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# SYMMETRIC FUNCTIONS OF BINARY PRODUCTS OF GAUSSIAN JACOBSTHAL LUCAS POLYNOMIALS AND CHEBYSHEV POLYNOMIALS

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**Abstract** In this paper, we define and study the Gaussian Jacobsthal polynomials, Gaussian Jacobsthal Lucas polynomials and Chebyshev polynomials. The main results show that after direct calculations and proofs, we give some new generating functions for the products of Gaussian Jacobsthal polynomials, Gaussian Pell polynomials and Chebyshev polynomials.

## 1 Introduction and some Properties

Recently, Fibonacci and Lucas numbers have investigated very largely and authors tried to developpe and give some directions to mathematical calculations using these type of special numbers. In 1996, the incomplete Fibonacci and the incomplete Lucas numbers are introduced by Filipponi in [14]. The year after, the authors [19] presented a systematic investigation of a new class of numbers associated with the familiar Lucas numbers. Next, Pintér and Srivastava [18] determined the generating functions of the incomplete Fibonacci and Lucas numbers. Also, Djordjevic and Srivastava [12] defined the incomplete generalized Jacobsthal and Jacobsthal Lucas numbers. The same authors in [13] gave some generalizations of the incomplete Fibonacci and the incomplete Lucas polynomials. Srivastava et al. [21] presented incomplete $q$-Fibonacci and incomplete $q$-Lucas polynomials and gave the generating functions of them.

As a brief background, the Gaussian polynomials and Chebyshev polynomials were introduced by authors [1, 3, 4, 10, 15, 20], and further the generating functions of these Gaussian polynomials and Chebyshev polynomials were presented by authors.

For $n \geq 2$, it is known that while the second-order linear recurrence polynomials $\{P_n(x)\}_{n\in\mathbb{N}}$ is defined by

$$\begin{cases} P_n(x) = aP_{n-1}(x) + bxP_{n-2}(x) \\ P_0(x) = p + qx,\ P_1(x) = r + sx \end{cases}.$$

The special cases of the polynomials $P_n(x)$ are listed as follows:

- For $a = r = 1$, $b = 2$, $p = \frac{i}{2}$ and $q = s = 0$ it yields the Gaussian Jacobsthal polynomials $GJ_n(x)$, (see [3]);
- For $a = r = 1$, $b = 2$, $p = 2 - \frac{i}{2}$, $q = 0$ and $s = 2i$ it reduces to the Gaussian Jacobsthal Lucas polynomials $Gj_n(x)$, (see [3]).

**Definition 1.1.** [15] The Gaussian Pell polynomials $GP_n(x)$ is defined by

$$\begin{cases} GP_n(x) = 2xGP_{n-1}(x) + GP_{n-2}(x),\ n \geq 2 \\ GP_0(x) = i,\ GP_1(x) = 1 \end{cases}.$$

**Definition 1.2.** Let $k$ and $n$ be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables the $k$-th elementary symmetric function $e_k\left(a_1, a_2, ..., a_n\right)$ is defined by

$$e_k\left(a_1, a_2, ..., a_n\right) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n} \qquad \left(0 \leq k \leq n\right),$$

with $i_1, i_2, ..., i_n = 0$ or 1.

**Definition 1.3.** Let $k$ and $n$ be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables the $k$-th complete homogeneous symmetric function $h_k(a_1, a_2, ..., a_n)$ is defined by

$$h_k(a_1, a_2, ..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n} \qquad (k \geq 0),$$

with $i_1, i_2, ..., i_n \geq 0$.

**Remark 1.4.** Set $e_0(a_1, a_2, ..., a_n) = 1$ and $h_0(a_1, a_2, ..., a_n) = 1$, by usual convention. For $k < 0$, we set $e_k(a_1, a_2, ..., a_n) = 0$ and $h_k(a_1, a_2, ..., a_n) = 0$.

Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients. Let $\mathbb{P}'$ be the algebraic linear dual of $\mathbb{P}$. We write $\langle u, p\rangle := u(p)$ ($u \in \mathbb{P}'$, $p \in \mathbb{P}$). A linear functional $u \in \mathbb{P}'$ is said to be regular [11, 16, 17] if it is quasi-definite, i.e., $\det\langle u, x^{i+j}\rangle_{i,j=1,...,n} \neq 0$ for $n \geq 0$. This is equivalent to the existence of a unique sequence of monic polynomials $\{p_n\}_{n\geq 0}$ of degree $n$ such that $\langle u, p_n p_m\rangle = r_n \delta_{n,m}$, $n, m \geq 0$, with $r_n \neq 0$ ($n \geq 0$). Then, the sequence $\{p_n\}_{n\geq 0}$ is said to be the sequence of monic orthogonal polynomials with respect to $u$.

**Proposition 1.5.** *(Favard's Theorem [11]). Let $\{P_n\}_{n\geq 0}$ be a monic polynomials sequence. Then $\{P_n\}_{n\geq 0}$ is orthogonal if and only if there exist two sequences of complex number $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$, such that $\gamma_n \neq 0$, $n \geq 1$ and satisfies the recurrence relation*

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \ n \geq 0. \end{cases}$$

An orthogonal polynomials sequence $\{P_n\}_{n\geq 0}$ is called classical, if $\{P_n^{[1]}\}_{n\geq 0}$ is also orthogonal (*Hermite, Laguerre, Bessel or Jacobi*), [2, 11, 22]. A second characterization of these polynomials is that they are the only polynomials solutions of the second-order differential equation (Bochner [5])

$$\phi(x)P''_{n+1}(x) - \psi(x)P'_{n+1}(x) = \mu_n P_{n+1}(x), \ n \geq 0,$$

where $\phi$, $\psi$ are polynomials, $\phi$ monic, $\deg \phi = t \leq 2$, $\deg \psi = 1$ and $\mu_n = (n+1)\big(\frac{1}{2}\phi''(0)n - \psi'(0)\big) \neq 0$, $n \geq 0$.

Next, we recall some properties of the classical orthogonal Chebyshev polynomials that we will need in the sequel. The Chebyshev polynomials $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$ of the first, second, third and fourth kinds are respectively defined by the following formulas:

$$T_n(\cos\theta) = \cos(n\theta),$$

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$

$$V_n(\cos\theta) = \frac{\cos(n+1/2)\theta}{\cos(\theta/2)},$$

$$W_n(\cos\theta) = \frac{\sin(n+1/2)\theta}{\sin(\theta/2)},$$

where $x = \cos\theta$ and $\theta \in [0, \pi]$.

**Corollary 1.6.** *[6] For $n \in \mathbb{N}$, we have*

$$GJ_n(x) = \frac{i}{2}S_n(e_1 + [-e_2]) + \left(1 - \frac{i}{2}\right)S_{n-1}(e_1 + [-e_2]), \tag{1.1}$$

$$\textit{with} \begin{cases} e_1 = \frac{1+\sqrt{1+8x}}{2} \\ e_2 = \frac{1-\sqrt{1+8x}}{2} \end{cases}.$$

**Corollary 1.7.** *[6] For $n \in \mathbb{N}$, we have*

$$Gj_n(x) = \left(2 - \frac{i}{2}\right) S_n(e_1 + [-e_2]) + \left(\left(2x + \frac{1}{2}\right) i - 1\right) S_{n-1}(e_1 + [-e_2]), \tag{1.2}$$

$$\text{with } \begin{cases} e_1 = \frac{1+\sqrt{1+8x}}{2} \\ e_2 = \frac{1-\sqrt{1+8x}}{2} \end{cases}.$$

**Corollary 1.8.** *[6] For $n \in \mathbb{N}$, we have*

$$GP_n(x) = iS_n(e_1 + [-e_2]) + (1 - 2ix) S_{n-1}(e_1 + [-e_2]), \tag{1.3}$$

$$\text{with } \begin{cases} e_1 = x + \sqrt{x^2+1} \\ e_2 = x - \sqrt{x^2+1} \end{cases}.$$

**Corollary 1.9.** *[10] For $n \in \mathbb{N}$, we have*

$$T_n(x) = S_n(2e_1 + [-2e_2]) - xS_{n-1}(2e_1 + [-2e_2]), \text{ with } \begin{cases} e_1 = x + \sqrt{x^2-1} \\ e_2 = x - \sqrt{x^2-1} \end{cases}. \tag{1.4}$$

**Corollary 1.10.** *[10] For $n \in \mathbb{N}$, we have*

$$U_n(x) = S_n(2e_1 + [-2e_2]), \text{ with } \begin{cases} e_1 = x + \sqrt{x^2-1} \\ e_2 = x - \sqrt{x^2-1} \end{cases}. \tag{1.5}$$

**Corollary 1.11.** *[1] For $n \in \mathbb{N}$, we have*

$$V_n(x) = S_n(2e_1 + [-2e_2]) - S_{n-1}(2e_1 + [-2e_2]), \text{ with } \begin{cases} e_1 = x + \sqrt{x^2-1} \\ e_2 = x - \sqrt{x^2-1} \end{cases}. \tag{1.6}$$

**Corollary 1.12.** *[1] For $n \in \mathbb{N}$, we have*

$$W_n(x) = S_n(2e_1 + [-2e_2]) + S_{n-1}(2e_1 + [-2e_2]), \text{ with } \begin{cases} e_1 = x + \sqrt{x^2-1} \\ e_2 = x - \sqrt{x^2-1} \end{cases}. \tag{1.7}$$

## 2 Main Results

In this part, we are now in a position to provide theorems. Also we derive the new generating functions of the products of some known polynomials.

**Theorem 2.1.** *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal polynomials with Chebyshev polynomials of the first kind is given by*

$$\sum_{n=0}^{\infty} GJ_n(x)T_n(y)z^n = \frac{i + 2y(1-i)z + \left(\left(1 + 2x - 4xy^2\right)i - 2\right)z^2 + 2xy(i-2)z^3}{2 - 4yz - (16xy^2 - 8x - 2)z^2 + 8xyz^3 + 8x^2z^4}.$$

*Proof.* By using the relationships (1.1) and (1.4), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} GJ_n(x)T_n(y)z^n &= \sum_{n=0}^{\infty}\left(\frac{i}{2}S_n(a_1+[-a_2])+\left(1-\frac{i}{2}\right)S_{n-1}(a_1+[-a_2])\right)\\
&\times \left(S_n(2e_1+[-2e_2])-yS_{n-1}(2e_1+[-2e_2])\right)z^n\\
&= \frac{i}{2}\sum_{n=0}^{\infty}S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n\\
&- \frac{yi}{2}\sum_{n=0}^{\infty}S_n(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n\\
&+ \left(1-\frac{i}{2}\right)\sum_{n=0}^{\infty}S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n\\
&- y\left(1-\frac{i}{2}\right)\sum_{n=0}^{\infty}S_{n-1}(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n\\
&= \frac{i\left(1+2xz^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)}\\
&- \frac{iy\left(z+4xyz^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)}\\
&+ \frac{(2-i)\left(2yz-z^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)}\\
&- \frac{y(2-i)\left(z+2xz^3\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)},
\end{aligned}$$

after a simple calculation, we have

$$\sum_{n=0}^{\infty} GJ_n(x)T_n(y)z^n = \frac{i+2y(1-i)z+\left(\left(1+2x-4xy^2\right)i-2\right)z^2+2xy(i-2)z^3}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4}.$$

This completes the proof. □

**Theorem 2.2.** *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal Lucas polynomials with Chebyshev polynomials of the first kind is given by*

$$\begin{aligned}
\sum_{n=0}^{\infty} Gj_n(x)T_n(y)z^n &= \frac{4-i+2y\left(i(1+2x)-3\right)z}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4}\\
&+ \frac{\left(\left(4xy^2-6x-1\right)i-16xy^2+8x+2\right)z^2}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4}\\
&+ \frac{2xy\left(2-i(4x+1)\right)z^3}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4}.
\end{aligned}$$

*Proof.* We have

$$\begin{aligned}
\sum_{n=0}^{\infty} Gj_n(x)T_n(y)z^n &= \sum_{n=0}^{\infty}\left(\left(2-\frac{i}{2}\right)S_n(a_1+[-a_2])+\left(\left(2x+\frac{1}{2}\right)i-1\right)S_{n-1}(a_1+[-a_2])\right)\\
&\times \left(S_n(2e_1+[-2e_2])-yS_{n-1}(2e_1+[-2e_2])\right)z^n\\
&= \left(2-\frac{i}{2}\right)\sum_{n=0}^{\infty}S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n\\
&- y\left(2-\frac{i}{2}\right)\sum_{n=0}^{\infty}S_n(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n\\
&+ \left(\left(2x+\frac{1}{2}\right)i-1\right)\sum_{n=0}^{\infty}S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n\\
&- y\left(\left(2x+\frac{1}{2}\right)i-1\right)\sum_{n=0}^{\infty}S_{n-1}(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n\\
&= \frac{(4-i)\left(1+2xz^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)}\\
&- \frac{y(4-i)\left(z+4xyz^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)}\\
&+ \frac{((4x+1)i-2)\left(2yz-z^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)}\\
&- \frac{y((4x+1)i-2)\left(z+2xz^3\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)}\\
&= \frac{4-i+2y(i(1+2x)-3)z}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4}\\
&+ \frac{\left(\left(4xy^2-6x-1\right)i-16xy^2+8x+2\right)z^2}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4}\\
&+ \frac{2xy(2-i(4x+1))z^3}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4}.
\end{aligned}$$

So, the proof is completed. □

**Theorem 2.3.** *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian pell polynomials with Chebyshev polynomials of the first kind is given by*

$$\sum_{n=0}^{\infty} GP_n(x)T_n(y)z^n = \frac{i+y(1-4ix)z+\left(\left(4x^2-2y^2+1\right)i-2x\right)z^2+y(2ix-1)z^3}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4}.$$

*Proof.* By using the relationships (1.3) and (1.4), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} GP_n(x)T_n(y)z^n &= \sum_{n=0}^{\infty} \left(iS_n(a_1+[-a_2]) + (1-2ix)\, S_{n-1}(a_1+[-a_2])\right) \\
&\quad \times \left(S_n(2e_1+[-2e_2]) - yS_{n-1}(2e_1+[-2e_2])\right) z^n \\
&= i\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&- iy\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n \\
&+ (1-2ix)\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&- y(1-2ix)\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n \\
&= \frac{i\left(1+z^2\right)}{1-4xyz-(4y^2-4x^2-2)\, z^2+4xyz^3+z^4} \\
&- \frac{iy\left(2xz+2yz^2\right)}{1-4xyz-(4y^2-4x^2-2)\, z^2+4xyz^3+z^4} \\
&+ \frac{(1-2ix)\left(2yz-2xz^2\right)}{1-4xyz-(4y^2-4x^2-2)\, z^2+4xyz^3+z^4} \\
&- \frac{y(1-2ix)\left(z+z^3\right)}{1-4xyz-(4y^2-4x^2-2)\, z^2+4xyz^3+z^4},
\end{aligned}$$

after a simple calculation, we have

$$\sum_{n=0}^{\infty} GP_n(x)T_n(y)z^n = \frac{i+y(1-4ix)\, z+\left(\left(4x^2-2y^2+1\right)i-2x\right)z^2+y(2ix-1)\, z^3}{1-4xyz-(4y^2-4x^2-2)\, z^2+4xyz^3+z^4}.$$

Thus, this completes the proof. □

**Theorem 2.4.** *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal polynomials with Chebyshev polynomials of the second kind is given by*

$$\sum_{n=0}^{\infty} GJ_n(x)\, U_n(y)\, z^n = \frac{i+2y(2-i)\, z+((2x+1)\, i-2)\, z^2}{2-4yz-(16xy^2-8x-2)\, z^2+8xyz^3+8x^2z^4}.$$

*Proof.* We have

$$\begin{aligned}
\sum_{n=0}^{\infty} GJ_n(x)\, U_n(y)\, z^n &= \sum_{n=0}^{\infty} \left(\frac{i}{2}S_n(a_1+[-a_2]) + \left(1-\frac{i}{2}\right) S_{n-1}(a_1+[-a_2])\right) S_n(2e_1+[-2e_2])z^n \\
&= \frac{i}{2}\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&+ \left(1-\frac{i}{2}\right)\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&= \frac{i\left(1+2xz^2\right)}{2\left(1-2yz-(8xy^2-4x-1)\, z^2+4xyz^3+4x^2z^4\right)} \\
&+ \frac{(2-i)\left(2yz-z^2\right)}{2\left(1-2yz-(8xy^2-4x-1)\, z^2+4xyz^3+4x^2z^4\right)} \\
&= \frac{i+2y(2-i)\, z+((2x+1)\, i-2)\, z^2}{2-4yz-(16xy^2-8x-2)\, z^2+8xyz^3+8x^2z^4}.
\end{aligned}$$

This completes the proof. □

**Theorem 2.5.** *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal Lucas polynomials with Chebyshev polynomials of the second kind is given by*

$$\sum_{n=0}^{\infty} Gj_n(x)\,U_n(y)\,z^n = \frac{4-i+2y\left((4x+1)\,i-2\right)z+\left(8x+2-i\,(6x+1)\right)z^2}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4}.$$

*Proof.* We have

$$\begin{aligned}
\sum_{n=0}^{\infty} Gj_n(x)\,U_n(y)\,z^n &= \sum_{n=0}^{\infty}\left(\left(2-\frac{i}{2}\right)S_n(a_1+[-a_2])+\left(\left(2x+\frac{1}{2}\right)i-1\right)S_{n-1}(a_1+[-a_2])\right)\\
&\times\; S_n(2e_1+[-2e_2])z^n\\
&= \left(2-\frac{i}{2}\right)\sum_{n=0}^{\infty}S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n\\
&+\; \left(\left(2x+\frac{1}{2}\right)i-1\right)\sum_{n=0}^{\infty}S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n\\
&= \frac{(4-i)\left(1+2xz^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)}\\
&+\; \frac{\left((4x+1)\,i-2\right)\left(2yz-z^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)}\\
&= \frac{4-i+2y\left((4x+1)\,i-2\right)z+\left(8x+2-i\,(6x+1)\right)z^2}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4}.
\end{aligned}$$

So, the proof is completed. □

**Theorem 2.6.** *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell polynomials with Chebyshev polynomials of the second kind is given by*

$$\sum_{n=0}^{\infty} GP_n(x)\,U_n(y)\,z^n = \frac{i+2y\,(1-2ix)\,z+\left(i\left(4x^2+1\right)-2x\right)z^2}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4}.$$

*Proof.* We have

$$\begin{aligned}
\sum_{n=0}^{\infty} GP_n(x)\,U_n(y)\,z^n &= \sum_{n=0}^{\infty}\left(iS_n(a_1+[-a_2])+(1-2ix)\,S_{n-1}(a_1+[-a_2])\right)S_n(2e_1+[-2e_2])z^n\\
&= i\sum_{n=0}^{\infty}S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n\\
&+\; (1-2ix)\sum_{n=0}^{\infty}S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n\\
&= \frac{i\left(1+z^2\right)}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4}\\
&+\; \frac{(1-2ix)\left(2yz-2xz^2\right)}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4}\\
&= \frac{i+2y\,(1-2ix)\,z+\left(i\left(4x^2+1\right)-2x\right)z^2}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4}.
\end{aligned}$$

Thus, this completes the proof. □

**Theorem 2.7.** *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal polynomials with Chebyshev polynomials of the third kind is given by*

$$\sum_{n=0}^{\infty} GJ_n(x)V_n(y)z^n = \frac{i + 2\left(2y - 1 - iy\right)z + \left(\left(2x - 4xy + 1\right)i - 2\right)z^2 + 2x\left(i - 2\right)z^3}{2 - 4yz - \left(16xy^2 - 8x - 2\right)z^2 + 8xyz^3 + 8x^2z^4}.$$

*Proof.* By using the relationships (1.1) and (1.6), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} GJ_n(x)V_n(y)z^n &= \sum_{n=0}^{\infty}\left(\frac{i}{2}S_n(a_1 + [-a_2]) + \left(1 - \frac{i}{2}\right)S_{n-1}(a_1 + [-a_2])\right) \\
&\quad \times \left(S_n(2e_1 + [-2e_2]) - S_{n-1}(2e_1 + [-2e_2])\right)z^n \\
&= \frac{i}{2}\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \\
&- \frac{i}{2}\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n \\
&+ \left(1 - \frac{i}{2}\right)\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \\
&- \left(1 - \frac{i}{2}\right)\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n \\
&= \frac{i\left(1 + 2xz^2\right)}{2\left(1 - 2yz - \left(8xy^2 - 4x - 1\right)z^2 + 4xyz^3 + 4x^2z^4\right)} \\
&- \frac{i\left(z + 4xyz^2\right)}{2\left(1 - 2yz - \left(8xy^2 - 4x - 1\right)z^2 + 4xyz^3 + 4x^2z^4\right)} \\
&+ \frac{\left(2 - i\right)\left(2yz - z^2\right)}{2\left(1 - 2yz - \left(8xy^2 - 4x - 1\right)z^2 + 4xyz^3 + 4x^2z^4\right)} \\
&- \frac{\left(2 - i\right)\left(z + 2xz^3\right)}{2\left(1 - 2yz - \left(8xy^2 - 4x - 1\right)z^2 + 4xyz^3 + 4x^2z^4\right)},
\end{aligned}$$

after a simple calculation, we have

$$\sum_{n=0}^{\infty} GJ_n(x)V_n(y)z^n = \frac{i + 2\left(2y - 1 - iy\right)z + \left(\left(2x - 4xy + 1\right)i - 2\right)z^2 + 2x\left(i - 2\right)z^3}{2 - 4yz - \left(16xy^2 - 8x - 2\right)z^2 + 8xyz^3 + 8x^2z^4}.$$

This completes the proof. □

**Theorem 2.8.** *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal Lucas polynomials with Chebyshev polynomials of the third kind is given by*

$$\begin{aligned}
\sum_{n=0}^{\infty} Gj_n(x)V_n(y)z^n &= \frac{4 - i + 2\left(\left(4xy - 2x + y\right)i - 2y - 1\right)z}{2 - 4yz - \left(16xy^2 - 8x - 2\right)z^2 + 8xyz^3 + 8x^2z^4} \\
&+ \frac{\left(\left(4xy - 6x - 1\right)i - 16xy + 8x + 2\right)z^2}{2 - 4yz - \left(16xy^2 - 8x - 2\right)z^2 + 8xyz^3 + 8x^2z^4} \\
&+ \frac{2x\left(2 - \left(4x + 1\right)i\right)z^3}{2 - 4yz - \left(16xy^2 - 8x - 2\right)z^2 + 8xyz^3 + 8x^2z^4}.
\end{aligned}$$

*Proof.* We have

$$\begin{aligned}
\sum_{n=0}^{\infty} Gj_n(x)V_n(y)z^n &= \sum_{n=0}^{\infty}\left(\left(2-\frac{i}{2}\right)S_n(a_1+[-a_2])+\left(\left(2x+\frac{1}{2}\right)i-1\right)S_{n-1}(a_1+[-a_2])\right) \\
&\quad\times\left(S_n(2e_1+[-2e_2])-S_{n-1}(2e_1+[-2e_2])\right)z^n \\
&= \left(2-\frac{i}{2}\right)\sum_{n=0}^{\infty}S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&- \left(2-\frac{i}{2}\right)\sum_{n=0}^{\infty}S_n(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n \\
&+ \left(\left(2x+\frac{1}{2}\right)i-1\right)\sum_{n=0}^{\infty}S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&- \left(\left(2x+\frac{1}{2}\right)i-1\right)\sum_{n=0}^{\infty}S_{n-1}(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n \\
&= \frac{(4-i)\left(1+2xz^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)} \\
&\quad-\frac{(4-i)\left(z+4xyz^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)} \\
&+ \frac{((4x+1)i-2)\left(2yz-z^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)} \\
&- \frac{((4x+1)i-2)\left(z+2xz^3\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)} \\
&= \frac{4-i+2\left((4xy-2x+y)i-2y-1\right)z}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4} \\
&+ \frac{\left((4xy-6x-1)i-16xy+8x+2\right)z^2}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4} \\
&+ \frac{2x\left(2-(4x+1)i\right)z^3}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4}.
\end{aligned}$$

So, the proof is completed. □

**Theorem 2.9.** *For $n\in\mathbb{N}$, the new generating function of the product of Gaussian Pell polynomials with Chebyshev polynomials of the third kind is given by*

$$\sum_{n=0}^{\infty} GP_n(x)V_n(y)z^n = \frac{i+(2y-1-4ixy)z+\left(\left(4x^2-2y+1\right)i-2x\right)z^2+(2ix-1)z^3}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4}.$$

*Proof.* We have

$$\begin{aligned}
\sum_{n=0}^{\infty} GP_n(x)V_n(y)z^n &= \sum_{n=0}^{\infty}\left(iS_n(a_1+[-a_2])+(1-2ix)S_{n-1}(a_1+[-a_2])\right) \\
&\quad\times\left(S_n(2e_1+[-2e_2])-S_{n-1}(2e_1+[-2e_2])\right)z^n \\
&= i\sum_{n=0}^{\infty}S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&- i\sum_{n=0}^{\infty}S_n(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n \\
&+ (1-2ix)\sum_{n=0}^{\infty}S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n
\end{aligned}$$

$$
\begin{aligned}
& -(1-2ix)\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n \\
= & \quad \frac{i\left(1+z^2\right)}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4} \\
- & \quad \frac{i\left(2xz+2yz^2\right)}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4} \\
+ & \quad \frac{(1-2ix)\left(2yz-2xz^2\right)}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4} \\
- & \quad \frac{(1-2ix)\left(z+z^3\right)}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4} \\
= & \quad \frac{i+(2y-1-4ixy)z+\left(\left(4x^2-2y+1\right)i-2x\right)z^2+(2ix-1)z^3}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4}.
\end{aligned}
$$

Thus, this completes the proof. □

**Theorem 2.10.** *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal polynomials with Chebyshev polynomials of the fourth kind is given by*

$$
\sum_{n=0}^{\infty} GJ_n(x)\,W_n(y)\,z^n = \frac{i+2(2y+1-iy)z+(i(4xy+2x+1)-2)z^2+2x(2-i)z^3}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4}.
$$

*Proof.* We have

$$
\begin{aligned}
\sum_{n=0}^{\infty} GJ_n(x)\,W_n(y)\,z^n &= \sum_{n=0}^{\infty}\left(\frac{i}{2}S_n(a_1+[-a_2])+\left(1-\frac{i}{2}\right)S_{n-1}(a_1+[-a_2])\right) \\
&\quad \times\left(S_n(2e_1+[-2e_2])+S_{n-1}(2e_1+[-2e_2])\right)z^n \\
&= \frac{i}{2}\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&+ \frac{i}{2}\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n \\
&+ \left(1-\frac{i}{2}\right)\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&+ \left(1-\frac{i}{2}\right)\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n \\
&= \frac{i\left(1+2xz^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)} \\
&+ \frac{i\left(z+4xyz^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)} \\
&+ \frac{(2-i)\left(2yz-z^2\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)} \\
&+ \frac{(2-i)\left(z+2xz^3\right)}{2\left(1-2yz-\left(8xy^2-4x-1\right)z^2+4xyz^3+4x^2z^4\right)} \\
&= \frac{i+2(2y+1-iy)z+(i(4xy+2x+1)-2)z^2+2x(2-i)z^3}{2-4yz-\left(16xy^2-8x-2\right)z^2+8xyz^3+8x^2z^4}.
\end{aligned}
$$

This completes the proof. □

**Theorem 2.11.** *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Jacobsthal Lucas polynomials with Chebyshev polynomials of the fourth kind is given by*

$$\begin{aligned}\sum_{n=0}^{\infty} Gj_n(x) W_n(y) z^n &= \frac{4-i+2((4xy+2x+y)i+1-2y)z}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4} \\ &+ \frac{(16xy+8x+2-i(4xy+6x+1))z^2+2x((4x+1)i-2)z^3}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}.\end{aligned}$$

*Proof.* By using the relationships (1.2) and (1.7), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} Gj_n(x) W_n(y) z^n &= \sum_{n=0}^{\infty}\left(\left(2-\frac{i}{2}\right)S_n(a_1+[-a_2])+\left(\left(2x+\frac{1}{2}\right)i-1\right)S_{n-1}(a_1+[-a_2])\right) \\ &\times \left(S_n(2e_1+[-2e_2])+S_{n-1}(2e_1+[-2e_2])\right)z^n \\ &= \left(2-\frac{i}{2}\right)\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\ &+\left(2-\frac{i}{2}\right)\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n \\ &+ \left(\left(2x+\frac{1}{2}\right)i-1\right)\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\ &+ \left(\left(2x+\frac{1}{2}\right)i-1\right)\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n \\ &= \frac{(4-i)\left(1+2xz^2\right)}{2\left(1-2yz-(8xy^2-4x-1)z^2+4xyz^3+4x^2z^4\right)} \\ &+ \frac{(4-i)\left(z+4xyz^2\right)}{2\left(1-2yz-(8xy^2-4x-1)z^2+4xyz^3+4x^2z^4\right)} \\ &+ \frac{((4x+1)i-2)\left(2yz-z^2\right)}{2\left(1-2yz-(8xy^2-4x-1)z^2+4xyz^3+4x^2z^4\right)} \\ &+ \frac{((4x+1)i-2)\left(z+2xz^3\right)}{2\left(1-2yz-(8xy^2-4x-1)z^2+4xyz^3+4x^2z^4\right)},\end{aligned}$$

after a simple calculation, we have

$$\begin{aligned}\sum_{n=0}^{\infty} Gj_n(x) W_n(y) z^n &= \frac{4-i+2((4xy+2x+y)i+1-2y)z}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4} \\ &+ \frac{(16xy+8x+2-i(4xy+6x+1))z^2+2x((4x+1)i-2)z^3}{2-4yz-(16xy^2-8x-2)z^2+8xyz^3+8x^2z^4}.\end{aligned}$$

So, the proof is completed. □

**Theorem 2.12.** *For $n \in \mathbb{N}$, the new generating function of the product of Gaussian Pell polynomials with Chebyshev polynomials of the fourth kind is given by*

$$\sum_{n=0}^{\infty} GP_n(x) W_n(y) z^n = \frac{i+(2y+1-4ixy)z+\left((4x^2+2y+1)i-2x\right)z^2+(1-2ix)z^3}{1-4xyz-(4y^2-4x^2-2)z^2+4xyz^3+z^4}.$$

*Proof.* We have

$$\begin{aligned}
\sum_{n=0}^{\infty} GP_n\left(x\right) W_n\left(y\right) z^n &= \sum_{n=0}^{\infty} \left(iS_n(a_1+[-a_2]) + \left(1-2ix\right) S_{n-1}(a_1+[-a_2])\right) \\
&\times \left(S_n(2e_1+[-2e_2]) + S_{n-1}(2e_1+[-2e_2])\right) z^n \\
&= i\sum_{n=0}^{\infty} S_n(a_1+[-a_2]) S_n(2e_1+[-2e_2]) z^n \\
&+ i\sum_{n=0}^{\infty} S_n(a_1+[-a_2]) S_{n-1}(2e_1+[-2e_2]) z^n
\end{aligned}$$

$$\begin{aligned}
&+ \left(1-2ix\right) \sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2]) S_n(2e_1+[-2e_2]) z^n \\
&+ \left(1-2ix\right) \sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2]) S_{n-1}(2e_1+[-2e_2]) z^n \\
&= \frac{i\left(1+z^2\right)}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4} \\
&+ \frac{i\left(2xz+2yz^2\right)}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4} \\
&+ \frac{\left(1-2ix\right)\left(2yz-2xz^2\right)}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4} \\
&+ \frac{\left(1-2ix\right)\left(z+z^3\right)}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4} \\
&= \frac{i+\left(2y+1-4ixy\right)z+\left(\left(4x^2+2y+1\right)i-2x\right)z^2+\left(1-2ix\right)z^3}{1-4xyz-\left(4y^2-4x^2-2\right)z^2+4xyz^3+z^4}.
\end{aligned}$$

This completes the proof. □

## 3 Conclusion

In this paper, by making use of theorems, we have derived some new generating functions for the products of Gaussian Jacobsthal, Gaussian Jacobsthal Lucas and Gaussian Pell polynomials with Chebyshev polynomials of the first, second, third and fourth kinds.

It would be interesting to apply the methods shown in the paper to families of other special polynomials.

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