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Commutativity of prime rings involving generalized derivations

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Abstract In this paper we investigate identities with two generalized derivations in prime rings. Let R be a 2-torsion free prime ring admitting two generalized derivations F and G, not both zero. Among others, we prove that if $F(xy) + G(yx) \in Z(R)$ for all $x, y \in R$, then R is a commutative. Also, if the ring R is equipped with an involution of the second kind and $F(xx^*) + G(x^*x) \in Z(R)$ for all $x \in R$, then R is commutative. The proved theorems give a rise to many corollaries which recover well-known results on (generalized) derivations and left multiplier maps on prime rings (resp. with involution). All along the paper, examples are given to discuss the necessity of our assumptions.

1 Introduction

Throughout this paper, R will denote an associative ring $\neq \{0\}$ with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ denotes the anti-commutator $xy + yx$. Recall that R is prime if, for any $x, y \in R$, $xRy = \{0\}$ implies that either $x = 0$ or $y = 0$, R is called semiprime if, for $x \in R$, $xRx = \{0\}$ implies that $x = 0$, and R is said to be 2-torsion free if $2x = 0$, $x \in R$, implies $x = 0$. A mapping $f : R \to R$ is said to be additive if $f(x + y) = f(x) + f(y)$ holds for all $x, y \in R$. An additive mapping d on R is called derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping f is called left multiplier if $f(xy) = f(x)y$ holds for all $x, y \in R$. In [2], Brešar (1991) introduced the notion of generalized derivation as follows: an additive mapping F on R is called a generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. The concept of generalized derivations includes strictly the concepts of derivations and left multiplier mappings. For $a, b \in R$, the mapping $F : R \to R$ defined by $F(x) = ax + xb$ for all $x \in R$ is an other example of a generalized derivation on R (associated with the inner derivation $d = [., b]$), called inner generalized derivation.
An additive mapping $x \mapsto x^*$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$ is called an involution on R. An element x in a ring R equipped with an involution $*$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. The involution $*$ is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the latter case, $S(R) \cap Z(R) \neq \{0\}$. The study of additive maps in rings with involution was initiated by Brešar et al. [3] to describe the centralizing maps on the skew-symmetric elements in prime rings.

In the last fifteen years, there has been ongoing interest concerning the relationship between the commutativity of rings (resp. with involution) and the existence of certain specific types of additive mappings of R (such as automorphisms, derivations, skew derivations, semi-derivations, and generalized derivations) acting on appropriate subsets of the rings (see for example [4, 6, 10, 12, 13]). In this paper we investigate identities with two generalized derivations in prime rings. Let R be a 2-torsion free prime ring admitting two generalized derivations F and G, not both zero. If $F(xy) + G(yx) \in Z(R)$ for all $x, y \in R$, then R is a commutative. If the ring R is equipped with an with involution of the second kind and $F(xx^*) + G(x^*x) \in Z(R)$ for all $x, y \in R$, then R is commutative. As consequence, many identities in term of commutator and anti-commutator implying commutativity are given. We also give examples to discuss our results on non prime rings. Note that the proved results recover many results on (generalized) derivations and left multiplier mappings on prime rings (resp. with involution) (see for example

[1, 5]).

2 A commutativity theorem for rings involving generalized derivations

We need the following lemma.

Lemma 2.1 ([7, Lemma 1]). *Let R be a 2-torsion free prime ring. If R admits a non-zero generalized derivation F such that $F([x,y]) \in Z(R)$ for all $x, y \in R$, then R is commutative.*

The main result of this section is the following theorem.

Theorem 2.2. *Let R be a 2-torsion free prime ring. If R admits two generalized derivations, not both zero, F and G such that $F(xy) + G(yx) \in Z(R)$ for all $x, y \in R$, then R is a commutative.*

Proof. Let f and g be the associate derivations to F and G, respectively.
By the assumption, we have

$$F(xy) + G(yx) \in Z(R) \qquad \text{for all } x, y \in R. \tag{2.1}$$

We claim that $Z(R) \neq \{0\}$. Otherwise, we have

$$F(xy) + G(yx) = 0 \qquad \text{for all } x, y \in R. \tag{2.2}$$

Replacing y by yx, we get

$$F(xy)x + G(yx)x + xyf(x) + yxg(x) = 0 \qquad \text{for all } x, y \in R, \tag{2.3}$$

and hence

$$xyf(x) + yxg(x) = 0 \qquad \text{for all } x, y \in R. \tag{2.4}$$

Substituting ry in place of y in (2.4), we obtain

$$xryf(x) + ryxg(x) = 0 \qquad \text{for all } r, x, y \in R. \tag{2.5}$$

Left multiplying (2.4) by r, where $r \in R$, we get

$$rxyf(x) + ryxg(x) = 0 \qquad \text{for all } r, x, y \in R. \tag{2.6}$$

Now, subtracting (2.5) from (2.6), we have

$$[r, x]yf(x) = 0 \qquad \text{for all } r, x, y \in R. \tag{2.7}$$

Since R is prime and $Z(R) = \{0\}$, we get immediately $f = 0$. Similarly, $g = 0$. Accordingly, F and G are left multiplier. Thus, (2.2) becomes

$$F(x)y + G(y)x = 0 \qquad \text{for all } x, y \in R. \tag{2.8}$$

Replacing y by yr, where $r \in R$, we get

$$F(x)yr + G(y)rx = 0 \qquad \text{for all } r, x, y \in R. \tag{2.9}$$

Right multiplying (2.8) by r, we obtain

$$F(x)yr + G(y)xr = 0 \qquad \text{for all } r, x, y \in R. \tag{2.10}$$

Subtracting (2.10) from (2.9), we have

$$G(y)[r, x] = 0 \qquad \text{for all } r, x, y \in R. \tag{2.11}$$

Since R is prime and $Z(R) = \{0\}$, we obtain $G = 0$. Hence, we have $F(x)y = 0$ for all $x, y \in R$, which means that $F = 0$, a contradiction. Consequently, $Z(R) \neq \{0\}$.
Now, let $y \in Z(R)\backslash\{0\}$ and replace x by x^2 in (2.1), we get

$$(F(x^2) + G(x^2))y + x^2(f(y) + g(y)) \in Z(R) \qquad \text{for all } x \in R. \tag{2.12}$$

Since $(F(x^2)+G(x^2))y \in Z(R)$ for all $x \in R$, we get that $x^2(f(y)+g(y)) \in Z(R)$. Then, since $f(y)+g(y) \in Z(R)$ for all $y \in Z(R)$, we conclude that either $x^2 \in Z(R)$ for all $x \in R$ or $(f+g)(y)=0$ for all $y \in Z(R)$. The first case implies that R is commutative. In the second case, our assumption implies that, for any $y \in Z(R)\backslash\{0\}$, we have

$$(F(x)+G(x))y \in Z(R) \qquad \text{for all } \ x \in R, \tag{2.13}$$

which means that

$$(F+G)(x) \in Z(R) \qquad \text{for all } \ x \in R. \tag{2.14}$$

Using Lemma 2.1, we deduce that either R is commutative or $F+G=0$. When $F+G=0$, our assumption becomes $F([x,y]) \in Z(R)$ for all $x,y \in R$ and $F \neq 0$, which implies, again by Lemma 2.1, that R is commutative.□
Next, we give a collection of consequences of the theorem above.

Corollary 2.3. *Let R be a 2-torsion free prime ring admitting two generalized derivations, not both zero, F and G. Then, the following assertions are equivalent:*

(i) $F(x \circ y)+G([x,y]) \in Z(R)$ *for all* $x,y \in R$.

(ii) $F(xy)+G([x,y]) \in Z(R)$ *for all* $x,y \in R$.

(iii) $F(xy)+G(x \circ y) \in Z(R)$ *for all* $x,y \in R$.

(iv) R *is commutative.*

Proof. We need only the prove the implications $(i) \Rightarrow (4)$ with $i \in \{1,2,3\}$.
$(1) \Rightarrow (4)$ Follows immediately from Theorem 2.2 since the generalized derivations $F+G$ and $F-G$ are not both zero and satisfy the relation

$$(F+G)(xy)+(F-G)(yx)=F(x \circ y)+G([x,y]) \in Z(R) \qquad \text{for all } \ x,y \in R.$$

The proofs of the implications $(2) \Rightarrow (4)$ and $(3) \Rightarrow (4)$ are similar to the first one by considering the generalized derivations $F+G$ and $-G$ and the generalized derivations $F+G$ and G, respectively. □

Corollary 2.4. *Let R be a 2-torsion free prime ring. If R admits two generalized derivations, not both zero, F and G such that $F(xy)=G(yx)$ for all $x,y \in R$, then R is commutative and $F=G$.*

Proof. The ring R is clearly commutative (by Theorem 2.2). Let d be the associate derivation to the generalized derivation $H=F-G$. We have $H(xy)=0$ for all $x,y \in R$. Thus, for all $x,y \in R$, we have $0=H(x^2y)=H(x^2)y+x^2d(y)=x^2d(y)$. Then, $d=0$. Consequently, $0=H(xy)=H(x)y$ for all $x,y \in R$, and so $H=0$. □

Note that in an arbitrary 2-torsion free ring R (even commutative), if F and G are two generalized derivations on R such that $F(xy)=G(yx)$ for all $x,y \in R$, then F and G are not necessary equal. To see this, consider the ring

$$R := \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a,b,c \in \mathbb{Z} \right\}.$$

We can verify easily that $xy=yx=0$ for all $x,y \in R$. Hence, for $F=$ and $G=0$, we have $F(xy)=G(yx)$. However, $F \neq G$. Unfortunately, R is not semiprime. So we can not be sure that, on a semi-prime ring, the relation $F(xy)=G(yx)$ for all $x,y \in R$ implies that $F=G$ or not.

Corollary 2.5. *Let R be a 2-torsion free prime ring admitting two generalized derivations, not both zero, F and G. Then, the following assertions are equivalent:*

(i) $F(x \circ y) = G([x, y])$ *for all* $x, y \in R$.

(ii) $F(xy) = G([x, y])$ *for all* $x, y \in R$.

(iii) R *is commutative and* $F = 0$.

Proof. $(1) \Rightarrow (3)$ Applying Corollary 2.4 to the generalized derivations $F - G$ and $-F - G$ which are not both zero and which satisfy $(F - G)(xy) = (-F - G)(yx)$, we conclude that R is commutative and that $F - G = -F - G$ (and so $F = 0$ since R is 2-torsion free).
$(1) \Rightarrow (3)$ It suffices to apply Corollary 2.4 to the generalized derivations $F - G$ and $-G$ (as in the proof of $(1) \Rightarrow (3)$).
The converse implications are trivial. □

Corollary 2.6. *Let* R *be a* 2*-torsion free prime ring. If* R *admits two generalized derivations, not both zero,* F *and* G *such that* $F(xy) = G(x \circ y)$ *for all* $x, y \in R$ *then* R *is commutative and* $F = 2G$.

Proof. Follows by applying Corollary 2.4 to the generalized derivations $F - G$ and G. □

Corollary 2.7. *Let* R *be a* 2*-torsion free prime ring. If* R *admits a generalized derivation* F *such that* $F(x \circ y) = 0$ *for all* $x, y \in R$ *then* $F = 0$.

Proof. Suppose that $F \neq 0$. Using Corollary 2.5 for the generalized derivations F and $G = 0$, we get that $F = 0$, a contradiction. Thus, $F = 0$. □
The following examples show that under the hypotheses of any result of this section we cannot hope to prove the commutativity of the ring R if this one is semiprime.

Example 2.8. Let R_1 be a 2-torsion free integral domain admitting a non zero derivation d_1 (take for example $R = \mathbb{R}[X]$ equipped with the usual derivation of polynomials) and R_2 be any noncommutative 2-torsion free prime ring. Set $R = R_1 \times R_2$ (which is a semiprime ring) and let $d : R \to R$ be the derivation defined by $d(x, y) = (d_1(x), 0)$ for all $(x, y) \in R$. Then, for all $(x, y), (x', y') \in R$,

$$d\left([(x, y), (x', y')]\right) = d\left([x, x'], [y, y']\right) = d\left(0, [y, y']\right) = (0, 0).$$

Let $F_0 : R \to R$ be the left multiplier defined by $F_0(x, y) = (x, 0)$ for all $(x, y) \in R$. Then, for all $(x, y), (x', y') \in R$,

$$F_0\left([(x, y), (x', y')]\right) = F_0\left([x, x'], [y, y']\right) = F\left(0, [y, y']\right) = (0, 0).$$

The ring R is not commutative. However,

(i) the (generalized) derivations $F = d$ (resp. $F = F_0$) and $G = 0$ satisfy the hypothesis of Theorem 2.2.

(ii) the (generalized) derivations $F = 0$ and $G = d$ (resp. $G = F_0$) satisfy the hypothesis of Corollary 2.3. .

(iii) the (generalized) derivations $F = G = d$ (resp. $F = G = F_0$) satisfy the hypothesis of Corollary 2.4.

(iv) the (generalized) derivations $F = 0$ and $G = d$ (resp. $G = F_0$) satisfy the hypothesis of Corollary 2.5.

(v) the (generalized) derivations $F = 2G = 2d$ (resp. $F = 2G = 2F_0$) satisfy the hypothesis of Corollary 2.6.

3 A commutativity theorem for rings with involution involving generalized derivations

We need the following lemma.

Lemma 3.1 ([7, Theorem 1]). *Let R be a 2-torsion free prime ring with involution of the second kind. If R admits a non-zero generalized derivation F such that $F([x, x^*]) \in Z(R)$ for all $x \in R$, then R is commutative.*

The main result of this section is as follows.

Theorem 3.2. *Let R be a 2-torsion free prime ring with involution of the second kind. If R admits two generalized derivations, not both zero, F and G such that $F(xx^*) + G(x^*x) \in Z(R)$ for all $x, y \in R$, then R is commutative.*

Proof. Let f and g be the associate derivations to F and G, respectively.
By assumption, we have

$$F(xx^*) + G(x^*x) \in Z(R) \qquad \text{for all } x \in R. \tag{3.1}$$

Linearizing (3.1), we get

$$F(xy^*) + F(yx^*) + G(x^*y) + G(y^*x) \in Z(R) \qquad \text{for all } x, y \in R, \tag{3.2}$$

which can be rewritten as

$$F(xy) + F(y^*x^*) + G(x^*y^*) + G(yx) \in Z(R) \qquad \text{for all } x, y \in R. \tag{3.3}$$

Let $h \in Z(R) \cap H(R) \backslash \{0\}$. Replacing y by hy, we get

$$(xy + y^*x^*)f(h) + (yx + x^*y^*)g(h) \in Z(R) \qquad \text{for all } x, y \in R. \tag{3.4}$$

In particular, for $y = h$ we have

$$(x + x^*)(f + g)(h)h \in Z(R) \qquad \text{for all } x \in R. \tag{3.5}$$

Then, since $(f + g)(h)h \in Z(R)$ and R is prime, we get either $x + x^* \in Z(R)$ for all $x \in R$ or $(f + g)(h) = 0$ for all $h \in Z(R) \cap H(R)$. In the first case, we have $[x, x^*] = 0$ for all $x \in R$, and so R is commutative (by Lemma 3.1). Now, if $(f + g)(h) = 0$ for all $h \in Z(R) \cap H(R)$ then $(f + g)(s^2) = 0$ for all $s \in Z(R) \cap S(R)$, which means that $(f + g)(s) = 0$. Therefore, $(f + g)(Z(R)) = 0$.
Replacing y in (3.3) by $h \in Z(R) \cap H(R) \backslash \{0\}$, we deduce that

$$F(x) + F(x^*) + G(x^*) + G(x) \in Z(R) \qquad \text{for all } x \in R, \tag{3.6}$$

and replacing y in (3.3) by $s \in Z(R) \cap S(R) \backslash \{0\}$, we also conclude that

$$F(x) - F(x^*) - G(x^*) + G(x) \in Z(R) \qquad \text{for all } x \in R. \tag{3.7}$$

Thus, from equations (3.6) and (3.7), we get

$$(F + G)(x) \in Z(R) \qquad \text{for all } x \in R. \tag{3.8}$$

By Lemma 2.1, either R is commutative or $F + G = 0$. In the second case, our assumption becomes $F([x, x^*]) \in Z(R)$ for all $x \in R$, which implies, also by Lemma 3.1, that R is commutative (since $F \neq 0$). □
Next, we give a collection of corollaries of the theorem above. The proofs of theses corollaries are easily obtained by replacing Theorem 2.2 by Theorem 3.2 and y by x^* in the proofs of Corollaries 2.3, 2.4, 2.5, 2.6, and 2.7.

Corollary 3.3. *Let R be a 2-torsion free prime ring with involution of the second kind and admitting two generalized derivations, not both zero, F and G. Then, the following assertions are equivalent:*

(i) $F(x \circ x^*) + G([x, x^*]) \in Z(R)$ *for all* $x \in R$.

(ii) $F(xx^*) + G([x, x^*]) \in Z(R)$ *for all* $x \in R$.

(iii) $F(xx^*) + G(x \circ x^*) \in Z(R)$ *for all* $x \in R$.

(iv) R *is commutative.*

Corollary 3.4. *Let* R *be a* 2*-torsion free prime ring with involution of the second kind. If* R *admits two generalized derivations, not both zero,* F *and* G *such that* $F(xx^*) = G(x^*x)$ *for all* $x \in R$*, then* R *is commutative and* $F = G$.

Corollary 3.5. *Let* R *be a* 2*-torsion free prime ring with involution of the second kind and admitting two generalized derivations, not both zero,* F *and* G*. Then, the following assertions are equivalent:*

(i) $F(x \circ x^*) = G([x, x^*])$ *for all* $x \in R$.

(ii) $F(xx^*) = G([x, x^*])$ *for all* $x \in R$.

(iii) R *is commutative and* $F = 0$.

Corollary 3.6. *Let* R *be a* 2*-torsion free prime ring with involution of the second kind. If* R *admits two generalized derivations, not both zero,* F *and* G *such that* $F(xx^*) = G(x \circ x^*)$ *for all* $x \in R$ *then* R *is commutative and* $F = 2G$.

Corollary 3.7. *Let* R *be a* 2*-torsion free prime ring with involution of the second kind. If* R *admits a generalized derivation* F *such that* $F(x \circ x^*) = 0$ *for all* $x \in R$ *then* $F = 0$.

The following examples show that the condition "$*$ is of the second kind" is necessary in all results of this section.

Example 3.8. Let $R := M_2(\mathbb{Z})$ (the matrix ring over $\mathbb{Z}$) equipped with the involution defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

It is straightforward to check that R is a prime and that $*$ is of the first kind. Moreover, for all $x \in R$, we have

$$[x, x^*] = 0 \quad \text{and} \quad xx^* = x^*x \in Z(R).$$

The map d of R defined by

$$d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$$

is a derivation which satisfies $d(xx^*) = 0$ for all $x \in R$.

(i) For any nonzero generalized derivation F of R, the generalized derivations F and $-F$ satisfy the conditions of Theorem 3.2. However, R is not commutative.

(ii) The generalized derivations $F = 0$ and $G =$ satisfy the assertions (1), (2), and (3) of Corollary 3.3. However, R is not commutative. We can also take F and G any two derivations on R.

(iii) The generalized derivations $F = d$ and $G = 0$ satisfy the hypothesis of Corollary 3.4, 3.6, and 3.7 and the assertions (1) and (2) of Corollary 3.5. However, neither R is commutative nor $F = 0$.

The following examples prove that our results cannot be extended to semi-prime rings.

Example 3.9. Let R, $*$, and d be as in Examples 3.8. Let $S := R \times \mathbb{C}$ equipped with the involution of the second kind defined by $(x, z)^\star = (x^*, \overline{z})$ where $\overline{z}$ is the conjugate of the complex number z. Consider also the derivation $(d, 0)$ of S defined by $(d, 0)(x, z) = (d(x), 0)$. It is easy to see that

$$[(x, z), (x, z)^\star] = (0, 0) \quad \text{and} \quad (x, z)(x, z)^\star = (x, z)^\star(x, z) = (x^*x, 2z\overline{z}) \in Z(S)$$

(i) For any nonzero generalized derivation F of S, the generalized derivations F and $-F$ satisfy the conditions of Theorem 3.2. However, S is not commutative.

(ii) The generalized derivations $F = 0$ and $G =$ satisfy the conditions (1), (2), and (3) of Corollary 3.3. However, S is not commutative. We can also take F and G any two derivations of S.

(iii) The generalized derivations $F = (d, 0)$ and $G = 0$ satisfy the hypothesis of Corollary 3.4, 3.6, and 3.7 and the assertions (1) and (2) of Corollary 3.5. However, neither S is commutative nor $F = 0$.

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