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A NOTE ON UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING TWO VALUES WITH THEIR DIFFERENCES

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Abstract In this paper, we study the uniqueness of meromorphic functions that share two non-zero complex constants CM with their differences. We also investigate the uniqueness problem of entire functions which share a non-zero complex constant CM and a non-zero complex constant IM with their differences.

1 Introduction, Definitions and Results

Let f be a non-constant meromorphic function in the open complex plane $\mathbb{C}$ and c be a non-zero complex number. We denote $n(r,\infty;f)$ the number of poles of f in $|z|<r$, the poles are counted according to their multiplicities. The quantity

$$N(r,f)=N(r,\infty;f)=\int_0^r \frac{n(t,\infty;f)-n(0,\infty;f)}{t}dt+n(0,\infty;f)\log r$$

is called the integrated counting function or simply the counting function of poles of f. Also, the function

$$N(r,a;f)=\int_0^r \frac{n(t,a;f)-n(0,a;f)}{t}dt+n(0,a;f)\log r$$

is called the counting function of a-points of f.

The proximity function for the poles of f is defined as

$$m(r,\infty;f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta,$$

where

$$\log^+ x=\begin{cases}0 & \text{if } 0\le x<1,\\ \log x & \text{if } x\ge 1.\end{cases}$$

The quantity $m(r,f)+N(r,f)$ is called Nevanlinna's characteristic function of the meromorphic function f and is denoted by $T(r,f)$.

Let us denote by $\overline{n}(r,a;f)$ the number of distinct a-points of f in $|z|<r$, where $a\in\mathbb{C}\cup\{\infty\}$. The quantity

$$\overline{N}(r,a;f)=\int_0^r \frac{\overline{n}(t,a;f)-\overline{n}(0,a;f)}{t}dt+\overline{n}(0,a;f)\log r$$

denotes the reduced counting function of a-points of f.

The order of growth of f is defined as follows

$$\rho(f)=\lim_{r\to\infty} sup\frac{\log T(r,f)}{\log r}.$$

If $\rho(f) < \infty$, then we say that f is a meromorphic function of finite order.

$S(r, f)$ is a quantity which satisfies $S(r, f) = \circ\{T(r, f)\}$, as $r \to \infty$, possibly outside a set of finite linear measure. A meromorphic function $a = a(z)$ defined in $\mathbb{C}$ is called a small function of f if $T(r, a) = S(r, f)$. We use $S(f)$ to denote the family of all small functions with respect to f.

Let f and g be two non-constant meromorphic functions and $a = a(z)$ be a polynomial. f and g share a CM if $f - a$ and $g - a$ have the same zeros with same multiplicities. On the other hand, f and g share a IM if $f - a$ and $g - a$ have the same zeros ignoring multiplicities. Especially, if f and g share a IM, then we denote by $N_{(p,q)}\left(r, \frac{1}{f-a}\right)$ $(\overline{N}_{(p,q)}\left(r, \frac{1}{f-a}\right))$ the counting function (the reduced counting function) of zeros of $f - a$ with respect to all the ponts such that they are zeros of $f - a$ with multiplicity p and zeros of $g - a$ with multiplicity q.

The shift of a meromorphic function f is defined by $f_c(z) = f(z + c)$, and its first order difference is defined by $\Delta_c f(z) = f(z + c) - f(z)$.

The n^{th} order difference of f is defined by $\Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)),\ n \in \mathbb{N}, n \geq 2$.

For standard definitions and results of the value distribution theory we refer the reader to [5, 8, 17, 18].

The uniqueness theory of meromorphic functions has been started from Nevanlinna's five values uniqueness theorem. He proved that any non-constant meromorphic function can be uniquely determined by five values. After a long research these five values were reduced to two values. The uniqueness of an entire function f sharing values with its derivative $f^{'}$ was firstly investigated by Rubel and Yang [15], Mues and Steinmetz [13, 14] and Gundersen [3] improved their results.

The uniqueness of meromorphic functions sharing values with their shifts or differences has become a subject of great interest recently. At first Heittokangas et al. [7] considered the value sharing problems for shifts of the uniqueness of meromorphic functions. The uniqueness of entire functions sharing values with their difference operators and proved some meaningful results by Chen and Yi [1], Li and Gao [10], Liu and Yang [12]. We mention some of these results here.

The investigation of uniqueness of meromorphic function sharing three values has been introduced by Heittokangas et al. [7] in 2009 in the following way.

Theorem A. [7]. Let f be a meromorphic function of finite order and $c \in S(f) \cup \{\infty\}$. If $f(z)$ and $f(z + c)$ share three distinct periodic functions $a_1, a_2, a_3 \in S(f) \cup \{\infty\}$ with period c CM, then

$$f(z) = f(z + c),$$

for all $z \in \mathbb{C}$.

In 2011 Heittokangas et al. [6] improved Theorem A by replacing "sharing three small functions CM" by "2 CM + 1 IM" and proved the following theorem.

Theorem B. [6]. Let f be a meromorphic function of finite order and $c \in S(f) \cup \{\infty\}$. Also let $a_1, a_2, a_3 \in S(f) \cup \{\infty\}$ be three distinct periodic functions with period c. If $f(z)$ and $f(z + c)$ share a_1, a_2 CM and a_3 IM, then

$$f(z) = f(z + c),$$

for all $z \in \mathbb{C}$.

Considering three shared values IM, in 2016 Li and Yi [11] proved the following result.

Theorem C. [11]. Let f be a non-constant entire function of finite order and c be a non-zero complex number. Also let a_1, a_2, a_3 be three distinct finite values. If $f(z)$ and $\Delta_c f(z)$ share a_1, a_2, a_3 IM, then

$$2f(z) = f(z + c),$$

for all $z \in \mathbb{C}$.

Our aim in this paper is what results we can get if the condition that $f(z)$ and $f(z+c)$ share three values CM or $f(z)$ and $\Delta_c f(z)$ share three values IM is relaxed to two values CM or one value CM and another one IM and if $f(z+c)$ or $\Delta_c f(z)$ is replaced by $\Delta_c^n f(z)$, where $n \in \mathbb{N}$.

In this paper we consider the following problems:
(i) $f(z)$ and $\Delta_c^n f(z)$ share a, b CM, where $f(z)$ is a non-constant meromorphic function with $N(r, f(z)) = S(r, f)$ and
(ii) $f(z)$ and $\Delta_c^n f(z)$ share a CM and b IM, where $f(z)$ is a non-constant entire function with $m\left(r, \frac{1}{f(z)-a}\right) = S(r, f)$.

We now state the following two theorems, which are the main results of this paper.

Theorem 1.1. Let f be a non-constant meromorphic function of finite order and c be a non-zero complex number. Also let a, b be two non-zero distinct finite complex constants. If
(i) $f(z)$ and $\Delta_c^n f(z)$ $(n \geq 1)$ share a, b CM
and
(ii) $N(r, f(z)) = S(r, f)$,
then

$$\Delta_c^n f(z) \equiv f(z).$$

If we taking an entire function in Theorem 1.1, we get the following corollary.

Corollary 1.1. Let f be a non-constant entire function of finite order and c be a non-zero complex number. Also let a, b be two non-zero distinct finite complex constants. If $f(z)$ and $\Delta_c^n f(z)$ $(n \geq 1)$ share a, b CM, then

$$\Delta_c^n f(z) \equiv f(z).$$

Theorem 1.2. Let f be a non-constant entire function of finite order and c be a non-zero complex number. Also let a, b be two non-zero distinct finite complex constants. If
(i) $f(z)$ and $\Delta_c^n f(z)$ $(n \geq 1)$ share a CM,
(ii) $f(z)$ and $\Delta_c^n f(z)$ $(n \geq 1)$ share b IM
and
(iii) $m\left(r, \frac{1}{f(z)-a}\right) = S(r, f)$,
then

$$\Delta_c^n f(z) \equiv f(z).$$

2 Lemmas

For the proof of our main results, we need the following lemmas.

Lemma 2.1. [16]. Let f be a non-constant meromorphic function and $a_j (j = 1, 2, \cdots, q)$ be q distinct complex numbers. Then

$$m\left(r, \sum_{j=1}^{q} \frac{1}{f - a_j}\right) = \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + O(1).$$

Lemma 2.2. [2, 4]. Let f be a meromorphic function of finite order and c be a non-zero complex constant. Then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Lemma 2.3. [10]. Let $c \in \mathbb{C}$, $n \in \mathbb{N}$ and f be a meromorphic function of finite order. Then for any small periodic function $a(z)$ with period c with respect to $f(z)$,

$$m\left(r, \frac{\Delta_c^n f(z)}{f(z) - a(z)}\right) = S(r, f),$$

where the exceptional set associated with $S(r, f)$ is of at most finite logarithmic measure.

In particular, the Lemma 2.2 and Lemma 2.3 are the difference analogue of the logarithmic derivative lemma.

Lemma 2.4. Let f be a non-constant meromorphic function in $\mathbb{C}$. Also let $a_1, a_2, \cdots, a_n$ $(n \geq 1)$ be distinct complex numbers. Then we get

$$\sum_{j=1}^{n} m\left(r, \frac{1}{f(z)-a_j}\right) \leq m\left(r, \frac{1}{f'(z)}\right) + S(r,f).$$

Proof. From Lemma 2.1, we have

$$\begin{aligned}
\sum_{j=1}^{n} m\left(r, \frac{1}{f(z)-a_j}\right) &= m\left(r, \sum_{j=1}^{n} \frac{1}{f(z)-a_j}\right) + O(1) \\
&= m\left(r, \frac{1}{f'(z)} \cdot \sum_{j=1}^{n} \frac{f'(z)}{f(z)-a_j}\right) + S(r,f) \\
&\leq m\left(r, \frac{1}{f'(z)}\right) + m\left(r, \sum_{j=1}^{n} \frac{f'(z)}{f(z)-a_j}\right) + S(r,f) \\
&= m\left(r, \frac{1}{f'(z)}\right) + \sum_{j=1}^{n} m\left(r, \frac{f'(z)}{f(z)-a_j}\right) + S(r,f).
\end{aligned}$$

Now using the Lemma on the Logarithmic derivative, we get

$$\sum_{j=1}^{n} m\left(r, \frac{1}{f(z)-a_j}\right) \leq m\left(r, \frac{1}{f'(z)}\right) + S(r,f).$$

The proof of Lemma 2.4 is completed. □

The Lemma 2.3 motivates us to prove the following:

Lemma 2.5. Let $c \in \mathbb{C}$, $n \in \mathbb{N}$ and f be a meromorphic function of finite order. Also let $a_1, a_2, \cdots, a_k$ $(k \geq 1)$ be distinct complex numbers. Then we have

$$\sum_{i=1}^{k} m\left(r, \frac{1}{f(z)-a_i}\right) \leq m\left(r, \frac{1}{\Delta_c^n f(z)}\right) + S(r,f).$$

Proof. Using Lemma 2.1, we obtain

$$\begin{aligned}
\sum_{i=1}^{k} m\left(r, \frac{1}{f(z)-a_i}\right) &= m\left(r, \sum_{i=1}^{k} \frac{1}{f(z)-a_i}\right) + O(1) \\
&= m\left(r, \frac{1}{\Delta_c^n f(z)} \cdot \sum_{i=1}^{k} \frac{\Delta_c^n f(z)}{f(z)-a_i}\right) + S(r,f) \\
&\leq m\left(r, \frac{1}{\Delta_c^n f(z)}\right) + m\left(r, \sum_{i=1}^{k} \frac{\Delta_c^n f(z)}{f(z)-a_i}\right) + S(r,f) \\
&= m\left(r, \frac{1}{\Delta_c^n f(z)}\right) + \sum_{i=1}^{k} m\left(r, \frac{\Delta_c^n f(z)}{f(z)-a_i}\right) + S(r,f).
\end{aligned}$$

By Lemma 2.3, we have

$$\sum_{i=1}^{k} m\left(r, \frac{1}{f(z)-a_i}\right) \leq m\left(r, \frac{1}{\Delta_c^n f(z)}\right) + S(r,f).$$

This proves the Lemma. □

Lemma 2.6. [17]. Suppose that f is a non-constant meromorphic function and $P(f) = a_0 f^p + a_1 f^{p-1} + \cdots + a_p (a_0 \neq 0)$ is a polynomial in f of degree p with constant coefficients $a_j(j = 0, 1, \cdots, p)$. Suppose furthermore that $b_j(j = 1, 2, \cdots, q)(q > p)$ are distinct values. Then

$$m\left(r, \frac{P(f)f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right) = S(r, f).$$

Lemma 2.7. [18]. If f_1 and f_2 are meromorphic functions in $|z| < R(R \leq \infty)$. Then

$$\begin{aligned} N(r, f_1(z)f_2(z)) - N\left(r, \frac{1}{f_1(z)f_2(z)}\right) &= N(r, f_1(z)) + N(r, f_2(z)) - N\left(r, \frac{1}{f_1(z)}\right) \\ &\quad -N\left(r, \frac{1}{f_2(z)}\right), \end{aligned}$$

where $0 < r < R$.

Lemma 2.8. [17]. Let f be a non-constant meromorphic function in the complex plane and $R(f) = \frac{P(f)}{Q(f)}$, where

$$P(f) = \sum_{k=0}^{p} a_k f^k$$

and

$$Q(f) = \sum_{j=0}^{q} b_j f^j$$

are two mutually prime polynomials in f. If the coefficients $\{a_k(z)\}$ and $\{b_j(z)\}$ are small functions of f and $a_p(z) \not\equiv 0$, $b_q(z) \not\equiv 0$, then

$$T(r, R(f)) = max\{p, q\}T(r, f).$$

Lemma 2.9. [9]. Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form

$$U(z, f)P(z, f) = Q(z, f),$$

where $U(z, f)$, $P(z, f)$ and $Q(z, f)$ are difference polynomials such that the total degree $degU(z, f) = n$ in $f(z)$ and its shifts and $degQ(z, f) \leq n$. If $U(z, f)$ contains exactly one term of maximal total degree in $f(z)$ and its shifts, then for each $\varepsilon > 0$,

$$m(r, P(z, f)) = O\left(r^{\rho-1+\varepsilon}\right) + S(r, f),$$

possible outside of an exceptional set of finite logarithmic measure.

3 Proof of the theorem 1.1

Let us suppose, on the contrary, that $\Delta_c^n f(z) \not\equiv f(z)$. To prove the Theorem 1.1, we consider a function defined as follow:

$$\psi(z) = \frac{(\Delta_c^n f(z))'}{\Delta_c^n f(z) - a} - \frac{f'(z)}{f(z) - a}. \tag{3.1}$$

By Lemma 2.3 and the Lemma on the Logarithmic Derivative, we have

$$\begin{aligned} m(r, \psi(z)) &= m(r, \frac{(\Delta_c^n f(z))'}{\Delta_c^n f(z) - a} - \frac{f'(z)}{f(z) - a}) \\ &\leq m\left(r, \frac{(\Delta_c^n f(z))'}{\Delta_c^n f(z) - a}\right) + m\left(r, \frac{f'(z)}{f(z) - a}\right) + \log 2 \\ &= S(r, f). \end{aligned} \tag{3.2}$$

We know that

$$\begin{aligned} T(r,\Delta_c^n f(z)) &= m(r,\Delta_c^n f(z)) + N(r,\Delta_c^n f(z)) + S(r,f) \\ &\leq m\left(r, \frac{\Delta_c^n f(z)}{f(z)}\cdot f(z)\right) + (n+1)N(r,f(z)) + S(r,f) \\ &\leq m\left(r, \frac{\Delta_c^n f(z)}{f(z)}\right) + m(r,f(z)) + S(r,f) \\ &= m(r,f(z)) + S(r,f) \\ &= T(r,f(z)) + S(r,f). \end{aligned} \tag{3.3}$$

Now the Logarithmic Derivative of $\frac{\Delta_c^n f(z)-a}{f(z)-a}$ is $\psi(z)$, the poles of $\psi(z)$ derive from the zeros and poles of $\frac{\Delta_c^n f(z)-a}{f(z)-a}$. Since $f(z)$ and $\Delta_c^n f(z)$ share the non-zero complex number a CM, then $\frac{\Delta_c^n f(z)-a}{f(z)-a}$ has no zeros and has at most $N(r,f(z))$ poles. Hence

$$\begin{aligned} N(r,\psi(z)) &\leq N(r,f(z)) \\ &= S(r,f). \end{aligned} \tag{3.4}$$

Thus from (3.2) and (3.4), we get

$$T(r,\psi(z)) = S(r,f). \tag{3.5}$$

We suppose that

$$\psi(z) \not\equiv 0.$$

Then dividing both sides of (3.1) by $f(z)-b$, we have

$$\frac{\psi(z)}{f(z)-b} = \frac{(\Delta_c^n f(z))'}{(\Delta_c^n f(z)-a)(f(z)-b)} - \frac{f'(z)}{(f(z)-a)(f(z)-b)}. \tag{3.6}$$

From (3.5) and (3.6) and using the Lemma on the Logarithmic Derivative, we have

$$
\begin{aligned}
m\left(r, \frac{1}{f(z)-b}\right) &= m\left(r, \frac{\psi(z)}{f(z)-b}\cdot\frac{1}{\psi(z)}\right) \\
&\leq m\left(r, \frac{\psi(z)}{f(z)-b}\right) + m\left(r, \frac{1}{\psi(z)}\right) \\
&\leq m\left(r, \frac{(\Delta_c^n f(z))^{'}}{(\Delta_c^n f(z)-a)(f(z)-b)} - \frac{f^{'}(z)}{(f(z)-a)(f(z)-b)}\right) \\
&\quad +T(r,\psi(z)) \\
&\leq m\left(r, \frac{(\Delta_c^n f(z))^{'}}{(\Delta_c^n f(z)-a)(f(z)-b)}\right) + m\left(r, \frac{f^{'}(z)}{(f(z)-a)(f(z)-b)}\right) \\
&\quad +S(r,f) \\
&\leq m\left(r, \frac{(\Delta_c^n f(z))^{'}}{\Delta_c^n f(z)(\Delta_c^n f(z)-a)}\cdot\frac{\Delta_c^n f(z)}{f(z)-b}\right) \\
&\quad +m\left(r, \frac{f^{'}(z)}{a-b}\left(\frac{1}{f(z)-a} - \frac{1}{f(z)-b}\right)\right) + S(r,f) \\
&\leq m\left(r, \frac{(\Delta_c^n f(z))^{'}}{a}\left(\frac{1}{\Delta_c^n f(z)-a} - \frac{1}{\Delta_c^n f(z)}\right)\right) \\
&\quad +m\left(\frac{\Delta_c^n f(z)}{f(z)-b}\right) + m\left(r, \frac{1}{a-b}\right) + m\left(r, \frac{f^{'}(z)}{f(z)-a}\right) \\
&\quad +m\left(\frac{f^{'}(z)}{f(z)-b}\right) + S(r,f) \\
&\leq m\left(r, \frac{1}{a}\right) + m\left(r, \frac{(\Delta_c^n f(z))^{'}}{\Delta_c^n f(z)-a}\right) + m\left(r, \frac{(\Delta_c^n f(z))^{'}}{\Delta_c^n f(z)}\right) + S(r,f) \\
&= S(r,f). \qquad (3.7)
\end{aligned}
$$

From (3.7) and by the First Fundamental Theorem, we see that

$$
T(r,f(z)) = N\left(r, \frac{1}{f(z)-b}\right) + S(r,f). \qquad (3.8)
$$

Since $f(z)$ and $\Delta_c^n f(z)$ share the non-zero complex numbers a, b CM, then by the Second Fundamental Theorem, we obtain

$$
\begin{aligned}
T(r,f(z)) &\leq N(r,f(z)) + N\left(r, \frac{1}{f(z)-a}\right) + N\left(r, \frac{1}{f(z)-b}\right) + S(r,f) \\
&= N\left(r, \frac{1}{f(z)-a}\right) + N\left(r, \frac{1}{f(z)-b}\right) + S(r,f) \\
&= N\left(r, \frac{1}{\Delta_c^n f(z)-a}\right) + N\left(r, \frac{1}{\Delta_c^n f(z)-b}\right) + S(r,f) \\
&\leq N\left(r, \frac{1}{\Delta_c^n f(z)-f(z)}\right) + S(r,f). \qquad (3.9)
\end{aligned}
$$

From the hypothesis of Theorem 1.1 and using Lemma 2.3, we get

$$\begin{aligned} N\left(r,\frac{1}{\Delta_c^n f(z)-f(z)}\right)+S(r,f) &= N\left(r,\frac{f(z)}{\Delta_c^n f(z)-f(z)}\right)+S(r,f)\\ &= N\left(r,\frac{1}{\frac{\Delta_c^n f(z)}{f(z)}-1}\right)+S(r,f)\\ &\leq T\left(r,\frac{1}{\frac{\Delta_c^n f(z)}{f(z)}-1}\right)+S(r,f)\\ &= T\left(r,\frac{\Delta_c^n f(z)}{f(z)}\right)+S(r,f)\\ &= m\left(r,\frac{\Delta_c^n f(z)}{f(z)}\right)+N\left(r,\frac{\Delta_c^n f(z)}{f(z)}\right)+S(r,f)\\ &= N\left(r,\frac{\Delta_c^n f(z)}{f(z)}\right)+S(r,f)\\ &\leq N\left(r,\frac{1}{f(z)}\right)+N(r,\Delta_c^n f(z))+S(r,f)\\ &\leq N\left(r,\frac{1}{f(z)}\right)+(n+1)N(r,f(z))+S(r,f)\\ &= N\left(r,\frac{1}{f(z)}\right)+S(r,f). \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we have

$$\begin{aligned} T(r,f(z)) &= N\left(r,\frac{1}{f(z)}\right)+S(r,f)\\ &= N\left(r,\frac{1}{f(z)-a}\right)+N\left(r,\frac{1}{f(z)-b}\right)+S(r,f). \end{aligned} \tag{3.11}$$

Now from (3.8) and (3.11), we get

$$N\left(r,\frac{1}{f(z)-a}\right)=S(r,f).$$

By the hypothesis of Theorem 1.1 and from the above equality, we have

$$\begin{aligned} N\left(r,\frac{1}{\Delta_c^n f(z)-a}\right) &= N\left(r,\frac{1}{f(z)-a}\right)\\ &= S(r,f). \end{aligned} \tag{3.12}$$

Since $f(z)$ and $\Delta_c^n f(z)$ share the non-zero constant b CM and using (3.3), (3.7), we get

$$\begin{aligned} m\left(r,\frac{1}{\Delta_c^n f(z)-b}\right)+N\left(r,\frac{1}{\Delta_c^n f(z)-b}\right) &= T(r,\Delta_c^n f(z))+S(r,f)\\ &\leq T(r,f(z))+S(r,f)\\ &= m\left(r,\frac{1}{f(z)-b}\right)+N\left(r,\frac{1}{f(z)-b}\right)\\ &\quad +S(r,f)\\ &= N\left(r,\frac{1}{f(z)-b}\right)+S(r,f)\\ &= N\left(r,\frac{1}{\Delta_c^n f(z)-b}\right)+S(r,f). \end{aligned}$$

Therefore

$$m\left(r, \frac{1}{\Delta_c^n f(z)-b}\right) = S(r,f).$$

By Lemma 2.4, we obtain

$$\begin{aligned} m\left(r, \frac{1}{\Delta_c^n f(z)}\right) + m\left(r, \frac{1}{\Delta_c^n f(z)-a}\right) + m\left(r, \frac{1}{\Delta_c^n f(z)-b}\right) &\leq m\left(r, \frac{1}{(\Delta_c^n f(z))'}\right) \\ &+S(r,f). \end{aligned} \tag{3.13}$$

Using Lemma 2.5, we have

$$m\left(r, \frac{1}{f(z)-a}\right) + m\left(r, \frac{1}{f(z)-b}\right) \leq m\left(r, \frac{1}{\Delta_c^n f(z)}\right) + S(r,f). \tag{3.14}$$

From (3.12), we get

$$\begin{aligned} N\left(r, \frac{1}{\Delta_c^n f(z)-a}\right) + N\left(r, \frac{1}{\Delta_c^n f(z)-b}\right) &= N\left(r, \frac{1}{\Delta_c^n f(z)-b}\right) \\ &+S(r,f). \end{aligned} \tag{3.15}$$

Now from (3.11), we have

$$N\left(r, \frac{1}{f(z)-a}\right) + N\left(r, \frac{1}{f(z)-b}\right) = T(r, f(z)) + S(r,f). \tag{3.16}$$

Adding both sides of (3.13), (3.14), (3.15) and (3.16), we have

$$\begin{aligned} &T\left(r, \tfrac{1}{\Delta_c^n f(z)-a}\right) + T\left(r, \tfrac{1}{\Delta_c^n f(z)-b}\right) + T\left(r, \tfrac{1}{f(z)-a}\right) + T\left(r, \tfrac{1}{f(z)-b}\right) \\ &\leq m\left(r, \tfrac{1}{(\Delta_c^n f(z))'}\right) + N\left(r, \tfrac{1}{\Delta_c^n f(z)-b}\right) + T(r, f(z)) + S(r,f) \\ &\leq T\left(r, \tfrac{1}{(\Delta_c^n f(z))'}\right) + N\left(r, \tfrac{1}{\Delta_c^n f(z)-b}\right) + T(r, f(z)) + S(r,f). \end{aligned}$$

On the other hand, using (3.3), we can easily see that

$$\begin{aligned} T\left(r, \frac{1}{(\Delta_c^n f(z))'}\right) &= m(r, (\Delta_c^n f(z))') + N(r, (\Delta_c^n f(z))') + O(1) \\ &= m\left(r, \frac{(\Delta_c^n f(z))'}{\Delta_c^n f(z)} \cdot \Delta_c^n f(z)\right) + N(r, \Delta_c^n f(z)) \\ &\quad +\overline{N}(r, \Delta_c^n f(z)) + S(r,f) \\ &\leq m\left(r, \frac{(\Delta_c^n f(z))'}{\Delta_c^n f(z)}\right) + m(r, \Delta_c^n f(z)) + (n+1)N(r, f(z)) \\ &\quad +(n+1)\overline{N}(r, f(z)) + S(r,f) \\ &= T(r, \Delta_c^n f(z)) + S(r,f) \\ &\leq T(r, f(z)) + S(r,f). \end{aligned}$$

Combining above two inequality and by the First Fundamental Theorem, we obtain

$$T(r, f(z)) = S(r,f),$$

which is a contradiction.

Therefore

$$\psi(z) \equiv 0.$$

Hence from (3.1), we get

$$\frac{(\Delta_c^n f(z))'}{\Delta_c^n f(z) - a} \equiv \frac{f'(z)}{f(z) - a}.$$

Integrating above identity, we have

$$\frac{\Delta_c^n f(z) - a}{f(z) - a} \equiv A, \tag{3.17}$$

where $A(\neq 0)$ is a constant.

Using the similar arguments as above, by the hypothesis $f(z)$ and $\Delta_c^n f(z)$ share the non-zero complex number b CM, we get

$$\frac{\Delta_c^n f(z) - b}{f(z) - b} \equiv B, \tag{3.18}$$

where $B(\neq 0)$ is a constant.

If $A = 1$ and $B = 1$, then from (3.17) and (3.18), we have

$$\Delta_c^n f(z) \equiv f(z),$$

which is a contradiction.

We now verify that $A \neq 1$ and $B \neq 1$.
Then from (3.17) and (3.18), we get

$$(A - B)f(z) = b - a + Aa - Bb. \tag{3.19}$$

If $A \neq B$, then f is a constant. Which leads towards a contradiction.
Hence

$$A = B.$$

Thus from (3.19), we obtain

$$A(a - b) = a - b.$$

This implies $A = B = 1$. Which is again a contradiction.

Therefore indeed we get

$$\Delta_c^n f(z) \equiv f(z).$$

This completes the proof of Theorem 1.1.

4 Proof of the theorem 1.2

Let us suppose, on the contrary, that $\Delta_c^n f(z) \not\equiv f(z)$. To prove the Theorem 1.2, let us consider two functions defined as follow:

$$\alpha(z) = \frac{f'(z)(\Delta_c^n f(z) - f(z))}{(f(z) - a)(f(z) - b)} \tag{4.1}$$

and

$$\beta(z) = \frac{(\Delta_c^n f(z))^{'}(\Delta_c^n f(z) - f(z))}{(\Delta_c^n f(z) - a)(\Delta_c^n f(z) - b)}. \tag{4.2}$$

We know from the hypothesis of Theorem 1.2 that $\alpha(z)$ and $\beta(z)$ are entire functions. Then by Lemma 2.3 and the Lemma on the Logarithmic Derivative, we have

$$\begin{aligned} T(r, \alpha(z)) &= m(r, \alpha(z)) \\ &= m\left(r, \frac{f^{'}(z)(\Delta_c^n f(z) - f(z))}{(f(z) - a)(f(z) - b)}\right) \\ &\le m\left(r, \frac{f^{'}(z)}{f(z) - b}\right) + m\left(r, \frac{\Delta_c^n f(z) - f(z)}{f(z) - a}\right) \\ &\le m\left(r, \frac{\Delta_c^n f(z)}{f(z) - a}\right) + m\left(r, \frac{f(z)}{f(z) - a}\right) + S(r, f) \\ &= m\left(r, \left(1 + \frac{a}{f(z) - a}\right)\right) + S(r, f) \\ &\le m\left(r, \frac{1}{f(z) - a}\right) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{4.3}$$

Since $f(z)$ and $\Delta_c^n f(z)$ share the non-zero complex number a CM, then we get

$$\frac{\Delta_c^n f(z) - a}{f(z) - a} = e^{\gamma(z)}, \tag{4.4}$$

where $\gamma(z)$ is a polynomial.

From the hypothesis of Theorem 1.2 and by Lemma 2.3, we have

$$\begin{aligned} T(r, e^{\gamma(z)}) &= m(r, e^{\gamma(z)}) \\ &= m\left(r, \frac{\Delta_c^n f(z) - a}{f(z) - a}\right) \\ &\le m\left(r, \frac{\Delta_c^n f(z)}{f(z) - a}\right) + m\left(r, \frac{a}{f(z) - a}\right) + \log 2 \\ &\le m\left(r, \frac{1}{f(z) - a}\right) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{4.5}$$

Now from (4.4), we get

$$\Delta_c^n f(z) = e^{\gamma(z)} f(z) + a(1 - e^{\gamma(z)}).$$

From (4.5) and the above equality, we obtain

$$\begin{aligned} T(r, \Delta_c^n f(z)) &= T(r, e^{\gamma(z)} f(z) + a(1 - e^{\gamma(z)})) \\ &\le T(r, e^{\gamma(z)} f(z)) + T(r, a(1 - e^{\gamma(z)})) + \log 2 \\ &\le T(r, e^{\gamma(z)}) + T(r, f(z)) + T(r, a) + T(r, (1 - e^{\gamma(z)})) + S(r, f) \\ &= T(r, f(z)) + S(r, f). \end{aligned} \tag{4.6}$$

By the assumption $m\left(r, \frac{1}{f(z)-a}\right) = S(r,f)$ and Lemma 2.3, Lemma 2.6 and for any $d \in \mathbb{C} \setminus \{a, b\}$, we have

$$\begin{aligned}
m\left(r, \frac{1}{f(z)-d}\right) &= m\left(r, \frac{f'(z)(\Delta_c^n f(z) - f(z))}{(f(z)-a)(f(z)-b)(f(z)-d)\alpha(z)}\right) \\
&\leq m\left(r, \frac{\Delta_c^n f(z) - f(z)}{(f(z)-a)}\right) + m\left(r, \frac{f'(z)}{(f(z)-b)(f(z)-d)}\right) \\
&\quad + m\left(r, \frac{1}{\alpha(z)}\right) \\
&\leq m\left(r, \frac{\Delta_c^n f(z)}{f(z)-a}\right) + m\left(r, \frac{f(z)}{f(z)-a}\right) + S(r,f) \\
&= m\left(r, \left(1 + \frac{a}{f(z)-a}\right)\right) + S(r,f) \\
&\leq m\left(r, \frac{1}{f(z)-a}\right) + S(r,f) \\
&= S(r,f). \qquad (4.7)
\end{aligned}$$

By Nevanlinna's Second Fundamental Theorem, we see that

$$\begin{aligned}
T(r, f(z)) &\leq \overline{N}\left(r, \frac{1}{f(z)-a}\right) + \overline{N}\left(r, \frac{1}{f(z)-b}\right) + \overline{N}(r, f(z)) + S(r,f) \\
&\leq \overline{N}\left(r, \frac{1}{f(z)-a}\right) + \overline{N}\left(r, \frac{1}{f(z)-b}\right) + S(r,f). \qquad (4.8)
\end{aligned}$$

Now from (4.1), (4.3) and Lemma 2.3, we have

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{f(z)-a}\right) + \overline{N}\left(r, \frac{1}{f(z)-b}\right) &= N\left(r, \frac{f'(z)}{(f(z)-a)(f(z)-b)}\right) + S(r,f) \\
&= N\left(r, \frac{\alpha(z)}{\Delta_c^n f(z) - f(z)}\right) + S(r,f) \\
&\leq N\left(r, \frac{1}{\Delta_c^n f(z) - f(z)}\right) + N(r, \alpha(z)) \\
&\quad + S(r,f) \\
&\leq T(r, \Delta_c^n f(z) - f(z)) + S(r,f) \\
&= m(r, \Delta_c^n f(z) - f(z)) + N(r, \Delta_c^n f(z) - f(z)) \\
&\quad + S(r,f) \\
&\leq m\left(r, f(z) \cdot \left(\frac{\Delta_c^n f(z)}{f(z)} - 1\right)\right) + N(r, \Delta_c^n f(z)) \\
&\quad + N(r, f(z)) + S(r,f) \\
&\leq m(r, f(z)) + m\left(r, \frac{\Delta_c^n f(z)}{f(z)} - 1\right) \\
&\quad + (n+1)N(r, f(z)) + S(r,f) \\
&= T(r, f(z)) + S(r,f).
\end{aligned}$$

From (4.8) and and the above inequality, we obtain

$$\overline{N}\left(r, \frac{1}{f(z)-a}\right) + \overline{N}\left(r, \frac{1}{f(z)-b}\right) = T(r, f(z)) + S(r, f). \tag{4.9}$$

Since $f(z)$ and $\Delta_c^n f(z)$ share the non-zero complex number a CM and the non-zero complex number b IM, by using (4.6) and (4.9), together with the Second Fundamental Theorem, we can deduce that

$$\begin{aligned}
2T(r, f(z)) &= 2T(r, \Delta_c^n f(z)) + S(r, f) \\
&\leq \overline{N}\left(r, \frac{1}{\Delta_c^n f(z)-a}\right) + \overline{N}\left(r, \frac{1}{\Delta_c^n f(z)-b}\right) + \overline{N}\left(r, \frac{1}{\Delta_c^n f(z)-d}\right) \\
&\quad + S(r, f) \\
&\leq \overline{N}\left(r, \frac{1}{f(z)-a}\right) + \overline{N}\left(r, \frac{1}{f(z)-b}\right) \\
&\quad + T\left(r, \frac{1}{\Delta_c^n f(z)-d}\right) - m\left(r, \frac{1}{\Delta_c^n f(z)-d}\right) + S(r, f) \\
&\leq T(r, f(z)) + T(r, \Delta_c^n f(z)) - m\left(r, \frac{1}{\Delta_c^n f(z)-d}\right) + S(r, f) \\
&= 2T(r, f(z)) - m\left(r, \frac{1}{\Delta_c^n f(z)-d}\right) + S(r, f).
\end{aligned}$$

This implies

$$m\left(r, \frac{1}{\Delta_c^n f(z)-d}\right) = S(r, f). \tag{4.10}$$

By Lemma 2.7 and from (4.6), (4.7), (4.10), we can easily see that

$$\begin{aligned}
m\left(r,\frac{f(z)-d}{\Delta_c^n f(z)-d}\right)-m\left(r,\frac{\Delta_c^n f(z)-d}{f(z)-d}\right) &= T\left(r,\frac{f(z)-d}{\Delta_c^n f(z)-d}\right)-N\left(r,\frac{f(z)-d}{\Delta_c^n f(z)-d}\right)\\
&\quad -T\left(r,\frac{\Delta_c^n f(z)-d}{f(z)-d}\right)+N\left(r,\frac{\Delta_c^n f(z)-d}{f(z)-d}\right)\\
&= N\left(r,\frac{\Delta_c^n f(z)-d}{f(z)-d}\right)-N\left(r,\frac{f(z)-d}{\Delta_c^n f(z)-d}\right)\\
&\quad +O(1)\\
&\leq N(r,\Delta_c^n f(z)-d)+N\left(r,\frac{1}{f(z)-d}\right)\\
&\quad -N(r,f(z)-d)-N\left(r,\frac{1}{\Delta_c^n f(z)-d}\right)\\
&\quad +S(r,f)\\
&= N\left(r,\frac{1}{f(z)-d}\right)-N\left(r,\frac{1}{\Delta_c^n f(z)-d}\right)\\
&\quad +S(r,f)\\
&= T\left(r,\frac{1}{f(z)-d}\right)-m\left(r,\frac{1}{f(z)-d}\right)\\
&\quad -T\left(r,\frac{1}{\Delta_c^n f(z)-d}\right)+m\left(r,\frac{1}{\Delta_c^n f(z)-d}\right)\\
&\quad +S(r,f)\\
&= T\left(r,\frac{1}{f(z)-d}\right)-T\left(r,\frac{1}{\Delta_c^n f(z)-d}\right)\\
&\quad +S(r,f)\\
&= T(r,f(z))-T(r,\Delta_c^n f(z))+S(r,f)\\
&= T(r,f(z))-T(r,f(z))+S(r,f)\\
&= S(r,f). \qquad (4.11)
\end{aligned}$$

Now from (4.7), (4.11) and using Lemma 2.3, we obtain

$$\begin{aligned}
m\left(r,\frac{f(z)-d}{\Delta_c^n f(z)-d}\right) &\leq m\left(r,\frac{\Delta_c^n f(z)-d}{f(z)-d}\right)+S(r,f)\\
&\leq m\left(r,\frac{\Delta_c^n f(z)}{f(z)-d}\right)+m\left(r,\frac{d}{f(z)-d}\right)+S(r,f)\\
&\leq m\left(r,\frac{1}{f(z)-d}\right)+S(r,f)\\
&= S(r,f). \qquad (4.12)
\end{aligned}$$

By Lemma 2.6 and from (4.2), (4.12), we have

$$\begin{aligned}
T(r,\beta(z)) &= m(r,\beta(z)) \\
&= m\left(r, \frac{(\Delta_c^n f(z))'(\Delta_c^n f(z)-f(z))}{(\Delta_c^n f(z)-a)(\Delta_c^n f(z)-b)}\right) \\
&= m\left(r, \frac{(\Delta_c^n f(z))'(\Delta_c^n f(z)-f(z))(\Delta_c^n f(z)-d)}{(\Delta_c^n f(z)-a)(\Delta_c^n f(z)-b)(\Delta_c^n f(z)-d)}\right) \\
&\le m\left(r, \frac{(\Delta_c^n f(z))'(\Delta_c^n f(z)-d)}{(\Delta_c^n f(z)-a)(\Delta_c^n f(z)-b)}\right) + m\left(r, \frac{\Delta_c^n f(z)-f(z)}{\Delta_c^n f(z)-d}\right) \\
&= m\left(r, \left(1-\frac{f(z)-d}{\Delta_c^n f(z)-d}\right)\right) + S(r,f) \\
&\le m\left(r, \frac{f(z)-d}{\Delta_c^n f(z)-d}\right) + S(r,f) \\
&= S(r,f). \qquad (4.13)
\end{aligned}$$

Now we consider z_0 be any zero of $f(z)-b$ and $\Delta_c^n f(z)-b$ with multiplicities p and q, respectively.

From (4.1) and (4.2), we have

$$\alpha(z_0) = \frac{p}{b-a} \cdot \left(\frac{\Delta_c^n f(z)-f(z)}{z-z_0}\right)|_{z=z_0} \qquad (4.14)$$

and

$$\beta(z_0) = \frac{q}{b-a} \cdot \left(\frac{\Delta_c^n f(z)-f(z)}{z-z_0}\right)|_{z=z_0}. \qquad (4.15)$$

From (4.14) and (4.15), we get

$$q\alpha(z_0) = p\beta(z_0).$$

Again we let z_1 be any zero of $f(z)-a$ and $\Delta_c^n f(z)-a$ with multiplicities p and q, respectively. Then similarly, we can prove that

$$q\alpha(z_1) = p\beta(z_1).$$

We shall the following two cases.

Case 1. First we suppose that

$$q\alpha(z) - p\beta(z) \not\equiv 0.$$

By the reasoning as mentioned above, we deduce that z_j is a zero of $f(z)-b$ and $\Delta_c^n f(z)-b$ or $f(z)-a$ and $\Delta_c^n f(z)-a$ with multiplicities p and q must be the zero of $q\alpha(z)-p\beta(z)$.

It follows from this and the fact that $\alpha(z)$ and $\beta(z)$ are small functions of $f(z)$, we have

$$\begin{aligned}
\overline{N}_{(p,q)}\left(r, \frac{1}{f(z)-a}\right) + \overline{N}_{(p,q)}\left(r, \frac{1}{f(z)-b}\right) &\le \overline{N}\left(r, \frac{1}{q\alpha(z)-p\beta(z)}\right) \\
&\le T(r, q\alpha(z)-p\beta(z)) \\
&\le T(r,\alpha(z)) + T(r,\beta(z)) + S(r,f) \\
&= S(r,f). \qquad (4.16)
\end{aligned}$$

By using (4.6) and (4.16) toghether with Second Fundamental Theorem, we obtain

$$\begin{aligned}
T(r,f(z)) &\leq \overline{N}\left(r,\frac{1}{f(z)-a}\right)+\overline{N}\left(r,\frac{1}{f(z)-b}\right)+S(r,f)\\
&= \sum_{p,q}\left(\overline{N}_{(p,q)}\left(r,\frac{1}{f(z)-a}\right)+\overline{N}_{(p,q)}\left(r,\frac{1}{f(z)-b}\right)\right)\\
&\quad +S(r,f)\\
&= \sum_{p+q<6}\left(\overline{N}_{(p,q)}\left(r,\frac{1}{f(z)-a}\right)+\overline{N}_{(p,q)}\left(r,\frac{1}{f(z)-b}\right)\right)\\
&\quad +\sum_{p+q\geq 6}\left(\overline{N}_{(p,q)}\left(r,\frac{1}{f(z)-a}\right)+\overline{N}_{(p,q)}\left(r,\frac{1}{f(z)-b}\right)\right)\\
&\quad +S(r,f)\\
&\leq \frac{1}{6}\sum_{p+q\geq 6}\left(N_{(p,q)}\left(r,\frac{1}{f(z)-a}\right)+N_{(p,q)}\left(r,\frac{1}{\Delta_c^n f(z)-a}\right)\right)\\
&\quad +\frac{1}{6}\sum_{p+q\geq 6}\left(N_{(p,q)}\left(r,\frac{1}{f(z)-b}\right)+N_{(p,q)}\left(r,\frac{1}{\Delta_c^n f(z)-b}\right)\right)\\
&\quad +S(r,f)\\
&\leq \frac{1}{6}\left(N_{(p,q)}\left(r,\frac{1}{f(z)-a}\right)+N_{(p,q)}\left(r,\frac{1}{\Delta_c^n f(z)-a}\right)\right)\\
&\quad +\frac{1}{6}\left(N_{(p,q)}\left(r,\frac{1}{f(z)-b}\right)+N_{(p,q)}\left(r,\frac{1}{\Delta_c^n f(z)-b}\right)\right)\\
&\quad +S(r,f)\\
&\leq \frac{1}{3}T(r,f(z))+\frac{1}{3}T(r,\Delta_c^n f(z))+S(r,f)\\
&= \frac{1}{3}T(r,f(z))+\frac{1}{3}T(r,f(z))+S(r,f)\\
&= \frac{2}{3}T(r,f(z))+S(r,f).
\end{aligned}$$

This implies that

$$T(r,f(z))=S(r,f),$$

which is a contradiction.

Case 2. Next we suppose that

$$q\alpha(z)-p\beta(z)\equiv 0.$$

Then from (4.1) and (4.2), we have

$$q\frac{f'(z)}{(f(z)-a)(f(z)-b)}\equiv p\frac{(\Delta_c^n f(z))'}{(\Delta_c^n f(z)-a)(\Delta_c^n f(z)-b)}.$$

Therefore

$$q\left(\frac{f'(z)}{f(z)-a}-\frac{f'(z)}{f(z)-b}\right)\equiv p\left(\frac{(\Delta_c^n f(z))'}{\Delta_c^n f(z)-a}-\frac{(\Delta_c^n f(z))'}{\Delta_c^n f(z)-b}\right). \tag{4.17}$$

Integrating both sides of (4.17), we get

$$\left(\frac{f(z)-a}{f(z)-b}\right)^q\equiv C\left(\frac{\Delta_c^n f(z)-a}{\Delta_c^n f(z)-b}\right)^p, \tag{4.18}$$

where $C(\neq 0)$ is a constant.

From (4.6) and (4.18) and using Lemma 2.8, we obtain

$$\begin{aligned} qT(r, f(z)) &= pT(r, \Delta_c^n f(z)) + O(1) \\ &= pT(r, f(z)) + S(r, f), \end{aligned}$$

This implies that

$$p = q.$$

Now from (4.18), we get

$$\left(\frac{f(z)-a}{f(z)-b}\right) \equiv D\left(\frac{\Delta_c^n f(z)-a}{\Delta_c^n f(z)-b}\right), \tag{4.19}$$

where $D(\neq 0)$ is a constant.
Since $\Delta_c^n f(z) \not\equiv f(z)$, by using (4.19), we have $D \neq 1$.
From (4.19), we get

$$f(z)[(1-D)\Delta_c^n f(z) + aD - b] = (a - bD)\Delta_c^n f(z) + ab(D-1).$$

By Lemma 2.9 and using above equality, we obtain

$$m(r, \Delta_c^n f(z)) = S(r, f).$$

Since $f(z)$ is an entire function, it follows that

$$T(r, \Delta_c^n f(z)) = S(r, f).$$

On the other hand, we can easily see that

$$\begin{aligned} T(r, f(z)) &= m(r, f(z)) \\ &= m\left(r, \frac{f(z)}{\Delta_c^n f(z)} \cdot \Delta_c^n f(z)\right) \\ &\leq m\left(r, \frac{f(z)}{\Delta_c^n f(z)}\right) + m(r, \Delta_c^n f(z)) \\ &= S(r, f), \end{aligned}$$

which is again a contradiction.
Therefore indeed we have

$$\Delta_c^n f(z) \equiv f(z).$$

This completes the proof of Theorem 1.2.

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