Palestine Journal of Mathematics

Vol. 10(2)(2021) , 440–442

CONVOLUTION IDENTITIES FOR TRIBONACCI NUMBERS VIA THE DIAGONAL OF A BIVARIATE GENERATING FUNCTION

Helmut Prodinger

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11B39; Secondary 11B83.

Keywords and phrases: Tribonacci numbers, convolution, generating function, diagonal.

Abstract Convolutions for Tribonacci numbers involving binomial coefficients are treated with ordinary generating functions and the diagonalization method of Hautus and Klarner. In this way, the relevant generating function can be established, which is rational. The coefficients can also be expressed. It is sketched how to extend this to Tetranacci numbers and similar quantities.

1 Introduction

Komatsu [2] treats

$$\sum_{k=0}^{n}\binom{n}{k}T_kT_{n-k},$$

with Tribonacci numbers T_m, using exponential generating functions. This sequence, however has a rational generating function, which one does not see from this treatment.

We present a method, based on ordinary generating functions, and the diagonal method of Hautus and Klarner [1], that provides this rational generating function.

As a warm-up, we will discuss the analogous question with Fibonacci numbers first.

2 A warm-up

We consider the double sequence

$$\sum_{k=0}^{n}\binom{n}{k}F_kF_{m-k},$$

with Fibonacci numbers F_i, and eventually specialize to $m = n$ (hence the name diagonal method). We set up and compute a double generating function:

$$\begin{aligned}H(x,y) &= \sum_{n\ge0}x^n\sum_{k=0}^{n}\binom{n}{k}F_k\sum_{m\ge k}y^mF_{m-k}\\ &= \frac{y}{1-y-y^2}\sum_{n\ge0}x^n\sum_{k=0}^{n}\binom{n}{k}F_ky^k\\ &= \frac{y}{1-y-y^2}\sum_{k\ge0}\frac{x^k}{(1-y)^{k+1}}F_ky^k\\ &= \frac{xy^2}{(1-2x+x^2-xy-x^2y+x^2y^2)(1-y-y^2)}.\end{aligned}$$

Now, in order to pull out the diagonal, following Hautus and Klarner, we consider

$$H(xt,\tfrac{1}{t})\tfrac{1}{t} = \frac{z}{(1-2zt+z^2t^2-z+tz^2-z^2)(t^2-t-1)}$$

and look at the poles

$$\frac{1 \pm \sqrt{5}}{2}; \qquad \frac{1}{z} - \frac{1 \pm \sqrt{5}}{2}.$$

One needs to consider the residues, but the third and fourth poles go to infinity for $z \to 0$. So using Cauchy's integral theorem, these poles lie outside and do not need to be considered.

We find

$$\operatorname{Res}(H(xt, \tfrac{1}{t})\tfrac{1}{t}; t = \tfrac{1+\sqrt{5}}{2}) + \operatorname{Res}(H(xt, \tfrac{1}{t})\tfrac{1}{t}; t = \tfrac{1-\sqrt{5}}{2}) = \frac{z^2}{(1-z)(1-2z-4z^2)},$$

which is the rational function of interest. We can do a little bit more. Writing

$$\alpha; \beta = \frac{1 \pm \sqrt{5}}{2}$$

as usual, we can decompose

$$\frac{z^2}{(1-z)(1-2z-4z^2)} = -\frac{2}{5}\frac{1}{1-z} + \frac{1}{5}\frac{1}{1-2\alpha z} + \frac{1}{5}\frac{1}{1-2\beta z}.$$

From this we conclude that

$$\sum_{k=0}^{n} \binom{n}{k} F_k F_{n-k} = -\frac{2}{5} + \frac{1}{5} 2^n L_n,$$

with Lucas numbers $L_m = \alpha^m + \beta^m$.

3 Tribonacci numbers

Now consider

$$\sum_{k=0}^{n} \binom{n}{k} T_k T_{n-k},$$

where the Tribonacci numbers are given via the generating function

$$\sum_{n \geq 0} T_n z^n = \frac{1}{1-z-z^2-z^3}.$$

Exactly as in the warm-up section, we find the bivariate generating function

$$G(x, y) = \frac{y}{1-y-y^2-y^3}\frac{1}{1-x}\frac{w}{1-w-w^2-w^3}\Big|_{w=\frac{xy}{1-x}}.$$

We look at $G(zt, \frac{1}{t})\frac{1}{t}$ and its poles. There are 6 poles, but only 3 of them need to be considered, namely the roots of $1 - t - t^2 - t^3$, call them s_1, s_2, s_3. While they are not particulary appealing, the computer can handle them. After heavy simplifications with a computer, we get

$$\begin{aligned}\operatorname{Res}(G(zt, \tfrac{1}{t})\tfrac{1}{t}; t = s_1) + \operatorname{Res}(G(zt, \tfrac{1}{t})\tfrac{1}{t}; t = s_2) + \operatorname{Res}(G(zt, \tfrac{1}{t})\tfrac{1}{t}; t = s_3)\\ = \frac{1}{11}\frac{1+z+10z^2}{1-2z-4z^2-8z^3} - \frac{1}{11}\frac{1+z-8z^2}{1-2z+2z^3}.\end{aligned}$$

The coefficients of the first term are easy:

$$\frac{1}{11}\Big(2^{n+1}T_{n+1} + \frac{1}{2}2^n T_n + \frac{5}{2}2^{n-1}T_{n-1}\Big).$$

For the second term we note that

$$\frac{z^3}{1-z-z^2-z^3}\Big|_{z=\frac{-x}{1-x}} = \sum_{n \geq 2} T_{n-2}\Big(\frac{-x}{1-x}\Big)^n = \frac{-x^3}{1-2x+2x^3}.$$

Setting

$$\frac{1}{1-2z+2z^3}=\sum_{n\ge0}U_nz^n,$$

we find that

$$\begin{aligned}U_{m-3}&=-[x^m]\sum_{n\ge2}T_{n-2}\Big(\frac{-x}{1-x}\Big)^n=-[x^{m-n}]\sum_{n\ge2}T_{n-2}(-1)^n\Big(\frac{1}{1-x}\Big)^n\\&=-[x^{m-n}]\sum_{n\ge2}T_{n-2}(-1)^n\sum_{k\ge0}\binom{n+k-1}{k}x^k\\&=\sum_{n\ge2}T_{n-2}(-1)^{n-1}\binom{m-1}{n-1},\end{aligned}$$

or, nicer

$$U_m=\sum_{k\ge1}T_{k-1}(-1)^k\binom{m+2}{k}.$$

Then we can write the second contribution as

$$-\frac{1}{11}[z^n]\frac{1+z-8z^2}{1-2z+2z^3}=-\frac{1}{11}(U_n+U_{n-1}-8U_{n-2}).$$

Komatsu [2] also considered arbitrary initial conditions for the Tribonacci numbers. That can be done with generating functions as well; only the numerator changes, and the 3 relevant poles stay the same. The reader is invited to do a few experiments herself.

4 Tetranacci and more

Without giving details of the computations, the rational generating functions for Tetranacci number (generating function $z/(1-z-z^2-z^3-z^4)$) is

$$\frac{2z^2\left(-z^3-2z^4+8z^5+6z^6+4z^7+1-2z-z^2\right)}{\left(16z^4+8z^3+4z^2+2z-1\right)\left(z^6+6z^5-4z^4-3z^3-z^2+3z-1\right)}.$$

The next instance is

$$\frac{-2z^2(-z^3-z^4-25z^6+19z^8+52z^{10}+40z^9-1+3z)}{(32z^5+16z^4+8z^3+4z^2+2z-1)(4z^{10}-4z^9-15z^8-12z^7+25z^6-2z^4-4z^3-3z^2+4z-1)},$$

but after that the computations become too heavy to be reported here.

References

[1] M. L. J. Hautus and D. A. Klarner. The diagonal of a double power series. *Duke Math. J.*, 38:229–235, 1971.

[2] Takao Komatsu. Convolution identities for Tribonacci-type numbers with arbitrary initial values. *Palest. J. Math.*, 8(2):413–417, 2019.

Author information

Helmut Prodinger, Department of Mathematics, University of Stellenbosch 7602, Stellenbosch, South Africa.
E-mail: hproding@sun.ac.za

Received: October 15, 2019.
Accepted: December 10, 2019.