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On Pointwise Convergence of Nonlinear Integrals in L_p Spaces $(1 < p < \infty)$

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Abstract In this short note, Fatou type (pointwise) convergence theorems for integral operators equipped with infinite sum designated as

$$\Psi_\nu(f;x) = \sum_{m=1}^{\infty} \int_a^b f^m(t) H_{\nu,m}(t-x)dt,\ \nu \in \Lambda,$$

where Λ is a non-empty index set involving non-negative real numbers ν, to the function $f \in L_p(a,b)$ $(1 < p < \infty)$ at its generalized type characteristic points are given. Here, (a,b) is an arbitrary finite interval in $\mathbb{R}$ or $(a,b) = \mathbb{R}$.

1 Introduction

Approximation features of various types of linear and nonlinear integral operators have been an hot interest of many researchers for years. The fundamental form of linear integral operators may be given as

$$L(f;x,\lambda) = \int_{-\pi}^{\pi} f(t)K(t-x,\lambda)dt,\ x \in (-\pi,\pi), \tag{1.1}$$

where λ is a non-negative parameter and $K(.,\lambda)$ is a $2\pi-$periodic kernel function satisfying some properties. The convergence properties of operators of type (1.1) and its modifications obtained by changing the domain of integration were detaily prensented in the famous monograph by Butzer and Nessel [16]. The operators of type (1.1) were also considered as two-parameter integral operators by Taberski [17] presenting pointwise convergence theorems including a generalization of well-known Natanson lemma [10]. Then, many generalizations of the operators of type (1.1) were studied by many authors including Gadjiev [2], Rydzewska [3], Karsli and Ibikli [9] and Esen Almali [19]. Musielak [11] considered the nonlinear analogues of the operators of type (1.1) and gave a solution method for the pointwise approximation problem of this case. For further reading concerning this approach, we refer the reader to [5, 6, 7, 12, 23]. Also, for some different approaches, we refer the reader to [1, 20].

As a continuation of [8], [15], [21] and [22], the main purpose of this manuscript is to obtain Fatou type (pointwise) convergence of nonlinear integrals equipped with infinite sum in the following setting:

$$\Psi_\nu(f;x) = \sum_{m=1}^{\infty} \int_a^b f^m(t) H_{\nu,m}(t-x)dt,\ \ \nu \in \Lambda,\ \ x \in (a,b), \tag{1.2}$$

where Λ is a non-empty index set involving non-negative real numbers ν, to the function $f \in L_p(a,b)$ $(1 < p < \infty)$, where (a,b) is an arbitrary finite interval in $\mathbb{R}$, as (x,ν) tends to (x_0,ν_0). Here, ν_0 denotes either accumulation point of Λ or ∞. Similar results are also obtained for the case $f \in L_p(\mathbb{R})$. Here, $H_{\nu,m} : \mathbb{R} \to \mathbb{R}_0^+$, $\nu \in \Lambda$ and $m = 1, 2, ...$ and f^m represents $m-th$ power of f. The operators of type (1.2) are obtained by incorporating the operators and terminologies used in [17] and [21].

Remark 1.1. Let (a,b) denote any interval on $\mathbb{R}$ and $(f^m(t)H_{\nu,m}(t-x))$ be a sequence of measurable functions defined almost everywhere on (a,b) for each fixed $x \in (a,b)$ and $\nu \in \Lambda$. If we suppose that $\sum\limits_{m=1}^{\infty} \int\limits_{b}^{a} |f^m(t)H_{\nu,m}(t-x)|\, dt < \infty$ for each fixed $x \in (a,b)$ and $\nu \in \Lambda$, then by a corollary of Lebesgue dominated convergence theorem (see page 29 in [24]), we see that the series $\sum\limits_{m=1}^{\infty} f^m(t)H_{\nu,m}(t-x)$ converges for almost all t to a function in $L_1(a,b)$ for each fixed $x \in (a,b)$ and $\nu \in \Lambda$, and

$$\sum_{m=1}^{\infty} \int_a^b f^m(t)H_{\nu,m}(t-x)dt = \int_a^b \sum_{m=1}^{\infty} f^m(t)H_{\nu,m}(t-x)dt$$

holds for each fixed $x \in (a,b)$ and $\nu \in \Lambda$. If, in addition, one assumes that all kernel functions are equal, that is, $H_{\nu,m} = H_\nu$, $m = 1, 2, ...,$ and $\sum\limits_{m=1}^{\infty} f^m(t)$ is summable to a function in $L_1(a,b)$, then the operators of type (1.2) may be considered as the operators of type (1.1). Using similar ideas, one may set a relationship between the operators of type (1.2) and nonlinear counterparts of the operators of type (1.1). In this work, we will consider similar problems constructed for the operators of type (1.1) from another point of view, that is, we will mainly follow the steps in the previous works, such as [21, 2, 17].

Definition 1.2. Let $\delta_1 > 0$ be a given fixed real number and $\delta_1 > h > 0$. A point $x_0 \in (a,b)$ (or $x_0 \in \mathbb{R}$) is called $\mu-$generalized Lebesgue point of the function $f \in L_p(a,b)$ (or $f \in L_p(\mathbb{R})$) if the following relation:

$$\lim_{h\to 0} \left(\frac{1}{\mu(h)} \int_{x_0}^{x_0 \pm h} |f(t) - f(x_0)|^p \, dt \right)^p = 0, \quad 1 \le p < \infty,$$

holds, where the function μ is increasing and absolutely continuous on $[0, \delta_1]$ with $\mu(0) = 0$ (see, for example, [2, 3, 19]).

The following definition, which gives a characterization of class, is adopted from [21].

Definition 1.3. $(Class\ \mathcal{A})$ Let Λ be a non-empty index set involving non-negative real numbers ν with accumulation point ν_0. For $m = 1, ...,$ a family $\{H_{\nu,m}\}_{\nu\in\Lambda}$ consisting of the globally integrable functions $H_{\nu,m} : \mathbb{R} \to \mathbb{R}_0^+$, for each fixed $m = 1, ...,$ and $\nu \in \Lambda$ is named as $Class\ A$, if there hold the following features:

$a.$
$$\int_{-\infty}^{\infty} H_{\nu,m}(t)dt = I_m,$$
where $I_m > 0$ are certain finite real numbers which are independent of $\nu \in \Lambda$ with $\sum\limits_{m=1}^{\infty} I_m < \infty$.

$b.$ For every $\xi > 0$,
$$\lim_{\nu\to\nu_0} \sum_{m=1}^{\infty} \left[\sup_{|t|>\xi} H_{\nu,m}(t) \right] = 0.$$

$c.$ For every $\xi > 0$,
$$\lim_{\nu\to\nu_0} \sum_{m=1}^{\infty} \left[\int_{|t|>\xi} H_{\nu,m}(t)dt \right] = 0.$$

$d.$ Let δ_0 be a certain positive real number satisfying $0 < \delta_0 \le \delta_1$ such that $H_{\nu,m}(t)$ is non-decreasing on $[-\delta_0, 0]$ and non-increasing on $[0, \delta_0]$ with respect to t for each fixed $m = 1, ...,$ and $\nu \in \Lambda$.

From now on, we suppose that $H_{\nu,m}$ is taken from the $Class\ \mathcal{A}$.

2 Main Results

Let (a,b) be finite interval in $\mathbb{R}$. First main theorem, which is similar to that of given in [2, 17, 21], is

Theorem 2.1. *If $x_0 \in (a,b)$ is a $\mu-$generalized Lebesgue point of $f \in L_p(a,b)$ $(1<p<\infty)$ and f is bounded on (a,b) with $\sup\limits_{t\in(a,b)} |f(t)| = A$, then*

$$\lim_{(x,\nu)\to(x_0,\nu_0)} \Psi_\nu(f;x) = \sum_{m=1}^{\infty} I_m f^m(x_0)$$

provided that the function given by

$$\sum_{m=1}^{\infty} \left\{ \int_{x_0-\delta}^{x_0+\delta} H_{\nu,m}(t-x) \left| \{\mu(|x_0-t|)\}'_t \right| dt + 2H_{\nu,m}(0)\mu(|x_0-x|) \right\},$$

is bounded on a set S consisting of $(x,\nu) \in (a,b)\times\Lambda$, as (x,ν) tends to (x_0,ν_0), where $0<\delta<\delta_0$, and $\sup\limits_m (m^p A^{mp})$, $m \in \{1,2,...\}$ is finite for each fixed $1 \le p < \infty$.

Proof. First of all, notice that, the sum $\sum\limits_{m=1}^{\infty} I_m f^m(x_0)$ achives a finite sum in view of condition (a) of $Class\ \mathcal{A}$ and hypothesis of the theorem. Now, we set $\sigma_\nu = \left| \Psi_\nu(f;x) - \sum\limits_{m=1}^{\infty} I_m f^m(x_0) \right|$. Let $g^m(t) := f^m(t)$ on (a,b) and $g^m(t) := 0$ on $\mathbb{R}\backslash(a,b)$. By condition (a) of $Class\ \mathcal{A}$ and the inequality given by $(w_1+w_2)^p \le 2^p w_1^p + 2^p w_2^p$, provided that w_1 and w_2 are certain positive real numbers (see, for example, [24]), we can write

$$\begin{aligned}
\sigma_\nu^p &\le 2^{2p} \left(\sum_{m=1}^{\infty} \int_a^b |f^m(t) - f^m(x_0)| \, H_{\nu,m}(t-x)dt \right)^p \\
&\quad + 2^p \left(\sum_{m=1}^{\infty} |f^m(x_0)| \left| \int_{-\infty}^{\infty} H_{\nu,m}(t-x)dt - I_m \right| \right)^p \\
&\quad + 2^{2p} \left(\sum_{m=1}^{\infty} |f^m(x_0)| \int_{\mathbb{R}\backslash(a,b)} H_{\nu,m}(t-x)dt \right)^p \\
&= 2^{2p}\sigma_1^p + 2^p\sigma_2^p + 2^{2p}\sigma_3^p.
\end{aligned}$$

We proceed further for the case $f \in L_p(a,b)$ with $1<p<\infty$. The case $f \in L_1(a,b)$ is analogues.

Suppose that $\delta < b - x_0$, $x_0 - \delta > a$ and $0 < x_0 - x < \frac{\delta}{2}$.

By a simple observation , we see that $\sup\limits_m |f^m(x_0)| \le \sup\limits_m (mA^m) < \infty$ by the hypothesis, and by conditions (a) and (c) of $Class\ \mathcal{A}$, σ_2^p converges to 0 and σ_3^p converges to 0 as (x,ν) tends to (x_0,ν_0), respectively.

Now, we consider σ_1. Applying Hölder's inequality integral part of it (see, for example, [24]), we have

$$\sigma_1 \le \sum_{m=1}^{\infty} \left(\int_a^b |f^m(t) - f^m(x_0)|^p \, H_{\nu,m}(t-x)dt \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} H_{\nu,m}(t)dt \right)^{\frac{1}{q}}.$$

Now, we apply Hölder's inequality for infinite sums (see, for example, [14]) as follows:

$$\begin{aligned}
&\sum_{m=1}^{\infty}\left(\int_{a}^{b}|f^{m}(t)-f^{m}(x_0)|^{p}H_{\nu,m}(t-x)dt\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}H_{\nu,m}(t)dt\right)^{\frac{1}{q}}\\
\leq&\left(\sum_{m=1}^{\infty}\left(\int_{a}^{b}|f^{m}(t)-f^{m}(x_0)|^{p}H_{\nu,m}(t-x)dt\right)\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}H_{\nu,m}(t)dt\right)^{\frac{1}{q}}\\
=&\ \sigma_{11}^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}H_{\nu,m}(t)dt\right)^{\frac{1}{q}},
\end{aligned}$$

where $\sum\limits_{m=1}^{\infty}\int\limits_{-\infty}^{\infty}H_{\nu,m}(t)dt<\infty$ by condition (a) of $Class\ \mathcal{A}$ and p and q are conjugate exponents such that $\frac{1}{p}+\frac{1}{q}=1$.

For a fixed real number δ satisfying $0<\delta<\delta_0$, we split the integral σ_{11} into four terms as follows:

$$\begin{aligned}
\sigma_{11}&=\sum_{m=1}^{\infty}\left[\int_{a}^{x_0-\delta}+\int_{x_0-\delta}^{x_0}+\int_{x_0}^{x_0+\delta}+\int_{x_0+\delta}^{b}\right]|f^{m}(t)-f^{m}(x_0)|^{p}H_{\nu,m}(t-x)dt\\
&=\sigma_{111}+\sigma_{121}+\sigma_{131}+\sigma_{141}.
\end{aligned}$$

In view of condition (d) of $Class\ \mathcal{A}$, there holds:

$$\begin{aligned}
\sigma_{111}&\leq 2^{p}\sup_{m}\left\{\int_{a}^{x_0-\delta}|f^{m}(t)|^{p}dt+\int_{a}^{x_0-\delta}|f^{m}(x_0)|^{p}dt\right\}\sum_{m=1}^{\infty}\left[\sup_{|u|>\frac{\delta}{2}}H_{\nu,m}(u)\right]\\
&\leq 2^{p}\left\{\sup_{m}\|f^{m}\|_{L_p(a,b)}^{p}+(b-a)\sup_{m}A^{mp}\right\}\sum_{m=1}^{\infty}\left[\sup_{|u|>\frac{\delta}{2}}H_{\nu,m}(u)\right]
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{141}&\leq 2^{p}\sup_{m}\left\{\int_{x_0+\delta}^{b}|f^{m}(t)|^{p}dt+\int_{x_0+\delta}^{b}|f^{m}(x_0)|^{p}dt\right\}\sum_{m=1}^{\infty}\left[\sup_{|u|>\frac{\delta}{2}}H_{\nu,m}(u)\right]\\
&\leq 2^{p}\left\{\sup_{m}\|f^{m}\|_{L_p(a,b)}^{p}+(b-a)\sup_{m}A^{mp}\right\}\sum_{m=1}^{\infty}\left[\sup_{|u|>\frac{\delta}{2}}H_{\nu,m}(u)\right].
\end{aligned}$$

By condition (b) of $Class\ \mathcal{A}$ and hypothesis, $\sigma_{111}\to 0$ and $\sigma_{141}\to 0$ as (x,ν) tends to (x_0,ν_0).

To continue proof, we need the following identity (see, for example, [13]):

$$(q_1^{n}-q_2^{n})=(q_1-q_2)\left(q_1^{n-1}+q_1^{n-2}q_2+...+q_2^{n-1}\right),$$

where $q_1,q_2\in\mathbb{R}$, $n\in\mathbb{N}$. Using this fact and boundedness of f on (a,b), we easily see that there are finite real numbers $m^{p}A^{(m-1)p}>0$ so that the inequality

$$\begin{aligned}
|f^{m}(t)-f^{m}(x_0)|^{p}&=\left|(f(t)-f(x_0))\left(f(t)^{m-1}+f(t)^{m-2}f(x_0)+...+f(x_0)^{m-1}\right)\right|^{p}\\
&\leq m^{p}A^{(m-1)p}|f(t)-f(x_0)|^{p},
\end{aligned}$$

where $m=1,...$ holds.

Using this inequality in σ_{121} and σ_{131}, we obtain

$$|\sigma_{121}|\leq\sum_{m=1}^{\infty}m^{p}A^{(m-1)p}\int_{x_0-\delta}^{x_0}|f(t)-f(x_0)|^{p}H_{\nu,m}(t-x)dt$$

and

$$|\sigma_{131}| \leq \sum_{m=1}^{\infty} m^p A^{(m-1)p} \int_{x_0}^{x_0+\delta} |f(t) - f(x_0)|^p H_{\nu,m}(t-x)dt.$$

Now, consider first the inequality in σ_{121}. Let $x_0 \in (a,b)$ be a $\mu-$generalized Lebesgue point of $f \in L_p(a,b)$. By definition, for all $\varepsilon > 0$ there exists $\delta > 0$ satisfying $0 < \delta < \delta_0$ such that

$$\int_{x_0-h}^{x_0} |f(t) - f(x_0)|^p dt \leq \varepsilon^p \mu(h) \tag{2.1}$$

hold provided that $0 < h \leq \delta$, respectively. Define the following new function as

$$F(t) = \int_{t}^{x_0} |f(z) - f(x_0)|^p dz. \tag{2.2}$$

Then, for every t satisfying the conditions $0 < x_0 - t \leq \delta$, we have

$$|F(t)| \leq \varepsilon^p \mu(x_0 - t). \tag{2.3}$$

Hence, from (2.2) and (2.3), we can write

$$|\sigma_{121}| = \sum_{m=1}^{\infty} \left| m^p A^{(m-1)p} \left[(\mathbf{LS}) \int_{x_0-\delta}^{x_0} H_{\nu,m}(t-x) d(-F(t)) \right] \right|$$

where (LS) denotes Lebesgue- Stieltjes integral. Applying integration by parts method to the Lebesgue-Stieltjes integral, we have

$$\begin{aligned} |\sigma_{121}| &\leq \varepsilon^p \sum_{m=1}^{\infty} m^p A^{(m-1)p} \{ H_{\nu,m}(x_0 - \delta - x)\mu(\delta) \\ &+ \left. \int_{x_0-\delta}^{x_0} \mu(x_0 - t) \left(\frac{\partial}{\partial t} H_{\nu,m}(t-x) \right) dt \right\}. \end{aligned}$$

Now, we define the variations:

$$V_m(t) = \begin{cases} \bigvee_{x_0-x-\delta}^{t} H_{\nu,m}(s) & , \quad x_0 - x - \delta < t \leq x_0 - x \\ 0 & , \quad t = x_0 - x - \delta \end{cases}. \tag{2.4}$$

Taking above variations and applying integration by parts method to last inequality, we get

$$\begin{aligned} |\sigma_{121}| &\leq \varepsilon^p \sum_{m=1}^{\infty} m A^{(m-1)p} \{ H_{\nu,m}(x_0 - \delta - x)\mu(\delta) + \\ &+ \left. \int_{x_0-x-\delta}^{x_0-x} V_m(t) \{\mu(x_0 - x - t)\}_t' dt \right\}. \end{aligned}$$

Let us consider the definition of V_m (see (2.4)) function and condition (d) of $Class\ \mathcal{A}$. Firstly, we shall write

$$\begin{aligned} |\sigma_{121}| &\leq \varepsilon^p \sum_{m=1}^{\infty} m A^{(m-1)p} \{ H_{\nu,m}(x_0 - \delta - x)\mu(\delta) + \\ &+ \int_{x_0-x-\delta}^{0} \left(\bigvee_{x_0-x-\delta}^{t} H_{\nu,m}(s) \right) \{\mu(x_0 - x - t)\}_t' dt \\ &+ \left. \int_{0}^{x_0-x} \left(\bigvee_{x_0-x-\delta}^{0} H_{\nu,m}(s) + \bigvee_{0}^{t} H_{\nu,m}(s) \right) \{\mu(x_0 - x - t)\}_t' dt \right\} \end{aligned}$$

and then

$$\begin{aligned} |\sigma_{121}| &= \varepsilon^p \sum_{m=1}^{\infty} m A^{(m-1)p} \left\{ 2H_{\nu,m}(0)\mu(|x_0 - x|) \right. \\ &\left. + \int_{x_0-\delta}^{x_0} H_{\nu,m}(t-x) \left\{\mu(x_0 - t)\right\}_t^{'} dt \right\}. \end{aligned} \tag{2.5}$$

We can use a similar method for estimating σ_{131}. Then we find the inequality

$$|\sigma_{131}| \leq \varepsilon^p \sum_{m=1}^{\infty} m A^{(m-1)p} \int_{x_0}^{x_0+\delta} H_{\nu,m}(t-x) \left\{\mu(t-x_0)\right\}_t^{'} dt. \tag{2.6}$$

Consequently, from (2.5) and (2.6), we can write the following inequality:

$$\begin{aligned} |\sigma_{121}| + |\sigma_{131}| &\leq \varepsilon^p \sup_{m\in\mathbb{N}} \left(m A^{(m-1)p} \right) \sum_{m=1}^{\infty} \left\{ \int_{x_0-\delta}^{x_0+\delta} H_{\nu,m}(t-x) \left| \left\{\mu(|x_0 - t|)\right\}_t^{'} \right| dt \right. \\ &\left. + 2H_{\nu,m}(0)\mu(|x_0 - x|) \right\}. \end{aligned}$$

Since $A^p \sup_m \left(m^p A^{(m-1)p} \right) = \sup_m \left(m^p A^{mp} \right) < \infty$ and the right hand side of the above inequality is bounded on S by the hypotheses, the assertion follows, that is,

$$\lim_{(x,\nu)\to(x_0,\nu_0)} \Psi_\nu(f;x) = \sum_{m=1}^{\infty} I_m f^m(x_0).$$

Note that same result is valid for the case $0 < x - x_0 < \frac{\delta}{2}$. Thus, the proof is completed. □

Let $(a,b) = (-\infty, \infty)$. Our next result is

Theorem 2.2. *If $x_0 \in \mathbb{R}$ is a $\mu-$generalized Lebesgue point of function $f \in L_p(\mathbb{R})$ $(1 < p < \infty)$ and f is bounded on $\mathbb{R}$ with $\sup_{t\in\mathbb{R}} |f(t)| = B$, then*

$$\lim_{(x,\nu)\to(x_0,\nu_0)} \Psi_\nu(f;x) = \sum_{m=1}^{\infty} I_m f^m(x_0)$$

provided that the function given by

$$\sum_{m=1}^{\infty} \left\{ \int_{x_0-\delta}^{x_0+\delta} H_{\nu,m}(t-x) \left| \left\{\mu(|x_0 - t|)\right\}_t^{'} \right| dt + 2H_{\nu,m}(0)\mu(|x_0 - x|) \right\},$$

is bounded on some sets $S^{'}$ consisting of $(x,\nu) \in \mathbb{R} \times \Lambda$, as (x,ν) tends to (x_0,ν_0), where $0 < \delta < \delta_0$, and $\sup_m (m^p B^{mp})$ and $\sup_m \|f^m(t)\|_{L_p(\mathbb{R})}^p$, $m \in \{1,2,...\}$ are finite for each fixed $1 \leq p < \infty$.

Proof. We set $\gamma_\nu = \left| \Psi_\nu(f;x) - \sum_{m=1}^{\infty} I_m f^m(x_0) \right|$. Following the similar strategy as in the Theorem 2.1, for the case $f \in L_p(\mathbb{R})$ $(1 < p < \infty)$, we have

$$\begin{aligned} \gamma_\nu^p &\leq 2^p \left(\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} |f^m(t) - f^m(x_0)| \, H_{\nu,m}(t-x) dt \right)^p \\ &+ 2^p \left(\sum_{m=1}^{\infty} |f^m(x_0)| \left| \int_{-\infty}^{\infty} H_{\nu,m}(t-x) dt - I_m \right| \right)^p \\ &= 2^p \gamma_1^p + 2^p \gamma_2^p. \end{aligned}$$

The integral γ_2^p is calculated as in the previous proof. It is sufficient to examine the integral γ_1^p for proof. Now, let's show that γ_1 tend to zero as $(x,\nu)\to(x_0,\nu_0)$. Using the same way as in Theorem 2.1, we can write

$$\begin{aligned}\gamma_1 &\le \left(\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}|f^m(t)-f^m(x_0)|^p H_{\nu,m}(t-x)dt\right)^{\frac{1}{p}}\times\left(\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}H_{\nu,m}(t)dt\right)^{\frac{1}{q}}\\ &= \gamma_{11}^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}H_{\nu,m}(t)dt\right)^{\frac{1}{q}},\end{aligned}$$

where $\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}H_{\nu,m}(t)dt<\infty$ by condition (a) of $Class\ \mathcal{A}$ and also p and q are conjugate exponents such that $\frac{1}{p}+\frac{1}{q}=1$. For a fixed real number δ satisfying $0<\delta<\delta_0$, we can write

$$\begin{aligned}\gamma_{11} &= \sum_{m=1}^{\infty}\left[\int_{-\infty}^{x_0-\delta}+\int_{x_0-\delta}^{x_0}+\int_{x_0}^{x_0+\delta}+\int_{x_0+\delta}^{\infty}\right]|f^m(t)-f^m(x_0)|^p H_{\nu,m}(t-x)dt\\ &= \gamma_{111}+\gamma_{121}+\gamma_{131}+\gamma_{141}.\end{aligned}$$

Firstly, we consider the integrals γ_{111} and γ_{141}. By condition (d) of $Class\ \mathcal{A}$, we obtain

$$\gamma_{111}\le 2^p\sum_{m=1}^{\infty}\left\{\left[\sup_{|u|>\frac{\delta}{2}}H_{\nu,m}(u)\right]\sup_m\|f^m\|_{L_p(\mathbb{R})}^p+\sup_m(m^pB^{mp})\int_{-\infty}^{x_0-\delta}H_{\nu,m}(t-x)dt\right\}$$

and

$$\gamma_{141}\le 2^p\sum_{m=1}^{\infty}\left\{\left[\sup_{|u|>\frac{\delta}{2}}H_{\nu,m}(u)\right]\sup_m\|f^m\|_{L_p(\mathbb{R})}^p+\sup_m(m^pB^{mp})\int_{x_0+\delta}^{\infty}H_{\nu,m}(t-x)dt\right\}.$$

By condition (a) and (b) of $Class\ \mathcal{A}$ and hypothesis, $\gamma_{111}\to 0$ and $\gamma_{141}\to 0$ as (x,ν) tends to (x_0,ν_0). On the other hand, taking into account the previous theorem, we can write

$$|\gamma_{121}|+|\gamma_{131}|\le\varepsilon^p\sup_m(m^pB^{mp})\sum_{m=1}^{\infty}\left\{\int_{x_0-\delta}^{x_0+\delta}H_{\nu,m}(t-x)\left|\{\mu(|x_0-t|)\}_t^{'}\right|dt+2H_{\nu,m}(0)\mu(|x_0-x|)\right\}.$$

Consequently, the right hand side of the above inequality is bounded on $S^{'}$ by the hypotheses, this completes the proof. □

3 Rate of Convergence

Theorem 3.1. *Suppose that the hypotheses of Theorem 2.1 are satisfied. Let*

$$\Delta(\nu,\delta,x)=\sum_{m=1}^{\infty}\left\{\int_{x_0-\delta}^{x_0+\delta}H_{\nu,m}(t-x)\left|\{\mu(|x_0-t|)\}_t^{'}\right|dt+2H_{\nu,m}(0)\mu(|x_0-x|)\right\},$$

where $0<\delta<\delta_0$ *and* $m=1,2,...$ *and the following conditions are satisfied:*

i. *For* $\delta>0$, $\lim\limits_{(x,\nu)\to(x_0,\nu_0)}\Delta(\nu,\delta,x)=0.$

ii. *For every* $\xi>0$,

$$\lim_{\nu\to\nu_0}\sum_{m=1}^{\infty}\left[\sup_{|t|>\xi}H_{\nu,m}(t)\right]=o\left(\Delta(\nu,\delta,x)\right).$$

iii. *For every* $\xi > 0$,

$$\lim_{\nu\to\nu_0}\sum_{m=1}^{\infty}\left[\int_{|t|>\xi} H_{\nu,m}(t)dt\right] = o\left(\Delta(\nu,\delta,x)\right).$$

iv.

$$\lim_{\nu\to\nu_0}\left|\int_{-\infty}^{\infty} H_{\nu,m}(t)dt - I_m\right| = o\left(\Delta(\nu,\delta,x)\right),$$

where $I_m > 0$ *are certain finite real numbers which are independent of* $\nu \in \Lambda$ *with* $\sum_{m=1}^{\infty} I_m < \infty$.

Then, at each μ*-generalized Lebesgue point of function* $f \in L_p(a,b)$*, we have*

$$\left|\Psi_\nu(f;x) - \sum_{m=1}^{\infty} I_m f^m(x_0)\right|^p = o\left(\Delta(\nu,\delta,x)\right).$$

Proof. The claim is obvious by the hypothesis of Theorem 2.1. □

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