) = f(a_{kj}x_{ii}, a_{kj}y_{jk}) = f(0, a_{kj}y_{jk}) = 0.$$

Since β is an epimorphism, then we conclude that $\mathfrak{R}f(x_{ii}, y_{jk}) = 0$ which implies that $f(x_{ii}, y_{jk}) = 0$, by condition (ii) of the Theorem 2.1.

Similarly we prove that $f(y_{jk}, x_{ii}) = 0$, for $j \neq k$. □

Lemma 2.4. $f(x_{12}, y_{12}) = 0$.

Proof. For an arbitrary element $a_{1s} \in \mathfrak{R}_{1s}$, we have

$$f(x_{12}, y_{12})\alpha(a_{1s}) = f(x_{12}a_{1s}, y_{12}a_{1s}) = f(0,0) = 0.$$

Also, for an arbitrary element $a_{2s} \in \mathfrak{R}_{1s}$, we have

$$\begin{aligned} f(x_{12}, y_{12})\alpha(a_{2s}) &= f(x_{12}a_{2s}, y_{12}a_{2s}) \\ &= f(x_{12}(a_{2s} + y_{12}a_{2s}), e_1(a_{2s} + y_{12}a_{2s})) \\ &= f(x_{12}, e_1)\alpha(a_{2s} + y_{12}a_{2s}) = 0, \end{aligned}$$

by Lemma 2.3. It follows that $f(x_{12}, y_{12})\mathfrak{R} = 0$ which implies that $f(x_{12}, y_{12}) = 0$. □

Lemma 2.5. $f(x_{11}, y_{11}) = 0$.

Proof. For an arbitrary element $a_{12} \in \mathfrak{R}_{12}$, we have

$$f(x_{11}, y_{11})\alpha(a_{12}) = f(x_{11}a_{12}, y_{11}a_{12}) = 0.$$

Also, for an arbitrary element $a = a_{11} + a_{12} + a_{21} + a_{22} \in \mathfrak{R}$, we have

$$\beta(e_1)f(x_{11}, y_{11})\alpha(e_1)\alpha\big(a(1 - e_1)\big) = \beta(e_1)f(x_{11}, y_{11})\alpha(e_1)\alpha(a_{12} + a_{22}) = f(x_{11}, y_{11})\alpha(a_{12}) = 0.$$

By condition (iii) of the Theorem 2.1, we obtain $f(x_{11}, y_{11}) = \beta(e_1)f(x_{11}, y_{11})\alpha(e_1) = 0$. □

Lemma 2.6. $f(x_{11}+x_{12}, y_{11}+y_{12}) = 0$.

Proof. For an arbitrary element $a_{ij} \in \mathfrak{R}_{ij}$, we have

$$f(x_{11}+x_{12}, y_{11}+y_{12})\alpha(a_{ij}) = f(x_{1i}a_{ij}, y_{1i}a_{ij}) = 0.$$

It follows that $f(x_{11}+x_{12}, y_{11}+y_{12})\mathfrak{R} = 0$ which implies that $f(x_{11}+x_{12}, y_{11}+y_{12}) = 0$. □

Proof of Theorem 2.1. Let r, x, y be arbitrary elements of $\mathfrak{R}$. By Lemmas 2.2 and 2.6 we have

$$\beta(e_1)\beta(r)f(x,y) = f(e_1rx, e_1ry) = 0.$$

It follow that $\beta(e_1)\mathfrak{R}f(x,y) = 0$. This allows us to conclude that $\beta(e_\gamma)\mathfrak{R}f(x,y) = 0$ for all $\gamma \in \Gamma$, since e_1 was chosen arbitrary. This results that $f(x,y) = 0$, by condition (ii) of the Theorem 2.1. The theorem is proved. □

3 Applications

Theorem 3.1. *Let $\mathfrak{R}$ be an associative ring containing a family $\{e_\gamma\}_{\gamma\in\Gamma}$ of non-trivial idempotents and $\alpha, \beta : \mathfrak{R} \to \mathfrak{R}$ be automorphisms which satisfy as follows:*

(i) If $x \in \mathfrak{R}$ is such that $x\mathfrak{R} = 0$, then $x = 0$;

(ii) If $x \in \mathfrak{R}$ is such that $\beta(e_\gamma)\mathfrak{R}x = 0$ for all $\gamma \in \Gamma$, then $x = 0$ (and hence $\mathfrak{R}x = 0$ implies $x = 0$);

(iii) For each $\gamma \in \Gamma$ and $x \in \mathfrak{R}$ if $\beta(e_\gamma)x\alpha(e_\gamma)\alpha\big(\mathfrak{R}(1-e_\gamma)\big) = 0$, then $\beta(e_\gamma)x\alpha(e_\gamma) = 0$.

Then every n-multiplicative (β,α)-derivation of $\mathfrak{R}$ is additive.

Proof. First, we observe that

$$\delta(0) = \delta(\underbrace{00\cdots 0}_{n \text{ terms}}) = \sum_{i=1}^{n} \underbrace{\beta(0)\cdots\beta(0)\delta(0)}_{i \text{ terms}}\alpha(0)\cdots\alpha(0) = \sum_{i=1}^{n} \underbrace{0\cdots 0\delta(0)}_{i \text{ terms}}0\cdots 0 = 0.$$

Set $f(x,y) = \delta(x+y) - \delta(x) - \delta(y)$ for all elements $x, y \in \mathfrak{R}$. Then $f(x,0) = 0 = f(0,x)$, for all element $x \in \mathfrak{R}$. This results that for arbitrary elements $x, y, a_1, \cdots, a_{n-1} \in \mathfrak{R}$,

$$\begin{aligned}
& \beta(a_1)\cdots\beta(a_{n-1})f(x,y) \\
= {} & \beta(a_1)\cdots\beta(a_{n-1})\big(\delta(x+y) - \delta(x) - \delta(y)\big) \\
= {} & \beta(a_1)\cdots\beta(a_{n-1})\delta(x+y) - \beta(a_1)\cdots\beta(a_{n-1})\delta(x) \\
& \qquad - \beta(a_1)\cdots\beta(a_{n-1})\delta(y) \\
= {} & \delta(a_1\cdots a_{n-1}(x+y)) \\
& \qquad - \sum_{i=1}^{n-1} \underbrace{\beta(a_1)\cdots\beta(a_{i-1})\delta(a_i)}_{i \text{ terms}}\alpha(a_{i+1})\cdots\alpha(a_{n-1})\alpha(x+y) \\
& - \delta(a_1\cdots a_{n-1}x) + \sum_{i=1}^{n-1} \underbrace{\beta(a_1)\cdots\beta(a_{i-1})\delta(a_i)}_{i \text{ terms}}\alpha(a_{i+1})\cdots\alpha(a_{n-1})\alpha(x) \\
& - \delta(a_1\cdots a_{n-1}y) + \sum_{i=1}^{n-1} \underbrace{\beta(a_1)\cdots\beta(a_{i-1})\delta(a_i)}_{i \text{ terms}}\alpha(a_{i+1})\cdots\alpha(a_{n-1})\alpha(y) \\
= {} & \delta(a_1\cdots a_{n-1}(x+y)) - \delta(a_1\cdots a_{n-1}x) - \delta(a_1\cdots a_{n-1}y) \\
= {} & f(a_1\cdots a_{n-1}x, a_1\cdots a_{n-1}y).
\end{aligned}$$

Similarly, we prove that

$$f(x,y)\alpha(a_1)\alpha(a_2)\cdots\alpha(a_{n-1}) = f(xa_1a_2\cdots a_{n-1}, ya_1a_2\cdots a_{n-1}).$$

By Theorem 2.1, we obtain $f(x,y) = 0$, for all elements $x, y \in \mathfrak{R}$. □

Corollary 3.2. *Let $\mathfrak{R}$ be a 2-torsion free associative prime ring containing a non-trivial idempotent e_1 and $\alpha, \beta : \mathfrak{R} \to \mathfrak{R}$ be automorphisms. Then every n-multiplicative (β, α)-derivation of $\mathfrak{R}$ is additive.*

Proof. It is evident that the conditions (i) and (ii), of the Theorem 2.1, are satisfied. Now, for an arbitrary element $x \in \mathfrak{R}$, suppose that $\beta(e_1)x\alpha(e_1)\alpha(\mathfrak{R}(1-e_1)) = 0$. Then, for an arbitrary element $r = r_{11} + r_{12} + r_{21} + r_{22} \in \mathfrak{R}$ we have

$$0 = \beta(e_1)x\alpha(e_1)\alpha(r(1-e_1)) = \beta(e_1)x\alpha(e_1)\alpha(r_{12} + r_{22}) = \beta(e_1)x\alpha(e_1)\alpha(r_{12}) \\ = \alpha((\alpha^{-1}\beta)(e_1)\alpha^{-1}(x)e_1r_{12})$$

This results that $(\alpha^{-1}\beta)(e_1)\alpha^{-1}(x)e_1r_{12} = 0$, for an arbitrary element $r_{12} \in \mathfrak{R}_{12}$. By [5, Lemma 2] we obtain $(\alpha^{-1}\beta)(e_1)\alpha^{-1}(x)e_1 = 0$ which implies that

$$\beta(e_1)x\alpha(e_1) = \alpha((\alpha^{-1}\beta)(e_1)\alpha^{-1}(x)e_1) = 0.$$

This allows us to conclude that every n-multiplicative (β, α)-derivation of $\mathfrak{R}$ is additive. □

The ideas that follow below are similar those presented by Wang [10].

Let $\mathfrak{X}$ be a Banach space. Denote by $\mathcal{B}(\mathfrak{X})$ the algebra of all bounded linear operators on $\mathfrak{X}$. A subalgebra of $\mathcal{B}(\mathfrak{X})$ is called a *standard operator algebra* if it contains all finite rank operators. It is well known that every standard operator algebra is prime. Moreover, if $\dim \mathfrak{X} \geq 2$, then there exists a non-trivial idempotent operator of rank one in $\mathcal{B}(\mathfrak{X})$. Therefore, it follows from Corollary 3.2 that

Corollary 3.3. *Let $\mathfrak{X}$ be a Banach space with* $\dim \mathfrak{X} \geq 2$, *$\mathfrak{A}$ be a standard operator algebra on $\mathfrak{X}$ and $\alpha, \beta : \mathfrak{R} \to \mathfrak{R}$ be automorphisms. Then any n-multiplicative (α, β)-derivation of $\mathfrak{A}$ is additive.*

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Received: November 3, 2019.
Accepted: July 9, 2020.