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RESULTS ON UNICITY OF MEROMORPHIC FUNCTIONS SHARING A SMALL FUNCTION REGARDING MULTIPLICITY

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Abstract In this note, we study the existence of uniqueness of two meromorphic or entire functions which is concerning about differential polynomials sharing a small function with regard to multiplicity. Our results generalize and improve the results obtained in [2] and also solves the open problems posed by M. B. Ahamed [2].

1 Introduction

In this paper, by meromorphic functions we always mean meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with standard notations of Nevanlinna theory as explained well in [10, 11], for a meromorphic function $f(z)$, we denote the proximity function as $m(r, f)$, the counting function by $N(r, f)$, the reduced counting function by $\overline{N}(r, f)$ and the characteristic function by $T(r, f)$. Two meromorphic functions f and g share the value a IM(ignoring multiplicities) if f and g have the same a-points counted by ignoring the multiplicities, we say that f and g share a CM(counting multiplicities), if $f - a$ and $g - a$ have the same zeros with same multiplicities. Also, we note that when $a = \infty$, the zeros of $f - a$ are the poles of f.

Also, a meromorphic function $a \equiv a(z)(\not\equiv 0, \infty)$ is said to be a small function of f provided $T(r, a) = S(r, f)$ i.e., $T(r, a) = O(T(r, f))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure.

In 2013, S. S. Bhoosnurmath and V. Pujari [7], obtained the following uniqueness results.

Theorem 1.1. *([7]) Let f and g be two non-constant meromorphic functions, $n \geq 11$ be an integer. If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share z CM, f and g share ∞ IM, then either $f \equiv g$ or*

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, f = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

where h is a non-constant meromorphic function.

Theorem 1.2. *([7]) Let f and g be two non-constant meromorphic functions, $n \geq 12$ be an integer. If $f^n(f-1)^2f^{'}$ and $g^n(g-1)^2g^{'}$ share z CM, f and g share ∞ IM, then $f \equiv g$.*

Theorem 1.3. *([7]) Let f and g be two non-constant entire functions, $n \geq 7$ be an integer. If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share z CM, then $f \equiv g$.*

In 2016, the authors Harina P. Waghamore and S. Anand [14] generalize theorems 1.1, 1.2 and 1.3 by considering the functions $f^n(f-1)^mf^{'}$ and $g^n(g-1)^mg^{'}$. They also proved that the second condition in Theorem A can be omitted. The results obtained are as follows.

Theorem 1.4. *([14]) Let f and g be two non-constant meromorphic functions, $n \geq m + 10$ be an integer. If $f^n(f-1)^mf^{'}$ and $g^n(g-1)^mg^{'}$ share z CM, f and g share ∞ IM, then $f \equiv g$.*

Theorem 1.5. *([14]) Let f and g be two non-constant entire functions, $n \geq m+6$ be an integer. If $f^n(f-1)^m f^{'}$ and $g^n(g-1)^m g^{'}$ share z CM, then $f \equiv g$.*

The author M. B. Ahamed [2] by introducing a general polynomial improved as well as extended the above mentioned results when they share $\alpha(z)$ CM. The following are the results obtained. But, first let me give the definition of the general polynomial of degree $n+m$ used in [6] which can also be expressed as a transformation.

Definition 1.6. ([6]) Let

$$\mathcal{P}(w) = w^{n+m} + ... + a_n w^n + ... + a_0 = a_{n+m} \prod_{i=1}^{s} (w - w_{p_i})^{p_i},$$

where $a_j (j = 0, 1, 2, ..., n+m-1)$ and $w_{p_i} (i = 1, 2, ..., s)$ are distinct finite complex numbers and $2 \leq s \leq n+m$ and $p_1, p_2, ..., p_s, s \geq 2$, n, m and k are all positive integers with $\sum_{i=1}^{s} p_i = n+m$. Also, let $p > \max_{p \neq p_i, i=1,2,...,r}\{p_i\}$, $r = s-1$, where r and s are two positive integers.

Let $\mathcal{L}(w_*) = \prod_{i=1}^{s-1} (w_* + w_p - w_{p_i})^{p_i} = b_q w_*^q + b_{q-1} w_*^{q-1} + ... + b_0$, where $w_* = w - w_p$, $q = n+m-p$. So it is clear that $\mathcal{P}(w) = w_*^p \mathcal{L}(w_*)$. In particular, if we choose $b_i = (-1)^{iq} C_i$, for $i = 0, 1, ..., q$. Then we get, easily $\mathcal{P}_*(w) = w_*^p (w_* - 1)^q$. Note that if $w_p = 0$ and $p = n$, then we get $w = w_*$ and $\mathcal{P}_*(w) = w^n (w-1)^m$.

Theorem 1.7. *([2]) Let f and g hence $f_* = f - w_p$ and $g_* = g - w_p$, $w_p \in \mathbb{C}$ be any two non-constant non-entire meromorphic functions, $n \geq q+9, q \in \mathbb{N}$, be an integer. If $\mathcal{P}_*(f) f_*^{'} = f_*^p (f_* - 1)^q f_*^{'}$ and $\mathcal{P}_*(g) g_*^{'} = g_*^p (g_* - 1)^q g_*^{'}$ share $\alpha \equiv \alpha(z) (\not\equiv 0, \infty)$ CM, f_* and g_* share ∞ IM, then $f \equiv g$.*

Theorem 1.8. *([2]) Let f and g hence $f_* = f - w_p$ and $g_* = g - w_p$, $w_p \in \mathbb{C}$ be any two non-constant entire functions, $n \geq q+5, q \in \mathbb{N}$, be an integer. If $\mathcal{P}_*(f) f_*^{'} = f_*^p (f_* - 1)^q f_*^{'}$ and $\mathcal{P}_*(g) g_*^{'} = g_*^p (g_* - 1)^q g_*^{'}$ share $\alpha \equiv \alpha(z) (\not\equiv 0, \infty)$ CM, then $f \equiv g$.*

In the same paper the author M. B. Ahamed [2] posed the following open questions.

Question 1.1. Is it possible to reduce further the lower bounds of p in Theorem 1.7 and Theorem 1.8?

Question 1.2. To get the uniqueness between f and g is it possible to replace $f_*^p (f_* - 1)^q f_*^{'}$ and $g_*^p (g_* - 1)^q g_*^{'}$ respectively by $f_*^p P_m(f_*) f_*^{'}$ and $g_*^p P_m(g_*) g_*^{'}$, where $P_m(f_*) = a_m f_*^m + a_{m-1} f_*^{m-1} + ... + a_1 f_* + a_0$ in Theorem 1.7 and Theorem 1.8?

Our aim in writing this paper is to give a positive answer to the above questions. By considering functions $f_*^p P_m(f_*) f_*^{'}$ and $g_*^p P_m(g_*) g_*^{'}$, where f_* and g_* are any two meromorphic functions with multiplicity atleast l. We obtain two results which improves and generalizes Theorems 1.7 and 1.8.

The main results of this article are as follows:

Theorem 1.9. *Let f, g and hence $f_* = f - w_p$ and $g_* = g - w_p$, $w_p \in \mathbb{C}$ be any two non-constant meromorphic functions with multiplicity atleast l, $p \geq \frac{2m-(m+1)l+10}{l}, m \in \mathbb{N}$, be an integer. If $f_*^p P_m(f_*) f_*^{'}$ and $g_*^p P_m(g_*) g_*^{'}$ share $\alpha \equiv \alpha(z) (\not\equiv 0, \infty)$ CM, f_* and g_* share ∞ IM, then $f \equiv g$.*

Theorem 1.10. *Let f, g and hence $f_* = f - w_p$ and $g_* = g - w_p$, $w_p \in \mathbb{C}$ be any two non-constant entire functions with multiplicity atleast l, $p \geq \frac{2m-(m+1)l+6}{l}, m \in \mathbb{N}$, be an integer. If $f_*^p P_m(f_*) f_*^{'}$ and $g_*^p P_m(g_*) g_*^{'}$ share $\alpha \equiv \alpha(z) (\not\equiv 0, \infty)$ CM, then $f \equiv g$.*

Remark 1.11. (i) If suppose we let $l = 1$, $P_m(f_*) = (f_* - 1)^q$, here $m = q$. Then our conditions in Theorem 1.9 and Theorem 1.10 will reduce to Theorem 1.7 and Theorem 1.8 respectively. That is, $p \geq q+9$ and $p \geq q+5$.

(ii) Let $l = 2$, then the condition in Theorem 1.9 will be $p \geq 4$ and the condition in Theorem 1.10 will be $p \geq 2$.

Therefore, we make a note that by introducing the concept of multiplicity, we reduce the lower bound of p. Also, as the multiplicity increases the condition value decreases.

2 Lemmas

The following lemmas are used in the sequel.

Lemma 2.1. *([15]) Let f_1, f_2 and f_3 be non-constant meromorphic functions such that $f_1 + f_2 + f_3 = 1$. If f_1, f_2 and f_3 are linearly independent, then*

$$T(r, f_1) < \sum_{i=1}^{3} N_2\left(r, \frac{1}{f_i}\right) + \sum_{i=1}^{3} \overline{N}(r, f) + o(T(r)),$$

where $T(r) = \max_{1\leq i\leq 3}\{T(r, f_i)\}$ and $r \notin E$.

Lemma 2.2. *([18]) Let f_1 and f_2 be non-constant meromorphic functions. If $c_1 f_1 + c_2 f_2 = c_3$, where $c_i, i = 1, 2, 3$ are non-zero constants, then*

$$T(r, f_1) \leq \overline{N}(r, f_1) + \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + S(r, f_1).$$

Lemma 2.3. *([18]) Let f be a non-constant meromorphic function and k be a non-negative integer, then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.4. *([20]) Suppose that f is a non-constant meromorphic function and $P(f) = a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0$, where $a_n(\not\equiv 0), a_{n-1}, ..., a_1, a_0$ are small meromorphic functions of $f(z)$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.5. *([16]) Let f_1, f_2 and f_3 be three meromorphic functions satisfying $\sum_{i=1}^{3} f_i = 1$, then the functions $g_1 = -\frac{f_3}{f_2}$, $g_2 = \frac{1}{f_2}$ and $g_3 = -\frac{f_1}{f_2}$ are linearly independent when f_1, f_2 and f_3 are linearly independent.*

Lemma 2.6. *([2]) Let $\Psi(z) = c^2(z^{p-q} - 1)^2 - 4b(z^{p-2q} - 1)(z^p - 1)$, where $b, c \in \mathbb{C} - \{0\}$, $\frac{c^2}{4b} = \frac{p(p-2q)}{(p-q)^2} \neq 1$, then $\Psi(z)$ has exactly one multiple zero of multiplicity 4 which is 1.*

Lemma 2.7. *Let f, g and hence $f_* = f - w_p$ and $g_* = g - w_p$, $w_p \in \mathbb{C}$ be any two non-constant meromorphic functions with multiplicity atleast l and $\alpha \equiv \alpha(z)(\not\equiv 0, \infty)$ be a small function of f and g. If $f_*^p P_m(f_*) f_*^{'}$ and $g_*^p P_m(g_*) g_*^{'}$ share α CM and $p \geq \frac{m-(m-2)l+5}{l}$, then*

$$T(r, g_*) \leq \left[\frac{l(p+m+2)}{(p+m-2)l - (m+4)}\right] T(r, f_*) + S(r, g_*)$$

Proof. First, by applying the second fundamental theorem on $g_*^p P_m(g_*) g_*^{'}$, we get

$$\begin{aligned} T(r, g_*^p P_m(g_*) g_*^{'}) &\leq \overline{N}(r, g_*^p P_m(g_*) g_*^{'}) + \overline{N}\left(r, \frac{1}{g_*^p P_m(g_*) g_*^{'}}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{g_*^p P_m(g_*) g_*^{'} - \alpha}\right) + S(r, g_*) \\ &\leq \overline{N}(r, g_*) + \overline{N}\left(r, \frac{1}{g_*^p P_m(g_*) g_*^{'}}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{g_*^p P_m(g_*) g_*^{'} - \alpha}\right) + S(r, g_*) \end{aligned} \tag{2.1}$$

Now, by applying first fundamental theorem, we get

$$\begin{aligned} (p+m)T(r, g_*) &\leq T(r, g_*^p P_m(g_*)) + S(r, g_*) \\ &\leq T(r, g_*^p P_m(g_*) g_*^{'}) + T\left(r, \frac{1}{g_*^{'}}\right) + S(r, g_*) \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we get

$$\begin{aligned}(p+m)T(r,g_*) &\leq \overline{N}(r,g_*)+\overline{N}\left(r,\frac{1}{g_*^p}\right)+\overline{N}(r,0;P_m(g_*))+\overline{N}\left(r,\frac{1}{g_*'}\right)\\ &+\overline{N}\left(r,\frac{1}{f_*^pP_m(f_*)f_*'-\alpha}\right)+T(r,g_*')+S(r,g_*)\end{aligned} \tag{2.3}$$

Since $S(r,g_*)=T(r,\alpha)=S(r,f_*)$, we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f_*^pP_m(f_*)f_*'-\alpha}\right) &\leq T\left(r,\frac{1}{f_*^pP_m(f_*)f_*'-\alpha}\right)+O(1)\\ &\leq T\left(r,\frac{1}{f_*^p}\right)+T\left(r,\frac{1}{P_m(f_*)}\right)+T\left(r,\frac{1}{f_*'}\right)+T(r,\alpha)+O(1)\\ &\leq (p+m+2)T(r,f_*)+S(r,g_*)\end{aligned} \tag{2.4}$$

Now taking (2.4) in (2.3), we get

$$\begin{aligned}(p+m)T(r,g_*) &\leq \overline{N}\left(r,\frac{1}{g_*}\right)+\overline{N}\left(r,\frac{1}{P_m(g_*)}\right)+\overline{N}(r,g_*)+\overline{N}\left(r,\frac{1}{g_*'}\right)\\ &+(p+m+2)T(r,f_*)+2T(r,g_*)+S(r,g_*)\end{aligned}$$

Now since zeros and poles of f_* and g_* are of multiplicities atleast l, we have

$$\overline{N}(r,f_*)\leq\frac{1}{l}N(r,f_*)\leq\frac{1}{l}T(r,f_*);\overline{N}\left(r,\frac{1}{f_*}\right)\leq\frac{1}{l}N\left(r,\frac{1}{f_*}\right)\leq\frac{1}{l}T(r,f_*). \tag{2.5}$$

Similarly, we have

$$\overline{N}(r,g_*)\leq\frac{1}{l}N(r,g_*)\leq\frac{1}{l}T(r,g_*);\overline{N}\left(r,\frac{1}{g_*}\right)\leq\frac{1}{l}N\left(r,\frac{1}{g_*}\right)\leq\frac{1}{l}T(r,g_*). \tag{2.6}$$

So, we get

$$\begin{aligned}(p+m)T(r,g_*) &\leq \frac{1}{l}T\left(r,\frac{1}{g_*}\right)+\frac{m}{l}T\left(r,\frac{1}{g_*}\right)+\frac{1}{l}T(r,g_*)+\frac{2}{l}T(r,g_*)\\ &+(p+m+2)T(r,f_*)+2T(r,g_*)+S(r,g_*)\\ &\leq\left(\frac{m+4+2l}{l}\right)T(r,g_*)+(p+m+2)T(r,f_*)+S(r,g_*)\end{aligned}$$

So,

$$\left[\frac{(p+m-2)l-(m+4)}{l}\right]T(r,g_*)\leq(p+m+2)T(r,f_*)+S(r,g_*)$$

Thus, we get

$$T(r,g_*)\leq\left[\frac{l(p+m+2)}{(p+m-2)l-(m+4)}\right]T(r,f_*)+S(r,g_*),$$

where $p\geq\frac{m-(m-2)l+5}{l}$.
Hence the proof. □

Lemma 2.8. *Let f, g and hence $f_*=f-w_p$ and $g_*=g-w_p$, $w_p\in\mathbb{C}$ be two non-constant entire functions with multiplicity atleast l. Let $\alpha\equiv\alpha(z)(\not\equiv 0,\infty)$ be a small function of f and g. If $f_*^pP_m(f_*)f_*'$ and $g_*^pP_m(g_*)g_*'$ share α CM and $p\geq\frac{m-(m-2)l+2}{l}$, then*

$$T(r,g_*)\leq\left[\frac{l(p+m+2)}{(p+m-2)l-(m+1)}\right]T(r,f_*)+S(r,g_*)$$

Proof. Since both the functions f,g and hence f_* and g_* are entire functions, so we have $\overline{N}(r,f)=0=\overline{N}(r,g)$; $\overline{N}(r,f_*)=0=\overline{N}(r,g_*)$. Now continuing the proof on lines of proof of Lemma 2.7, we prove Lemma 2.8. □

3 Proof of theorems

Proof. (Proof of Theorem 1.9) By hypothesis, $f_*^p P_m(f_*)f_*^{'}$ and $g_*^p P_m(g_*)g_*^{'}$ share $\alpha \equiv \alpha(z)$ CM, also f_* and g_* share ∞ IM, so let us suppose that

$$\mathcal{H} \equiv \frac{f_*^p P_m(f_*)f_*^{'} - \alpha}{g_*^p P_m(g_*)g_*^{'} - \alpha}. \tag{3.1}$$

From (3.1), we have

$$\begin{aligned} T(r,\mathcal{H}) &= T\left(r, \frac{f_*^p P_m(f_*)f_*^{'} - \alpha}{g_*^p P_m(g_*)g_*^{'} - \alpha}\right) \\ &\leq T(r, f_*^p P_m(f_*)f_*^{'} - \alpha) + T(r, g_*^p P_m(g_*)g_*^{'} - \alpha) + O(1) \\ &\leq (p+m+2)[T(r,f_*) + T(r,g_*)] + S(r,f_*) + S(r,g_*) \\ &\leq 2(p+m+2)T_*(r) + S_*(r), \end{aligned}$$

where $T_*(r) = \max\{T(r,f_*), T(r,g_*)\}$ and $S_*(r) = \max\{S(r,f_*), S(r,g_*)\}$.
i.e.,

$$T(r,\mathcal{H}) = O(T_*(r)). \tag{3.2}$$

By (3.1), again we see that the zeros and poles of $\mathcal{H}$ are multiple, hence

$$\overline{N}(r,\mathcal{H}) \leq \overline{N}_L(r,f_*), \overline{N}(r,\frac{1}{\mathcal{H}}) \leq \overline{N}_L(r,g_*). \tag{3.3}$$

Let $f_1 = \frac{f_*^p P_m(f_*)f_*^{'}}{\alpha}$, $f_2 = \mathcal{H}$ and $f_3 = -\mathcal{H}\frac{g_*^p P_m(g_*)g_*^{'}}{\alpha}$. Thus, we get $f_1 + f_2 + f_3 = 1$.

Let us now denote $T(r) = \max\{T(r,f_1), T(r,f_2), T(r,f_3)\}$. Then, we have

$$\begin{aligned} &T(r,f_1) = O(T(r,f_*)), \\ &T(r,f_2) = O(T(r,f_*) + T(r,g_*)) = T(r,f_3). \end{aligned}$$

So, $T(r,f_i) = O(T_*(r))$ for $i = 1,2,3$ and also $S(r,f_*) + S(r,g_*) = o(T_*(r))$.

We now study the following cases.

Case 1. Suppose that none of f_2 and f_3 are constant. If f_1 and f_2, f_3 are linearly independent, then by using Lemma 2.1 and Lemma 2.4, we get

$$\begin{aligned} T(r,f_1) &\leq \sum_{i=1}^{3} N_2\left(r, \frac{1}{f_i}\right) + \sum_{i=1}^{3} \overline{N}(r,f_i) + o(T(r)) \\ &\leq N_2\left(r, \frac{\alpha}{f_*^p P_m(f_*)f_*^{'}}\right) + N_2\left(r, \frac{1}{\mathcal{H}}\right) + N_2\left(r, \frac{\alpha}{\mathcal{H} g_*^p P_m(g_*)g_*^{'}}\right) \\ &+ \overline{N}(r, f_*^p P_m(f_*)f_*^{'}) + \overline{N}(r,\mathcal{H}) + \overline{N}(r, \mathcal{H} g_*^p P_m(g_*)g_*^{'}) + o(T(r)) \\ &\leq N_2\left(r, \frac{1}{f_*^p P_m(f_*)f_*^{'}}\right) + 2N_2\left(r, \frac{1}{\mathcal{H}}\right) + N_2\left(r, \frac{1}{g_*^p P_m(g_*)g_*^{'}}\right) \\ &+ \overline{N}(r,f_*) + 2\overline{N}(r,\mathcal{H}) + \overline{N}(r,g_*) + o(T(r)). \end{aligned} \tag{3.4}$$

Now, since $N_2\left(r, \frac{1}{\mathcal{H}}\right) \leq 2\overline{N}\left(r, \frac{1}{\mathcal{H}}\right) \leq 2\overline{N}_L(r,g_*)$ and $\overline{N}(r,\mathcal{H}) \leq \overline{N}_L(r,f_*)$.

Also, since $\overline{N}_L(r,f_*) = 0 = \overline{N}_L(r,g_*)$ and we note that $\overline{N}(r,f_*) = \overline{N}(r,g_*)$, so by using all these facts, we get from (3.4) that

$$\begin{aligned} T(r,f_1) &\le N_2\left(r,\frac{1}{f_*^pP_m(f_*)f_*'}\right)+N_2\left(r,\frac{1}{g_*^pP_m(g_*)g_*'}\right)+2\overline{N}(r,f_*)+o(T(r))\\ &\le N\left(r,\frac{1}{f_*^pP_m(f_*)f_*'}\right)-\left[N_{(3}\left(r,\frac{1}{f_*^pP_m(f_*)f_*'}\right)-2\overline{N}_{(3}\left(r,\frac{1}{f_*^pP_m(f_*)f_*'}\right)\right]\\ &+N\left(r,\frac{1}{g_*^pP_m(g_*)g_*'}\right)-\left[N_{(3}\left(r,\frac{1}{g_*^pP_m(g_*)g_*'}\right)-2\overline{N}_{(3}\left(r,\frac{1}{g_*^pP_m(g_*)g_*'}\right)\right]\\ &+2\overline{N}(r,f_*)+o(T(r)). \end{aligned} \tag{3.5}$$

Let z_0 be a zero of f_* of multiplicity r, then z_0 is also a zero of $f_*^pP_m(f_*)f_*'$ of multiplicity $pr+r-1\ge 3$. Then, we have

$$N_{(3}\left(r,\frac{1}{f_*^pP_m(f_*)f_*'}\right)-2\overline{N}_{(3}\left(r,\frac{1}{f_*^pP_m(f_*)f_*'}\right)\ge(p-2)N\left(r,\frac{1}{f_*}\right). \tag{3.6}$$

Similarly, we get

$$N_{(3}\left(r,\frac{1}{g_*^pP_m(g_*)g_*'}\right)-2\overline{N}_{(3}\left(r,\frac{1}{g_*^pP_m(g_*)g_*'}\right)\ge(p-2)N\left(r,\frac{1}{g_*}\right). \tag{3.7}$$

Let us consider

$$\mathcal{F}=\frac{a_m}{p+m+1}f_*^{p+m+1}+\frac{a_{m-1}}{p+m}f_*^{p+m}+...+\frac{a_1}{p+2}f_*^{p+2}+\frac{a_0}{p+1}f_*^{p+1}$$

and

$$\mathcal{G}=\frac{a_m}{p+m+1}g_*^{p+m+1}+\frac{a_{m-1}}{p+m}g_*^{p+m}+...+\frac{a_1}{p+2}g_*^{p+2}+\frac{a_0}{p+1}g_*^{p+1}.$$

Now, by using Lemma 2.4, we get

$$T(r,\mathcal{F})=(p+m+1)T(r,f_*)+S(r,f_*).$$

So, it is clear that $\mathcal{F}'=\alpha f_1$. We also have

$$m\left(r,\frac{1}{\mathcal{F}}\right)\le m\left(r,\frac{1}{\alpha f_1}\right)+m\left(r,\frac{\mathcal{F}'}{\mathcal{F}}\right)\le m\left(r,\frac{1}{f_1}\right)+S(r,f_*). \tag{3.8}$$

By using (3.8) and the first fundamental theorem, we get

$$\begin{aligned} T(r,\mathcal{F}) &= m\left(r,\frac{1}{\mathcal{F}}\right)+N\left(r,\frac{1}{\mathcal{F}}\right)\\ &\le T(r,f_1)-N\left(r,\frac{1}{f_1}\right)+N\left(r,\frac{1}{\mathcal{F}}\right)+S(r,f_*)\\ &\le T(r,f_1)+(p+1)N\left(r,\frac{1}{f_*}\right)+\sum_{i=1}^{m}N\left(r,\frac{1}{f_*-b_i}\right)-N\left(r,\frac{1}{f_1}\right), \end{aligned} \tag{3.9}$$

where $b_i(i=1,2,...,m)$ are the roots of the algebraic equation

$$\frac{a_m}{p+m+1}z^m+\frac{a_{m-1}}{p+m}z^{m-1}+\frac{a_{m-2}}{p+m-1}z^{m-2}+...+\frac{a_1}{p+2}z+\frac{a_0}{p+1}=0.$$

Substituting (3.5) to (3.8) and using (2.5), (2.6) in (3.9), we get

$$\begin{aligned} T(r,\mathcal{F}) &\le N\left(r,\frac{1}{f_*^pP_m(f_*)f_*'}\right)+(2-p)N\left(r,\frac{1}{f_*}\right)+N\left(r,\frac{1}{g_*^pP_m(g_*)g_*'}\right)\\ &+(2-p)N\left(r,\frac{1}{g_*}\right)+2\overline{N}(r,f_*)+(p+1)N\left(r,\frac{1}{f_*}\right)\\ &+\sum_{i=1}^{m}N\left(r,\frac{1}{f_*-b_i}\right)-N\left(r,\frac{1}{f_*^pP_m(f_*)f_*'}\right)+o(T(r)) \end{aligned}$$

$$(p+m+1)T(r,f_*) \le 3N\left(r,\frac{1}{f_*}\right) + 3N\left(r,\frac{1}{g_*}\right) + \overline{N}(r,g_*) + mN\left(r,\frac{1}{g_*}\right)$$
$$+ 2\overline{N}(r,f_*) + \sum_{i=1}^{m} N\left(r,\frac{1}{f_*-b_i}\right) + o(T(r))$$
$$\le \frac{3}{l}T\left(r,\frac{1}{f_*}\right) + \frac{3}{l}T\left(r,\frac{1}{g_*}\right) + \frac{1}{l}T(r,g_*) + \frac{m}{l}T\left(r,\frac{1}{g_*}\right)$$
$$+ \frac{2}{l}T(r,f_*) + \frac{m}{l}T\left(r,\frac{1}{f_*}\right) + o(T(r))$$

i.e.,

$$\left[\frac{(p+m+l)l-(m+5)}{l}\right] T(r,f_*) \le \left(\frac{m+4}{l}\right) T(r,g_*) + o(T(r)) \tag{3.10}$$

Let $g_1 = -\frac{f_3}{f_2} = \frac{g_*^p P_m(g_*) g_*^{'}}{\alpha}$, $g_2 = \frac{1}{f_2} = \frac{1}{\mathcal{H}}$ and $g_3 = -\frac{f_1}{f_2} = -\frac{f_*^p P_m(f_*) f_*^{'}}{\alpha \mathcal{H}}$. Then we get $g_1 + g_2 + g_3 = 1$.

By Lemma 2.5, we have g_1, g_2 and g_3 are linearly independent because f_1, f_2 and f_3 are linearly independent. Now, proceeding in the same lines as above, we obtain

$$\left[\frac{(p+m+l)l-(m+5)}{l}\right] T(r,g_*) \le \left(\frac{m+4}{l}\right) T(r,f_*) + o(T(r)) \tag{3.11}$$

Let $T_*(r) = \max\{T(r,f_*), T(r,g_*)\}$. Combining (3.10) and (3.11),

$$\left[\frac{(p+m+l)l-(m+5)}{l}\right] T_*(r) \le \left(\frac{m+4}{l}\right) T_*(r) + o(T(r))$$

i.e.,

$$\left[\frac{(p+m+l)l-(m+5)}{l} - \left(\frac{m+4}{l}\right)\right] T_*(r) \le o(T(r))$$

i.e.,

$$p \le \frac{2m-(m+1)l+9}{l},$$

which contradicts $p \ge \frac{2m-(m+1)l+10}{l}$.

Thus, f_1, f_2 and f_3 are linearly independent. Therefore there exists constants c_1, c_2 and c_3, atleast one of them is non-zero such that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0. \tag{3.12}$$

Subcase 1.1. If $c_1 = 0$, $c_2 \ne 0$ and $c_3 \ne 0$, then from (3.12), we get $f_3 = -\frac{c_2}{c_3} f_2$, which implies that $g_*^p P_m(g_*) g_*^{'} = \frac{c_2}{c_3}\alpha$.

On integrating, we get

$$\frac{a_m}{p+m+1} g_*^{p+m+1} + \frac{a_{m-1}}{p+m} g_*^{p+m} + ... + \frac{a_1}{p+2} g_*^{p+2} + \frac{a_0}{p+1} g_*^{p+1} = \frac{c_2}{c_3}\alpha + c, \tag{3.13}$$

where c is an arbitrary constant. Thus, we get

$$T\left(r, \frac{a_m}{p+m+1} g_*^{p+m+1} + \frac{a_{m-1}}{p+m} g_*^{p+m} + ... + \frac{a_1}{p+2} g_*^{p+2} + \frac{a_0}{p+1} g_*^{p+1}\right) \le T(r,\alpha) + O(1)$$

i.e., $(p+m+1)T(r,g_*) \le S(r,g_*)$.

Now, since $p \ge \frac{2m-(m+1)l+10}{l}$, we get a contradiction.

Subcase 1.2. Let $c_1 \ne 0$. Then by (3.12), we get $f_1 = \left(-\frac{c_2}{c_1}\right) f_2 + \left(-\frac{c_3}{c_1}\right) f_3$.

On substituting this in the relation $f_1 + f_2 + f_3 = 1$, we get $\left(1 - \frac{c_2}{c_1}\right) f_2 + \left(1 - \frac{c_3}{c_1}\right) f_3 = 1$, where $(c_1 - c_2)(c_1 - c_3) \neq 0$. Thus, we get

$$\left(1 - \frac{c_3}{c_1}\right) \frac{g_*^p P_m(g_*) g_*^{'}}{\alpha} + \frac{1}{\mathcal{H}} = \left(1 - \frac{c_2}{c_1}\right) \tag{3.14}$$

We now see that

$$\begin{aligned} T(r, g_*^p P_m(g_*) g_*^{'}) &\leq T\left(r, \frac{g_*^p P_m(g_*) g_*^{'}}{\alpha}\right) + T(r, \alpha) \\ &\leq T\left(r, \frac{g_*^p P_m(g_*) g_*^{'}}{\alpha}\right) + S(r, g_*). \end{aligned}$$

By using Lemma 2.2 to (3.14), we get

$$\begin{aligned} T\left(r, \frac{g_*^p P_m(g_*) g_*^{'}}{\alpha}\right) \leq \overline{N}\left(r, \frac{g_*^p P_m(g_*) g_*^{'}}{\alpha}\right) + \overline{N}\left(r, \frac{\alpha}{g_*^p P_m(g_*) g_*^{'}}\right) \\ + \overline{N}(r, \mathcal{H}) + S(r, g_*). \end{aligned}$$

Combining the above two, we get

$$T(r, g_*^p P_m(g_*) g_*^{'}) \leq \overline{N}\left(r, \frac{1}{g_*^p P_m(g_*) g_*^{'}}\right) + 2\overline{N}(r, g_*) + S(r, g_*). \tag{3.15}$$

By using Lemma 2.3, Lemma 2.4 and (3.15), we get

$$\begin{aligned} (p+m)T(r, g_*) &\leq T(r, g_*^p P_m(g_*)) + S(r, g_*) \\ &\leq T(r, g_*^p P_m(g_*) g_*^{'}) + T\left(r, \frac{1}{g_*^{'}}\right) + S(r, g_*) \\ &\leq \overline{N}\left(r, \frac{1}{g_*^p P_m(g_*) g_*^{'}}\right) + 2\overline{N}(r, g_*) + T\left(r, \frac{1}{g_*^{'}}\right) + S(r, g_*) \\ &\leq 8T(r, g_*) + S(r, g_*), \end{aligned}$$

which contradicts $p \geq \frac{2m-(m+1)l+10}{l}$.

Case 2. If $f_2 = k$, where k is a constant.

Subcase 2.1. If $k \neq 1$, then from the relation $f_1 + f_2 + f_3 = 1$, we get

$$\frac{f_*^p P_m(f_*) f_*^{'}}{\alpha} - k \frac{g_*^p P_m(g_*) g_*^{'}}{\alpha} = 1 - k. \tag{3.16}$$

By applying Lemma 2.2 to (3.16), we get

$$T\left(r, \frac{f_*^p P_m(f_*) f_*^{'}}{\alpha}\right) \leq \overline{N}(r, f_*) + \overline{N}\left(r, \frac{1}{f_*^p P_m(f_*) f_*^{'}}\right) + \overline{N}\left(r, \frac{1}{g_*^p P_m(g_*) g_*^{'}}\right) + S(r, g_*). \tag{3.17}$$

Again by applying Lemma 2.3, Lemma 2.4 and (3.17), we get

$$\begin{aligned} (p+m)T(r, f_*) &\leq T(r, f_*^p P_m(f_*)) + S(r, f_*) \\ &\leq T(r, f_*^p P_m(f_*) f_*^{'}) + T\left(r, \frac{1}{f_*^{'}}\right) + S(r, f_*) \\ &\leq T\left(r, \frac{f_*^p P_m(f_*) f_*^{'}}{\alpha}\right) + T\left(r, \frac{1}{f_*^{'}}\right) + S(r, f_*) \\ &\leq 7T(r, f_*) + 4T(r, g_*) + S(r, f_*) \end{aligned}$$

i.e.,

$$(p+m-7)T(r,f_*) \le 4T(r,g_*) + S(r,f_*).$$

Now, using Lemma 2.7, we get

$$(p+m-7)T(r,f_*) \le 4\left[\frac{l(p+m+2)}{(p+m-2)l-(m+4)}\right]T(r,f_*) + S(r,g_*),$$

which contradicts $p \ge \frac{2m-(m+1)l+10}{l}$.

Subcase 2.2. Let $k=1$ i.e., $\mathcal{H}=1$ and $f_*^p P_m(f_*)f_*^{'} = g_*^p P_m(g_*)g_*^{'}$.

On integrating both sides, we get

$$\frac{a_m}{p+m+1}f_*^{p+m+1} + \frac{a_{m-1}}{p+m}f_*^{p+m} + ... + \frac{a_1}{p+2}f_*^{p+2} + \frac{a_0}{p+1}f_*^{p+1} \equiv \frac{a_m}{p+m+1}g_*^{p+m+1} + \frac{a_{m-1}}{p+m}g_*^{p+m} + ... + \frac{a_1}{p+2}g_*^{p+2} + \frac{a_0}{p+1}g_*^{p+1} + c,$$

where c is an arbitrary constant. That is

$$\mathcal{F} \equiv \mathcal{G} + c. \tag{3.18}$$

Subcase 2.2.1. Let if possible $c \ne 0$. Then, we get

$$\Theta(0,\mathcal{F}) + \Theta(c,\mathcal{F}) + \Theta(\infty,\mathcal{F}) = \Theta(0,\mathcal{F}) + \Theta(0,\mathcal{G}) + \Theta(\infty,\mathcal{F}).$$

So, we have

$$\overline{N}\left(r,\frac{1}{\mathcal{F}}\right) = \overline{N}\left(r,\frac{1}{f_*}\right) + \overline{N}\left(r,\frac{1}{f_*-b_1}\right) + ... + \overline{N}\left(r,\frac{1}{f_*-b_m}\right) \le (m+1)T(r,f_*).$$

Similarly, we get

$$\overline{N}\left(r,\frac{1}{\mathcal{G}}\right) \le (m+1)T(r,g_*).$$

We also note that,

$$T(r,\mathcal{F}) = (p+m+1)T(r,f_*) + S(r,f_*)$$
$$T(r,\mathcal{G}) = (p+m+1)T(r,g_*) + S(r,g_*).$$

Thus, $\Theta(0,\mathcal{F}) = 1 - \overline{\lim}_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{\mathcal{F}}\right)}{T(r,\mathcal{F})} \ge 1 - \frac{(m+1)T(r,f_*)}{(p+m+1)T(r,f_*)} = 1 - \frac{m+1}{p+m+1}$.

Therefore, $\Theta(0,\mathcal{F}) \ge \frac{p}{p+m+1}$.

Thus, $\Theta(0,\mathcal{F}) + \Theta(c,\mathcal{F}) + \Theta(\infty,\mathcal{F}) \ge \frac{3p+m}{p+m+1} > 2$.

Since $p \ge \frac{2m-(m+1)l+10}{l}$, we get a contradiction.

Subcase 2.2.2. Thus, we get $c=0$. So,

$$\mathcal{F} \equiv \mathcal{G}. \tag{3.19}$$

Let $h = \frac{f_*}{g_*}$. Then taking h in (3.19), we get

$$\frac{a_m}{p+m+1}\left[g_*^{p+m+1}\left\{h^{p+m+1}-1\right\}\right] + \frac{a_{m-1}}{p+m}\left[g_*^{p+m}\left\{h^{p+m}-1\right\}\right] + ... + \frac{a_1}{p+2}\left[g_*^{p+2}\left\{h^{p+2}-1\right\}\right] + \frac{a_0}{p+1}\left[g_*^{p+1}\left\{h^{p+1}-1\right\}\right] = 0. \tag{3.20}$$

Subcase 2.2.2.1. If h is a non-constant, then by using Lemma 2.6 and proceeding exactly in the same lines as done in [13, p.1272], we get a contradiction.

Subcase 2.2.2.2. Let h be a constant, then from (3.20), we get

$$h^{p+m+1}-1=0,\ h^{p+m}-1=0, ..., h^{p+1}-1=0.$$

That is $h^d = 1$, where $d = gcd\{p+m+1, p+m, ..., p+1\} = 1$. That is $h = 1$.

Hence $f_* \equiv g_*$ (or) $f \equiv g$.

Case 3. Suppose $f_3 = c$, where c is a constant.

Subcase 3.1. If $c \neq 1$, then from the relation $f_1 + f_2 + f_3 = 1$, we get

$$\frac{f_*^p P_m(f_*) f_*^{'}}{\alpha} - \frac{c\alpha}{g_*^p P_m(g_*) g_*^{'}} = 1 - \mathcal{H}. \tag{3.21}$$

Now, by applying Lemma 2.2 to the above equation, we get

$$T(r, f_*^p P_m(f_*) f_*^{'}) \leq T\left(r, \frac{f_*^p P_m(f_*) f_*^{'}}{\alpha}\right) + S(r, f_*) \tag{3.22}$$

$$\leq \overline{N}\left(r, \frac{f_*^p P_m(f_*) f_*^{'}}{\alpha}\right) + \overline{N}\left(r, \frac{\alpha}{f_*^p P_m(f_*) f_*^{'}}\right) + \overline{N}\left(r, \frac{g_*^p P_m(g_*) g_*^{'}}{\alpha}\right)$$
$$\leq \overline{N}(r, f_*) + \overline{N}\left(r, \frac{1}{f_*^p P_m(f_*) f_*^{'}}\right) + \overline{N}(r, g_*) + S(r, g_*).$$

By using Lemma 2.3, Lemma 2.4 and (3.22), we have

$$\begin{aligned}(p+m)T(r, f_*) &\leq T(r, f_*^p P_m(f_*)) + S(r, f_*)\\ &\leq T(r, f_*^p P_m(f_*) f_*^{'}) + T\left(r, \frac{1}{f_*^{'}}\right) + S(r, f_*)\\ &\leq 7T(r, f_*) + \overline{N}(r, g_*) + S(r, f_*).\end{aligned}$$

Again using Lemma 2.7, we get

$$\begin{aligned}(p+m-7)T(r, f_*) &\leq T(r, g_*) + S(r, f_*)\\ &\leq \left[\frac{l(p+m+2)}{(p+m-2)l-(m+4)}\right] T(r, f_*) + S(r, f_*),\end{aligned}$$

which contradicts $p \geq \frac{2m-(m+1)l+10}{l}$.

Subcase 3.2. Let $c = 1$. Then from (3.21), we get

$$f_*^p P_m(f_*) f_*^{'} g_*^p P_m(g_*) g_*^{'} = \alpha^2. \tag{3.23}$$

Let z_0 be a zero of f_* of order r_0. Then from (3.23), we see that z_0 is a pole of g_* of order $s_0(say)$. Then, from (3.23), we get $pr_0 + r_0 - 1 = ps_0 + ms_0 + s_0 + 1$ i.e., $(p+1)(r_0 - s_0) = ms_0 + 2 \geq p+1$ i.e., $r_0 \geq \frac{p+m-1}{m}$.

Let z_1 be a zero of $P_m(f_*)$ of order r_1. Then from (3.23), we see that z_1 is a pole of g_* of order $s_1(say)$. So, we have $r_1 + r_1 - 1 = ps_1 + ms_1 + s_1 + 1$ i.e., $r_1 \geq \frac{p+m+3}{2}$.

Let z_2 be a zero of $f_*^{'}$ of order r_2 which are not the zeros of $f_* P_m(f_*)$, so from (3.23) we see that z_2 will be a pole of g_* of order $s_2(say)$. Then from (3.23), we get $r_2 = ps_2 + ms_2 + s_2 + 1$ i.e., $r_2 \geq p + m + 2$.

The similar explanations holds for the zeros of $g_*^p P_m(g_*) g_*^{'}$. Next, we see from (3.23) that

$$\overline{N}(r, f_*^p P_m(f_*) f_*^{'}) = \overline{N}\left(r, \frac{\alpha^2}{g_*^p P_m(g_*) g_*^{'}}\right)$$

i.e.,

$$\begin{aligned}\overline{N}(r,f_*) &\leq \overline{N}\left(r,\frac{1}{g_*}\right)+\overline{N}\left(r,\frac{1}{P_m(g_*)}\right)+\overline{N}\left(r,\frac{1}{g_*^{'}}\right)\\ &\leq \left(\frac{m}{p+m-1}\right)N\left(r,\frac{1}{g_*}\right)+\left(\frac{2}{p+m+3}\right)N\left(r,\frac{1}{P_m(g_*)}\right)\\ &+\left(\frac{1}{p+m+2}\right)N\left(r,\frac{1}{g_*^{'}}\right)+S(r,g_*)\\ &\leq \left(\frac{m}{p+m-1}+\frac{2}{p+m+3}+\frac{1}{p+m+2}\right)T(r,g_*)+S(r,g_*).\end{aligned}$$

By applying the second fundamental theorem, we get

$$T(r,f_*) \leq \overline{N}(r,f_*)+\overline{N}\left(r,\frac{1}{f_*}\right)+\overline{N}\left(r,\frac{1}{P_m(f_*)}\right)+S(r,f_*)$$

$$\begin{aligned}&\leq \left(\frac{m}{p+m-1}+\frac{2}{p+m+3}\right)T(r,f_*)\\ &+\left(\frac{m}{p+m-1}+\frac{2}{p+m+3}+\frac{2}{p+m+2}\right)T(r,g_*)+S(r,f_*)+S(r,g_*).\end{aligned} \tag{3.24}$$

Similarly, we get

$$\begin{aligned}T(r,g_*) &\leq \left(\frac{m}{p+m-1}+\frac{2}{p+m+3}\right)T(r,g_*)\\ &+\left(\frac{m}{p+m-1}+\frac{2}{p+m+3}+\frac{2}{p+m+2}\right)T(r,f_*)+S(r,f_*)+S(r,g_*).\end{aligned} \tag{3.25}$$

From (3.24) and (3.25), we get

$$T_*(r) \leq \left(\frac{2m}{p+m-1}+\frac{4}{p+m+3}+\frac{2}{p+m+2}\right)T_*(r)+S_*(r)$$

i.e.,

$$\left[1-\frac{2m}{p+m-1}-\frac{4}{p+m+3}-\frac{2}{p+m+2}\right]T_*(r) \leq S_*(r),$$

which contradicts $p \geq \frac{2m-(m+1)l+10}{l}$.

Hence the proof of theorem 1.9. □

Proof. (Proof of Theorem 1.10.) Since f_* and g_* are both non-constant entire functions, then we may consider the following two cases.

Case 1. Let f_* and g_* are two transcendental entire functions. Then it is clear that $\overline{N}(r,f_*) = S(r,f_*)$ and $\overline{N}(r,g_*) = S(r,g_*)$. With this the result of the proof is carried out in the same lines as in the proof of theorem 1.9.

Case 2. Let f_* and g_* be both polynomials. Since $f_*^pP_m(f_*)f_*^{'}$ and $g_*^pP_m(g_*)g_*^{'}$ share α CM, then we have

$$f_*^pP_m(f_*)f_*^{'}-\alpha = \kappa(g_*^pP_m(g_*)g_*^{'}-\alpha), \tag{3.26}$$

where κ is a non-zero constant.

Subcase 2.1. Suppose that $\kappa \neq 1$, then from (3.26), we get

$$\frac{f_*^pP_m(f_*)f_*^{'}}{\alpha}-\kappa\frac{g_*^pP_m(g_*)g_*^{'}}{\alpha} = 1-\kappa. \tag{3.27}$$

By Lemma 2.2, we get

$$\begin{aligned}T(r, f_*^p P_m(f_*)f_*^{'}) &\leq T\left(r, \frac{f_*^p P_m(f_*)f_*^{'}}{\alpha}\right) + S(r, f_*)\\ &\leq \overline{N}\left(r, \frac{f_*^p P_m(f_*)f_*^{'}}{\alpha}\right) + \overline{N}\left(r, \frac{\alpha}{f_*^p P_m(f_*)f_*^{'}}\right)\\ &+ \overline{N}\left(r, \frac{\alpha}{g_*^p P_m(g_*)g_*^{'}}\right) + S(r, f_*)\\ &\leq \overline{N}(r, f_*) + \overline{N}\left(r, \frac{\alpha}{f_*^p P_m(f_*)f_*^{'}}\right) + \overline{N}\left(r, \frac{\alpha}{g_*^p P_m(g_*)g_*^{'}}\right) + S(r, f_*).\end{aligned}$$

By using Lemma 2.3, Lemma 2.4 and (3.27) gives

$$\begin{aligned}(p+m)T(r, f_*) &\leq T(r, f_*^p P_m(f_*))\\ &\leq T(r, f_*^p P_m(f_*)f_*^{'}) + T\left(r, \frac{1}{f_*^{'}}\right) + S(r, f_*)\\ &\leq 4T(r, f_*) + 3T(r, g_*) + S(r, f_*)\end{aligned}$$

i.e.,

$$(p+m-4)T(r, f_*) \leq 3T(r, g_*) + S(r, f_*).$$

Again using Lemma 2.8, we get

$$(p+m-4)T(r, f_*) \leq 3\left[\frac{l(P+m+2)}{(P+m-2)l-(m+1)}\right]T(r, f_*) + S(r, f_*),$$

which contradicts $p \geq \frac{2m-(m+1)l+6}{l}$.

Subcase 2.2. Let $\kappa = 1$, from (3.27), we get

$$f_*^p P_m(f_*)f_*^{'} \equiv g_*^p P_m(g_*)g_*^{'}.$$

Now, proceeding in the same lines as in Subcase 2.2.2.1 and Subcase 2.2.2.2 in the proof of theorem 1.9, we get proof of theorem 1.10. □

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