**Palestine Journal of Mathematics**

**Vol. 10(2)(2021)** , 432–439 

# New idea of generalized of variant of d'Alembert functional equation

Hajira Dimou, Abdellatif Chahbi and Samir Kabbaj

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 39B52; Secondary 46E40.

Keywords and phrases: Functional equation; Variant of d'Alembert; Semigroup; Involutive automorphism; Multiplicative function; Hilbert valued function; Hadamard product.

**Abstract**. Let $(S,.)$ be a semigroup and let $\sigma \in Hom(S,S)$ satisfies $\sigma\circ\sigma = id$. In this paper we show that any solution $f : S \to \mathbb{C}$ of the functional equation

$$\chi_1(y)f(xy) + \chi_2(y)f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in S,$$

has the form $f = \frac{\mu+\chi_2\chi_1\mu\circ\sigma}{2}$, where $\mu$ is a multiplicative function on $S$ and $\chi_1,\chi_2 : S \to$

$(\mathbb{C}\backslash\{0\},.)$ be two characters on $S$ (i.e, $\chi_1(xy) = \chi_1(x)\chi_1(y)$ and $\chi_2(xy) = \chi_2(x)\chi_2(y)$ $for\ all\ x, y \in S$) such that $\chi_2(x\sigma(x)) = 1\ for\ all\ x \in S$. These results are applied to study the solutions of this equation defined on a semigroup and taking valued in a complex Hilbert space with the Hadamard product.

## 1 Introduction

We recall that a semigroup $S$ is a non-empty set equipped with an associative operation and we write the operation multiplicatively as a function $\mu : S \to \mathbb{C}$ such that

$$\mu(xy) = \mu(x)\mu(y)\ for\ all\ x, y \in S.$$

Let $(S,.)$ be a semigroup and $\sigma : S \to S$ a homomorphism such that $\sigma\circ\sigma = id$. We say that $\chi : S \to (\mathbb{C}\backslash\{0\},.)$ is a character on $S$ if $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in S$.
If $S$ is a topological space, then we let $C(S)$ denote the algebra of continuous functions from $S$ into $\mathbb{C}$.

The symmetrized multiplicative Cauchy equation is of the following form

$$f(xy) + f(yx) = 2f(x)f(y), \quad x, y \in S, \tag{1.1}$$

where $f : S \to \mathbb{C}$ is the unknown function. The complex-valued solutions of (1.1) are known to be the multiplicative functions on $S$ (see, for instance, [10] or [11, Theorem 3.21]).
Equation (1.1) is a special case of the following variant of d'Alembert's functional equation:

$$f(xy) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in S, \tag{1.2}$$

which was introduced and solved by Stetkær [12]. Some information, applications, and numerous references concerning (1.1), (1.2), the d'Alembert's functional equation

$$f(x + y) + f(x - y) = 2f(x)f(y), \quad x, y \in \mathbb{R},$$

and their further generalizations can be found, e.g, in [1, 9, 12].

H. Stetkær [13] obtained the complex valued solutions of the following version of d'Alembert's functional equation

$$f(xy) + \chi(y)f(xy^{-1}) = 2f(x)f(y), \quad x, y \in G, \tag{1.3}$$

where $\chi : G \to \mathbb{C}$ is a character of $G$ and $G$ is a group. The non-zero solutions of equation (1.3) are the normalized traces of certain representation of the group $G$ on $\mathbb{C}^2$.

E. Elqorachi and A. Redouani [8] studied the solution of the following variant of d'Alembert's functional equation

$$f(xy) + \chi(y)f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in G, \tag{1.4}$$

where $G$ is a group, $\chi : G \to \mathbb{C}\backslash\{0\}$ is a character on $G$ and $\sigma$ is an involutive automorphism of $G$ such that $\sigma \circ \sigma = id$ and $\chi(x\sigma(x)) = 1$ for all $x \in G$.

The purpose of the present paper is to solve the following functional equation

$$\chi_1(y)f(xy) + \chi_2(y)f(\sigma(y)x) = 2f(x)f(y)\ , x, y \in S, \tag{1.5}$$

where $S$ is a semigroup, $\chi_1, \chi_2 : S \to \mathbb{C}\backslash\{0\}$ be two characters on $S$ and $\sigma$ is an involutive homomorphism of $S$ such that $\sigma \circ \sigma = id$ and $\chi_2(x\sigma(x)) = 1$ $for\ all$ $x \in S$. This equation is a natural generalization of the equations (1.1), (1.2) and (1.4). As an application we study an extension of the equation (1.5) above to a situation where the unknown function $f$ map a semigroup into a complex Hilbert space $H$ with the Hadamard product. Our considerations refer mainly to result Stetkær [11], Zeglami [14], Dimou [4, 5].

Let $H$ be a separable Hilbert space with a orthonormal basis $\{e_n, n \in \mathbb{N}\}$. For two vectors $x, y \in H$, the Hadamard product, also known as the entrywise product on the Hilbert space $H$ is defined by

$$x * y = \sum_{n=0}^{\infty} \langle x, e_n\rangle\langle y, e_n\rangle e_n, \quad x, y \in H. \tag{1.6}$$

The Cauchy-Schwarz inequality together with the Parseval identity ensure that the Hadamard multiplication is well defined. In fact,

$$\|x * y\| \leq (\sum_{n=0}^{\infty} |\langle x, e_n\rangle|^2)^{\frac{1}{2}} (\sum_{n=0}^{\infty} |\langle y, e_n\rangle|^2)^{\frac{1}{2}} = ||x||||y||, \tag{1.7}$$

and second is to obtain a characterization, in terms of multiplicative functions, the continuous of the Hilbert space valued functional equation by Hadamard product:

$$\chi_1(y)f(xy) + \chi_2(y)f(\sigma(y)x) = 2f(x) * f(y)\ , x, y \in S, \tag{1.8}$$

where $f : S \to H$ are the unknown function.
In what follows $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ stand for the sets of all positive integers, real numbers and complex numbers, respectively. $S$ is a semigroup and $H$ is a separable complex Hilbert space with a fixed countable orthonormal basis $\{e_n, n \in \mathbb{N}\}$ and with the Hadamard product defined as in (1.6).

## 2 Solutions of the functional equation (1.5)

In this section, using elementary methods, we find all solutions of (1.5) on semigroups in terms of multiplicative functions. We also note that the sine addition law on semigroups given in [6, 11] is a key ingredient of the proof of Theorem 2.1. The following theorem is our main result which we describe the solutions of equation (1.5).

**Theorem 2.1.** *Let $S$ be a semigroup, $\sigma \in Hom(S, S)$ such that $\sigma \circ \sigma = id$ (where id denotes the identity map) and $\chi_1, \chi_2$ be two characters of $S$ with $\chi_2(x\sigma(x)) = 1$ for all $x \in S$. The solutions $f : S \to \mathbb{C}$ of (1.5) are the functions of the form $f = \frac{\mu + \chi_2\chi_1\mu\circ\sigma}{2}$, where $\mu : S \to \mathbb{C}$ is a multiplicative function such that :*
*(i) $\chi_1^2\mu = \mu$ and $\mu = \chi_2\mu \circ \sigma$, or*
*(ii) $\chi_1\mu = \mu$ and $\mu = \chi_2\mu \circ \sigma$.*
*If $S$ is a topological semigroup and $f \in C(S)$, then $\mu, \chi_2\chi_1\mu \circ \sigma \in C(S)$.*

**Proof.** The computations in our proof are similar to the ones found in Stetkær [12]. Let $x, y, z \in S$ be arbitrary. Replacing $x$ by $xy$ and $y$ by $z$ in (1.5), we get

$$\chi_1(z)f(xyz) + \chi_2(z)f(\sigma(z)xy) = 2f(xy)f(z). \tag{2.1}$$

If we replace $x$ by $\sigma(z)x$ in (1.5), then we obtain

$$\chi_1(y)f(\sigma(z)xy) + \chi_2(y)f(\sigma(yz)x) = 2f(\sigma(z)x)f(y). \tag{2.2}$$

Replacing $y$ by $z$ in (1.5), we obtain

$$\chi_1(z)f(xz) + \chi_2(z)f(\sigma(z)x) = 2f(x)f(z). \tag{2.3}$$

Using $\chi_2(z\sigma(z)) = 1$, we get from (2.3) that

$$f(\sigma(z)x) = \chi_2(\sigma(z))[2f(x)f(z) - \chi_1(z)f(xz)]. \tag{2.4}$$

It follows from (2.2) and (2.4) that

$$\chi_1(y)f(\sigma(z)xy) + \chi_2(y)f(\sigma(yz)x) = 2\chi_2(\sigma(z))f(y)[2f(x)f(z) - \chi_1(z)f(xz)]. \tag{2.5}$$

Replacing $y$ by $yz$ in (1.5) and using the condition $\chi_2(z\sigma(z)) = 1$, we obtain

$$\chi_2(y)f(\sigma(yz)x) = \chi_2(\sigma(z))[2f(x)f(yz) - \chi_1(yz)f(xyz)], \tag{2.6}$$

according to (2.5) and (2.6), we get

$$\chi_2(z)\chi_1(y)f(\sigma(z)xy)+[2f(x)f(yz)-\chi_1(yz)f(xyz)] = 2f(y)[2f(x)f(z)-\chi_1(z)f(xz)]. \tag{2.7}$$

Subtracting this equation from $\chi_1(y)$ and multiplying by equation (2.1), we get after some simplifications that

$$\chi_1(yz)f(xyz) - f(x)f(yz) = f(y)[\chi_1(z)f(xz) - f(x)f(z)] + f(z)[\chi_1(y)f(xy) - f(x)f(y)]. \tag{2.8}$$

With the notation $f_x(y) = \chi_1(y)f(xy) - f(x)f(y)$, we can reformulate (2.8) as the following

$$f_a(xy) = f_a(x)f(y) + f_a(y)f(x). \tag{2.9}$$

This shows that the pair $(f_a, f)$ satisfies the sine addition formula for any $a \in S$.
Thus, we study the following cases:
**Case 1:** If $f_a = 0$ for all $a \in S$, then $f$ satisfies the functional equation

$$\chi_1(y)f(xy) = f(x)f(y) \ for\ all\ x, y \in S. \tag{2.10}$$

This implies that

$$\chi_2(y)f(\sigma(y)x) = f(x)f(y) \ for\ all\ x, y \in S. \tag{2.11}$$

Making the substitutions $(y, \sigma(x))$ in (1.5), we obtain

$$\chi_1(\sigma(x))f(y\sigma(x)) + \chi_2(\sigma(x))f(xy) = 2f(y)f(\sigma(x)). \tag{2.12}$$

By similar method, we get

$$\chi_2(\sigma(x))f(xy) = \chi_1(\sigma(x))f(y\sigma(x)) = f(y)f(\sigma(x)). \tag{2.13}$$

Multiplying equation (2.13) by $\chi_2(x)$, we get

$$f(xy) = \chi_2(x)f(y)f(\sigma(x)) \tag{2.14}$$

and

$$\chi_1(y)f(xy) = \chi_1(y)\chi_2(x)f(y)f(\sigma(x)) = f(x)f(y) \ for\ all\ x, y \in S. \tag{2.15}$$

Since $f \neq 0$, we obtain

$$f(y) = \chi_1(y)f(y) = \chi_1^2(y)f(y) \textit{ and } \chi_2(x)f(\sigma(x)) = f(x) \textit{ for all } x, y \in S. \tag{2.16}$$

Hence, it is not hard, by a simple computations, to check that $f$ is a multiplicative function. This implies that $f = \frac{\varphi+\chi_2\chi_1\varphi\circ\sigma}{2}$, where $f = \varphi$ is multiplicative .

**Case 2:** If $f_a \neq 0$ for some $a \in S$, we get, from the known solution of the sine addition law (see, for instance [6] or [11, Theorem 4.1]), that there exist two multiplicative functions $\psi_1, \psi_2 : S \to \mathbb{C}$ such that

$$f = \frac{\psi_1 + \psi_2}{2}.$$

If $\psi_1 = \psi_2$, then letting $\eta := \psi_1$, we have $f = \eta$. Substituting $f = \eta$ into (1.5) we get that

$$\chi_1\eta + \chi_2\eta\circ\sigma = 2\eta.$$

Therefore, $\eta = \chi_1\eta = \chi_2\eta\circ\sigma$ (see, for instance, [11, Corollary 3.19]). Then, $f$ has the desired form with $f = \eta$.
If $\psi_1 \neq \psi_2$, substituting $f = \frac{\psi_1+\psi_2}{2}$ into (1.5), we find after a reduction that

$$\begin{aligned}&\psi_1(x)[\chi_1(y)\psi_1(y) + \chi_2(y)\psi_1\circ\sigma(y) - \psi_1(y) - \psi_2(y)]\\ &+\psi_2(x)[\chi_1(y)\psi_2(y) + \chi_2(y)\psi_2\circ\sigma(y) - \psi_1(y) - \psi_2(y)] = 0\end{aligned}$$

for all $x, y \in S$. Since $\psi_1 \neq \psi_2$, we get from the theory of multiplicative functions (see, for instance, [11, Theorem 3.18]) that both terms are 0, so

$$\begin{cases} \psi_1(x)[\chi_1(y)\psi_1(y) + \chi_2(y)\psi_1\circ\sigma(y) - \psi_1(y) - \psi_2(y)] = 0 \\ \\ \psi_2(x)[\chi_1(y)\psi_2(y) + \chi_2(y)\psi_2\circ\sigma(y) - \psi_1(y) - \psi_2(y)] = 0 \end{cases} \tag{2.17}$$

for all $x, y \in S$. Since $\psi_1 \neq \psi_2$, at last one of $\psi_1$ and $\psi_2$ is not zero.
**Subcase 2.1:** If $\psi_2 = 0$, then $\psi_1 \neq 0$. From (2.17), we infer that

$$\psi_1 = \chi_1\psi_1 + \chi_2\psi_1\circ\sigma.$$

Therefore, $\chi_1\psi_1 = 0$ or $\chi_2\psi_1\circ\sigma = 0$.
In either case, $\psi_1 = 0$ because $\sigma$ is surjective. But that contradicts $\psi_1 \neq 0$. Therefore, this subcase is void. The same is true for $\psi_1 = 0$ and $\psi_2 \neq 0$.
**Subcase 2.2:** $\psi_1 \neq 0$ and $\psi_2 \neq 0$. From (2.17), we have

$$\psi_1 + \psi_2 = \chi_1\psi_1 + \chi_2\psi_1\circ\sigma = \chi_1\psi_2 + \chi_2\psi_2\circ\sigma. \tag{2.18}$$

Using (2.18) and the fact that $\psi_1 \neq \psi_2$, we see that $\psi_1 = \chi_2\psi_1\circ\sigma = \chi_1\psi_2 \textit{ and } \psi_2 = \chi_1\psi_1 = \chi_2\psi_2\circ\sigma$. Thus $\psi_1 = \chi_1^2\psi_1 \textit{ and } \psi_2 = \chi_2\psi_1\circ\sigma$.

Next, we use (2.18) to get that $\psi_1 = \chi_1\psi_1 = \chi_2\psi_2\circ\sigma$ and $\psi_2 = \chi_1\psi_2 = \chi_2\psi_1\circ\sigma$, hence $\psi_1 = \chi_1\psi_1$ and $\psi_2 = \chi_2\psi_1\circ\sigma$. Therefore, we have in the solution stated in the theorem with $\psi_1 = \mu$.
Finally, in view of these cases, we deduce that $f$ has the form stated in Theorem 2.1.
The other direction of the proof is trivial to verify. The continuity statement follows from [11, Theorem 3.18 (d)].
□

## 3 Applications

**Corollary 3.1.** *[8] Let $S$ be a semigroup and $\sigma \in Hom(S,S)$ such that $\sigma \circ \sigma = id$ (where id denotes the identity map) and $\chi$ is a character of $S$ with $\chi(x\sigma(x)) = 1$ for all $x \in S$. The solutions $f : S \to \mathbb{C}$ of (1.4) are the functions of the form $f = \frac{\mu + \chi\mu\circ\sigma}{2}$, where $\mu : S \to \mathbb{C}$ is a multiplicative function.*
*If $S$ is a topological semigroup and $f \in C(S)$, then $\mu, \chi\mu \circ \sigma \in C(S)$.*

**Proof.** It suffices to take $\chi_1(y) = 1$ and $\chi_2(y) = \chi(y)$ for all $y \in S$ in Theorem 2.1. □

**Corollary 3.2.** *[12] Let $S$ be a semigroup and $\sigma \in Hom(S,S)$ such that $\sigma \circ \sigma = id$ (where id denotes the identity map). The solution $f : S \to \mathbb{C}$ of the functional equation (1.2) are the functions of the form $f = \frac{\mu + \mu\circ\sigma}{2}$, where $\mu : S \to \mathbb{C}$ is a mutiplicative function.*
*If $S$ is a topological semigroup and $f \in C(S)$, then $\mu, \mu \circ \sigma \in C(S)$.*

**Proof.** It suffices to take $\chi_1(y) = \chi_2(y) = 1$ for all $y \in S$ in Theorem 2.1. □

**Corollary 3.3.** *Let $S$ be a semigroup. The solution $f : S \to \mathbb{C}$ of the functional equation (1.1) is a multiplicative function.*
*If $S$ is a topological semigroup and $f \in C(S)$, then $\mu \in C(S)$. where $\mu : S \to \mathbb{C}$ is a multiplicative function.*

**Proof.** It suffices to take $\sigma(y) = y$ and $\chi_1(y) = \chi_2(y) = 1$ for all $y \in S$ in Theorem 2.1. □

**Example 3.4.** For a non-abelian example of a group, consider the set of complex $2 \times 2$ matrices under matrix multiplication

$$S := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} |a, b, c, d \in \mathbb{C}, |ad - bc| = 1 \right\},$$

and take as homomorphisms

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d}. \end{pmatrix}.$$

You may use similar method of [11, Exercise3.14] combined with [11, Example3.10], we can get the continuous non- zero multiplicative function on $S$ as follows:

$$\chi_2(X) = (det(X))^n,$$

where $n \in \mathbb{Z}$.
Simple computations show that $\chi_2(X(\sigma(X)) = 1 \ for\ all\ X \in S.$
Therefore, any continuous multiplicative function $\mu$ on $S$ satisfies $\chi_1\mu = \mu$ and $\chi_2\mu \circ \sigma = \mu$, with $\mu(X) = (det(X))^m$, $\mu \circ \sigma = (det(\bar{X}))^m$ where $n = 2m$ and

$$\chi_1(X) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In conclusion, using Theorem 2.1, the non-zero continuous solutions $f : S \to \mathbb{C}$ of (1.5) are

$$f \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)^m,$$

for all $a, b, c, d \in \mathbb{C}$.

**Remark 3.5.** In particular, if we take $S = SL(2,\mathbb{C})$ in the Example 3.4, then the unique non-zero continuous solution $f : S \to \mathbb{C}$ of (1.5) is $f \equiv 1$.

## 4 Solutions of the functional equation (1.8)

Now we shall extend results about equation (1.5) to the case about Hilbert space valued solutions in terms of multiplicative functions by Hadamard product.
We give a characterization of solutions for the following equation:

$$\chi_1(y)f(xy) + \chi_2(y)f(\sigma(y)x) = 2f(x) * f(y)\ , x, y \in S.$$

**Theorem 4.1.** *Let $H$ be a separable real Hilbert space, $S$ be a semigroup, let $\sigma \in Hom(S, S)$ such that $\sigma \circ \sigma = id$ (where id denotes the identity map) and $\chi_1, \chi_2$ be two characters of $S$ with $\chi_2(x\sigma(x)) = 1$ for all $x \in S$. Assume that the function $f : S \to H$ satisfy (1.8).*
*Then, there exists a positive integer $N$ such that*

$$f(x) = \sum_{n=1}^{N} \langle f(x), e_n\rangle e_n$$

*for all $x \in S$ such that*

$$f(x) = \frac{1}{2}\sum_{k=1}^{N} \epsilon_k(\mu_k(x) + \chi_2(x)\chi_1(x)\mu_k \circ \sigma(x))e_k,\ x \in S,$$

*where $\epsilon_k = 1$ or $0$ for every $k \in \{1, 2, ....., N\}$, for all $x \in S$, where $\mu_k$ is a non-zero multiplicative function of $S$.*

**Proof.** Let $\{e_k, k \in \mathbb{N}\}$ be an orthonormal basis for $H$. For every integer $k \geq 0$, consider the function $f_k : S \to \mathbb{C}$ defined by

$$f_k(x) = \langle f(x), e_k\rangle\ for\ x \in S.$$

Since $f$ satisfies (1.8), we have for all $x, y \in S$

$$\begin{aligned}\sum_{k=0}^{+\infty}\{\langle\chi_1(y)f(xy), e_k\rangle + \langle\chi_2(y)f(\sigma(y)x), e_k\rangle\}e_k &= \sum_{k=0}^{+\infty}\langle\{\chi_1(y)f(xy) + \chi_2(y)f(\sigma(y)x)\}, e_k\rangle e_k\\ &= \chi_1(y)f(xy) + \chi_2(y)f(\sigma(y)x)\\ &= 2f(x) * f(y)\\ &= 2\sum_{k=0}^{+\infty}\langle f(x), e_k\rangle\langle f(y), e_k\rangle\ e_k.\end{aligned}$$

This yields for all $k \in \mathbb{N}$,

$$\chi_1(y)f_k(xy) + \chi_2(y)f_k(\sigma(y)x) = 2f_k(x)f_k(y) \tag{4.1}$$

for all $x, y \in S$. In view of Theorem 2.1, one the following statements holds:
(a) $f_k = 0$.

(b) $f_k = \frac{\mu_k(x)+\chi_2(x)\chi_1(x)\mu_k\circ\sigma(x)}{2}$.
We have

$$\begin{aligned}f(x) &= \sum_{k=0}^{+\infty}\langle f(x), e_k\rangle e_k\\ &= \sum_{k=0}^{+\infty} f_k(x)e_k.\end{aligned}$$

The continuation of the proof depends on the dimension of $H$. In fact if $H$ is infinite dimensional, since

$$f_k(x) \to 0\ as\ k \to +\infty,$$

for every $x \in S$. The statement (b) is not possible for infinitely many positive integers $k$. Hence, there exists some positive integer $N$ such that $f_k = 0$ for every $k > N$.
Thus $f$ can be represented as

$$f(x) = \sum_{k=0}^{N} \langle f(x), e_k \rangle e_k.$$

In the case that, $H$ is of finite-dimensional type the proof is clear. Then the function $f$ satisfying (1.8). □

**Corollary 4.2.** *Let $H$ be a separable real Hilbert space, $S$ be a semigroup, let $\sigma \in Hom(S, S)$ such that $\sigma \circ \sigma = id$ (where id denotes the identity map) and $\chi_2$ be a character of $S$ with $\chi_2(x\sigma(x)) = 1$ for all $x \in S$.*

*Assume that the function $f : S \to H$ satisfy*

$$f(xy) + \chi_2(y)f(\sigma(y)x) = 2f(x) * f(y)\ , x, y \in S,$$

*where $f : S \to H$ are the unknown function. Then, there exists a positive integer $N$ such that*

$$f(x) = \textstyle\sum_{n=1}^{N} \langle f(x), e_n \rangle e_n$$

*for all $x \in S$ such that*

$$f(x) = \frac{1}{2} \sum_{k=1}^{N} \epsilon_k (\mu_k(x) + \chi_2(x)\mu_k \circ \sigma(x)) e_k,\ x \in S,$$

*where $\epsilon_k = 1$ or $0$ for every $k \in \{1, 2, ....., N\}$, where $\mu_k$ is a non-zero multiplicative function of $S$.*
*If $S$ is a topological semigroup and $f \in C(S)$, then $\mu_k, \chi_2(x)\mu_k \circ \sigma \in C(S)$.*

**Proof.** It suffices to take $\chi_1(y) = 1$ for all $y \in S$ in Theorem 4.1. □

**Corollary 4.3.** *Let $H$ be a separable real Hilbert space, $S$ be a semigroup, let $\sigma \in Hom(S, S)$ such that $\sigma \circ \sigma = id$ (where id denotes the identity map). Assume that the function $f : S \to H$ satisfy*

$$f(xy) + f(\sigma(y)x) = 2f(x) * f(y)\ , x, y \in S,$$

*where $f : S \to H$ are the unknown function. Then, there exists a positive integer $N$ such that*

$$f(x) = \textstyle\sum_{n=1}^{N} \langle f(x), e_n \rangle e_n$$

*for all $x \in S$ such that*

$$f(x) = \frac{1}{2} \sum_{k=1}^{N} \epsilon_k (\mu_k(x) + \mu_k \circ \sigma(x)) e_k,\ x \in S,$$

*where $\epsilon_k = 1$ or $0$ for every $k \in \{1, 2, ....., N\}$, where $\mu_k$ is a non-zero multiplicative function of $S$.*
*If $S$ is a topological semigroup and $f \in C(S)$, then $\mu_k, \mu_k \circ \sigma \in C(S)$.*

**Proof.** It suffices to take $\chi_1(y) = \chi_2(y) = 1$ for all $y \in S$ in Theorem 4.1. □

## References


[1] J. Aczél, J., Dhombres, *Functional Equations in Several Variables.* Cambridge University Press. New York (1989).

[2] A. Chahbi, B. Fadli, S. Kabbaj, *A generalization of the symmetrized multiplicative Cauchy equation.* Acta Math. Hung. **149**, 170–176 (2016).

[3] T.M.K. Davison, *D'Alembert's functional equation on topological monoids.* Publ. Math. Debrecen **75**, 41–66 (2009).

[4] H. Dimou and S. Kabbaj, *A generalization of variant of Wilson's type Hilbert space valued functional equations.* Proycecciones. **36**, 485–498 (2017).

[5] H. Dimou, M. Almahalebi, A. Chahbi, *A joint generalization of Van Vleck's type Hilbert space valued functional equations.* Asia Mathematika. **2**, 29–40 (2018).

[6] B.R. Ebanks, H. Stetkaer, *d'Alembert's other functional equation on monoids with an involution.* Aequ. Math. **89**, 187–206 (2015).

[7] I. El Fassi, A. Chahbi and S. Kabbaj, *The solution of a generalized variant of d'Alembert's functional equation.* Bol. Soc. Mat. Mex. **24**, 1–7 (2017).

[8] E. Elqorachi, A. Redouani, *Solutions and stability of variant of Wilson's functional equation.* Proyecciones. J. Math. **37**, 317–344 (2018).

[9] P.L. Kannapan, *Functional Equations and Inequalities with Applications.* Springer, New York. (2009).

[10] H. Stetkær, *On multiplicative maps.* Semigr. Forum **63**, 466–468 (2001).

[11] H. Stetkær, *Functional Equations on Groups.* World Scientific Publishing Company, Singapore, (2013).

[12] H. Stetkær, *A variant of d'Alembert's functional equation.* Aequationes Math. **89**, 657–662 (2015).

[13] H. Stetkær, *D'Alembert's functional equation on groups.* Banach Center Publ. **99**, 173–191 (2013).

[14] D. Zeglami, M. Tial and B. Fadli, *Wilson's type Hilbert space valued functional equations.* Adv. Pure Appl. Math. **7**, 189–196 (2016).

## Author information

Hajira Dimou, Abdellatif Chahbi, Department of Mathematics, Ibn Zohr University, Agadir, Morocco.
E-mail: dimouhajira@gmail.com (H. Dimou), abdellatifchahbi@gmail.com (A. Chahbi)

Samir Kabbaj, Department of Mathematics, Faculty of Sciences, Ibn Tofail University, BP14000, Kenitra, Morocco.
E-mail: samkabbaj@yahoo

Corresponding author: Hajira Dimou

Received: July 13, 2019
Accepted: December 28, 2019