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Generalized Meir-Keeler contraction mappings in controlled metric type spaces

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Abstract In this paper, we generalize the notion of Meir-Keeler contraction condition in controlled metric type spaces. We prove some fixed point theorems for this class of contractions which enables us to extend and generalize many results in the literature.

1 Introduction and preliminaries

After the brake through of Banach in 1922 [1], researchers around the world have been inspired for decades now, and the simple reason that his result has many applications in many fields. Since then researchers has been trying to generalizes his work so it can be applied to more fields to solve open problems. So, some will generalize the type of metric spaces while others generalizes the contraction principle, we encourage the reader to check the following work in the literature [4, 6, 7, 8, 9, 10, 11, 13, 15, 16, 17, 18, 19]. In this paper, we present a more general contraction so called Generalized Meir-Keeler which we will use in controlled metric type spaces, which is a generalization of $b-$metric spaces that has been introduced lately.

First, we present a brief background on the controlled type metric spaces. The notion of $b-$metric spaces was introduced by Bakhtin in [3], which an extension of metric spaces and it was defined as follows;

Definition 1.1. [2] Given a nonempty set X and $\theta : X \times X \to [1, \infty)$. The b-metric is a function $\zeta : X \times X \to [0, \infty)$ such that there exists $s \geq 1$ where for all $x, y, z \in X$ we have,

(i) $\zeta(x, y) = 0 \iff x = y$;

(ii) $\zeta(x, y) = \zeta(y, x)$;

(iii) $\zeta(x, y) \leq s[\zeta(x, z) + \zeta(z, y)]$.

Since then many generalization of $b-$metric spaces was introduced see [12, 14, 5]. Lately, the authors in [2] introduced the concept of extended $b-$metric spaces by replacing the constant $s > 1$ in the triangle inequality by a control function θ.

Definition 1.2. [2] Given a nonempty set X and $\theta : X \times X \to [1, \infty)$. An extended b-metric is a function $\Omega : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,

(i) $\Omega(x, y) = 0 \iff x = y$;

(ii) $\Omega(x, y) = \Omega(y, x)$;

(iii) $\Omega(x, y) \leq \theta(x, y)[\Omega(x, z) + \Omega(z, y)]$.

Among the generalizations of b-metric spaces [3, 4], Mlaiki et al. [5], introduced the concept of controlled metric type space via a control function, as follows:

Definition 1.3. [5] Consider the set $Y \neq \emptyset$ and $\tau : Y^2 \longrightarrow [1, +\infty)$. If the function $\xi_\tau : Y^2 \longrightarrow [0, +\infty)$ satisfies the following conditions;
(ξ_1) $\xi_\tau(g,h) = 0 \Leftrightarrow g = h$;
(ξ_2) $\xi_\tau(g,h) = \xi_\tau(h,g)$;
(ξ_3) $\xi_\tau(g,h) \leq \tau(g,z)\xi_\tau(g,z) + \tau(z,h)\xi_\tau(z,h)$, for all $g,h,z \in Y$,
then the pair (Y, ξ_τ) is called a controlled metric type space.

Next, we remind the reader of the topology of controlled metric type spaces.

Definition 1.4. [5]Let (Y, ξ_τ) be a controlled metric type space and $(g_n)_n$ be a sequence in Y.
(1) We say that the sequence (g_n) converges to some g in Y, if, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\xi_\tau(g_n, g) < \epsilon$ for all $n \geq N$. In this case, we write $\lim_{n \longrightarrow +\infty} g_n = g$.
(2)We say that the sequence (g_n) is $\xi_\tau-$Cauchy, if, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\xi_\tau(g_n, g_m) < \epsilon$ for all $n, m \geq N$.
(3) The controlled metric type space (Y, ξ_τ) is called complete if every Cauchy sequence is convergent.

Definition 1.5. Let (Y, ξ_τ) be a controlled metric type space and let T be a self mapping on Y. T is called ξ_τ-orbitally continuous if whenever

$$\lim_{n \to +\infty} \xi_\tau(Tg^n, z) = 0 \Rightarrow \lim_{n \to +\infty} \xi_\tau(TTg^n, Tz) = 0 \;\; \forall \;\; g, z \in Y. \tag{1.1}$$

Remark 1.6. Note that, continuous mappings are $\xi_\tau-$orbitally continuous. But, the converse is not necessary true. For example, consider the controlled metric type space defined by $\xi_\tau(g,h) = |g-h|$ for all $g,h \in Y$, where $Y = [0,1]$ and the map $T : Y \longrightarrow Y$ defined by

$$Tg = \begin{cases} \frac{g}{2} & \text{if} \quad 0 \leq g < 1 \\ 0 & \text{if} \quad \quad g = 1 \end{cases}$$

It is not difficult to see that T is not continuous, but T is ξ_τ-orbitally continuous.

Definition 1.7. Let Y be a nonempty set, $T : Y \longrightarrow Y$ be a mapping and $\alpha : Y \times Y \longrightarrow [0, +\infty)$ be a function. Then T is said to be $\alpha-$admissible if for all $g, h \in Y$, we have

$$\alpha(g,h) \geq 1 \Longrightarrow \alpha(Tg, Th) \geq 1.$$

Definition 1.8. A mapping $T : Y \longrightarrow Y$ is called triangular $\alpha-$admissible if it is $\alpha-$admissible and it satisfies the following condition:

$$\alpha(g,h) \geq 1 \;\text{ and }\; \alpha(h,z) \geq 1 \Longrightarrow \alpha(g,z) \geq 1.$$

Notation: Let T be a self mapping on a controlled metric type space (Y, ξ_τ). For $g, h \in Y$, set

$$M(g,h) = \max\{\xi_\tau(g,h), \xi_\tau(Tg,g), \xi_\tau(Th,h)\}, \tag{1.2}$$

and

$$N(g,h) = \max(\xi_\tau(g,h), \frac{\xi_\tau(g,Tg) + \xi_\tau(h,Th)}{2}). \tag{1.3}$$

Remark 1.9. Note that for all $g, h \in Y$, we have $N(g,h) \leq M(g,h)$.

2 Main result

Our first main result is as follows:

Theorem 2.1. *Let (Y, ξ_τ) be a complete controlled metric type space and T be a triangular $\alpha-$admissible mapping on Y. Suppose that the following conditions hold:*

(i) there exists $g_0 \in Y$ such that $\alpha(Tg_0, g_0) \geq 1$ and $\alpha(g_0, Tg_0) \geq 1$;

(ii) if $(g_n)_n$ is a sequence in Y that converges to z as $n \longrightarrow \infty$, and $\alpha(g_n, g_m) \geq 1$ for all $n, m \in \mathbb{N}$, then $\alpha(g_n, z) \geq 1$ for all $n \in \mathbb{N}$;

(iii) if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$2s\epsilon \leq \xi_\tau(h, Th)\frac{1+\xi_\tau(g, Tg)}{1+M(g,h)} + N(g,h) < s(2\epsilon+\delta),$$

then we have $\alpha(g,h)\xi_\tau(Tg,Th) < \epsilon$, where $s = \sup\{\tau(g_n, g_m),\ \ n,m \in \mathbb{N}\} > 1$ with $g_n = T^n(g_0)$;

(iv) for every $g \in Y$, $\lim_n \alpha(g, T^n g_0)$ and $\lim_n (T^n g_0, g)$ exist and are finite;

(iv) T is $\xi_\tau-$orbitally continuous.

Then T has a fixed point in Y.

Proof. From condition (iii), we can easily deduce that

$$\alpha(g,h)\xi_\tau(Tg,Th) < \frac{1}{2s}[\xi_\tau(h,Th)\frac{1+\xi_\tau(g,Tg)}{1+M(g,h)} + N(g,h)].$$

Let $g_0 \in Y$ satisfy condition (i) and define the sequence $(g_n)_n$ by

$$g_{n+1} = Tg_n, \quad n = 0, 1, \cdots.$$

First, note that if there exists $p \geq 0$ such that $g_{p+1} = g_p$, then we are done and the fixed point is g_p. Thus, we may assume that $g_{n+1} \neq g_n$ for all $n \in \mathbb{N}$. Using again condition (i) and the fact that T is a triangular $\alpha-$admissible mapping, we can easily deduce that $\alpha(g_n, g_m) \geq 1$ for all $n, m \in \mathbb{N}$.

Note that

$$M(g_n, g_{n+1}) := \max\{\xi_\tau(g_n, g_{n+1}), \xi_\tau(g_{n+1}, g_{n+2})\}.$$

Next, we are going to complete our proof in the following steps:

Step 1 : $M(g_n, g_{n+1}) = \xi_\tau(g_n, g_{n+1})$ for each $n \geq 0$.

If for some n, $M(g_n, g_{n+1}) = \xi_\tau(g_{n+1}, g_{n+2})$, then

$$\begin{aligned} 0 < \xi_\tau(g_{n+1}, g_{n+2}) &\leq \alpha(g_{n+1}, g_n)\xi_\tau(Tg_{n+1}, Tg_n) \\ &< \frac{1}{2s}[\xi_\tau(g_{n+1}, g_{n+2})\frac{1+\xi_\tau(g_n, g_{n+1})}{1+M(g_n, g_{n+1})} + N(g_n, g_{n+1})] \\ &\leq \frac{1}{2s}[\xi_\tau(g_{n+1}, g_{n+2}) + M(g_n, g_{n+1})] \\ &= \frac{1}{2s}[\xi_\tau(g_{n+1}, g_{n+2}) + \xi_\tau(g_{n+1}, g_{n+2})] \\ &\leq \frac{1}{s}\xi_\tau(g_{n+1}, g_{n+2}). \end{aligned}$$

It is a contradiction with the fact that $s > 1$. Then $M(g_n, g_{n+1}) = \xi_\tau(g_n, g_{n+1})$ for each n.

Step 2 : $(\xi_\tau(g_n, g_{n+1}))_n$ is a strictly decreasing sequence.

From step 1,

$$\xi_\tau(g_{n+1}, g_{n+2}) \leq M(g_n, g_{n+1}) = \xi_\tau(g_n, g_{n+1}),$$

for each $n \geq 0$. Thus, the sequence $(\xi_\tau(g_n, g_{n+1}))_n$ is decreasing.

Step 3 : $(\xi_\tau(g_n, g_{n+1}))_n$ converges to 0. The sequence $\{\xi_\tau(g_n, g_{n+1})\}_n$ is decreasing, so it converges to some number, say $r \geq 0$. Suppose that $r > 0$. Let $\epsilon = \frac{r}{s}(> 0)$ and choose $\delta > 0$ such that the condition (iii) is verified. Since $\lim_{n\to+\infty}(\xi_\tau(g_{n+1}, g_{n+2}) + \xi_\tau(g_n, g_{n+1})) = 2r$, it follows that there is $N_0 \in \mathbb{N}$ such that

$$2r < \xi_\tau(g_{N_0+1}, g_{N_0+2}) + \xi_\tau(g_{N_0}, g_{N_0+1}) < 2r + \delta.$$

Therefore,

$$\begin{aligned} 2s\epsilon &< \xi_\tau(g_{N_0+1}, g_{N_0+2}) + \xi_\tau(g_{N_0}, g_{N_0+1}) \\ &= \xi_\tau(g_{N_0+1}, Tg_{N_0+1})\frac{1+\xi_\tau(g_{N_0}, Tg_{N_0})}{1+M(g_{N_0}, g_{N_0+1})} + N(g_{N_0}, g_{N_0+1}) \\ &< 2s\epsilon + \delta < s(2\epsilon + \delta). \end{aligned}$$

From the condition (iii), we deduce that

$$\xi_\tau(g_{N_0+1}, g_{N_0+2}) \leq \alpha(g_{N_0}, g_{N_0+1})\xi_\tau(Tg_{N_0}, Tg_{N_0+1}) < \epsilon = \frac{r}{s} < r.$$

It is a contradiction with the fact that $r \leq \xi_\tau(g_n, g_{n+1})$ for each $n \in \mathbb{N}$. Thus, $r = 0$.

Step 4 : $\xi_\tau(g_m, g_n)$ tends to 0 as $n, m \to \infty$.

Given $\epsilon > 0$. Set $\epsilon' = \frac{\epsilon}{5s^2}$ and $\delta' = \min(\delta, \epsilon', 1)$. Since $\lim_n \xi_\tau(g_m, g_{m+1}) = 0$, there exists $k_0 \in \mathbb{N}$ such that $\xi_\tau(g_m, g_{m+1}) \leq \frac{\delta'}{4}$, for all $m \geq k_0$. Let $\mu = s(2\epsilon' + \frac{\delta'}{2})$ and consider the set

$$D[g_{k_0}, \mu] = \{g_i | i > k_0, \xi_\tau(g_i, g_{k_0}) < \mu\}.$$

We prove first that T maps $D[g_{k_0}, \mu]$ to itself. Since $\xi_\tau(g_{k_0}, g_{k_0+1}) = \xi_\tau(g_{k_0}, Tg_{k_0}) \leq \frac{\delta'}{4} < \mu$, it follows that $g_{k_0+1} \in D[g_{k_0}, \mu]$. Let $g_l \in D[g_{k_0}, \mu]$ with $l > k_0$, so $\xi_\tau(g_l, g_{k_0}) < \mu$. Case 1: Suppose that $2s\epsilon' \leq \xi_\tau(g_l, g_{k_0})$. Then

$$2s\epsilon' \leq \xi_\tau(g_l, g_{k_0}) < \mu.$$

We have

$$\begin{aligned} \epsilon' &\leq \frac{1}{2s}\xi_\tau(g_l, g_{k_0}) \\ &\leq \frac{1}{2s}[\xi_\tau(g_{k_0}, g_{k_0+1})\frac{1 + \xi_\tau(g_l, g_{l+1})}{1 + M(g_l, g_{k_0})} + \xi_\tau(g_l, g_{k_0})]. \end{aligned}$$

Since $\xi_\tau(g_l, g_{k_0}) \leq N(g_l, g_{k_0})$, it follows that

$$\epsilon' \leq \frac{1}{2s}[\xi_\tau(g_{k_0}, g_{k_0+1})\frac{1 + \xi_\tau(g_l, g_{l+1})}{1 + M(g_l, g_{k_0})} + N(g_l, g_{k_0})].$$

From the inequality $\xi_\tau(g_l, g_{l+1}) \leq M(g_l, g_{k_0})$, one gets

$$\epsilon' \leq \frac{1}{2s}[\xi_\tau(g_{k_0}, g_{k_0+1})\frac{1 + \xi_\tau(g_l, g_{l+1})}{1 + M(g_l, g_{k_0})} + N(g_l, g_{k_0})] \leq \frac{1}{2s}[\xi_\tau(g_{k_0}, g_{k_0+1}) + N(g_l, g_{k_0})].$$

Note that $N(g_l, g_{k_0}) = \max(\xi_\tau(g_l, g_{k_0}), \frac{\xi_\tau(g_l, g_{l+1}) + \xi_\tau(g_{k_0}, g_{k_0+1})}{2})$.

If for some $l > k_0$, we have $N(g_l, g_{k_0}) = \frac{\xi_\tau(g_l, g_{l+1}) + \xi_\tau(g_{k_0}, g_{k_0+1})}{2}$, then

$$N(g_l, g_{k_0}) \leq \xi_\tau(g_{k_0}, g_{k_0+1}) \leq \frac{\delta'}{4}$$

and consequently,

$$\begin{aligned} \frac{1}{2s}[\xi_\tau(g_{k_0}, g_{k_0+1})\frac{1 + \xi_\tau(g_l, g_{l+1})}{1 + M(g_l, g_{k_0})} + N(g_l, g_{k_0})] \leq & \frac{1}{2s}[\xi_\tau(g_{k_0}, g_{k_0+1}) + N(g_l, g_{k_0})] \\ &\leq \frac{1}{2s}[\frac{\delta'}{4} + \frac{\delta'}{4}] \\ &< \epsilon' + \frac{\delta'}{2}. \end{aligned}$$

Thus,

$$\xi_\tau(g_{k_0}, g_{k_0+1})\frac{1 + \xi_\tau(g_l, g_{l+1})}{1 + M(g_l, g_{k_0})} + N(g_l, g_{k_0}) < s(2\epsilon' + \delta').$$

Now, if for some $l > k_0$, we have $N(g_l, g_{k_0}) = \xi_\tau(g_l, g_{k_0})$, then since $\xi_\tau(g_l, g_{k_0}) < \mu$, it follows that

$$\xi_\tau(g_{k_0}, g_{k_0+1})\frac{1 + \xi_\tau(g_l, g_{l+1})}{1 + M(g_l, g_{k_0})} + N(g_l, g_{k_0}) < \frac{\delta'}{4} + s(2\epsilon' + \frac{\delta'}{2}) < s(2\epsilon' + \delta').$$

We deduce from above that

$$2s\epsilon' \leq \xi_\tau(g_{k_0}, Tg_{k_0})\frac{1+\xi_\tau(g_l, Tg_l)}{1+M(g_l, g_{k_0})} + N(g_l, g_{k_0}) < s(2\epsilon' + \delta').$$

Thus, by condition (iii), we have

$$\xi_\tau(Tg_l, Tg_{k_0}) \leq \alpha(g_l, g_{k_0})\xi_\tau(Tg_l, Tg_{k_0}) < \epsilon'.$$

The triangle inequality yields that

$$\begin{aligned}\xi_\tau(Tg_l, g_{k_0}) &\leq \tau(Tg_l, g_l)\xi_\tau(Tg_l, g_l) + \tau(g_l, g_{k_0})\xi_\tau(g_l, g_{k_0})\\ &= \tau(g_{l+1}, g_l)\xi_\tau(g_{l+1}, g_l) + \tau(g_l, g_{k_0})\xi_\tau(g_l, g_{k_0})\\ &\leq s\frac{\delta'}{4} + s\epsilon' \leq s(2\epsilon' + \frac{\delta'}{2}) = \mu,\end{aligned}$$

which implies that $Tg_l = g_{l+1} \in D[g_{k_0}, \mu]$.
Case 2: Assume that $\xi_\tau(g_l, g_{k_0}) < 2s\epsilon'$. Then

$$\begin{aligned}\xi_\tau(Tg_l, g_{k_0}) &\leq \tau(Tg_l, Tg_{k_0})\xi_\tau(Tg_l, Tg_{k_0}) + \tau(Tg_{k_0}, g_{k_0})\xi_\tau(Tg_{k_0}, g_{k_0})\\ &\leq s\xi_\tau(Tg_l, Tg_{k_0}) + s\xi_\tau(Tg_{k_0}, g_{k_0})\\ &\leq s\alpha(g_l, g_{k_0})\xi_\tau(Tg_l, Tg_{k_0}) + s\xi_\tau(Tg_{k_0}, g_{k_0})\\ &\leq s(\frac{1}{2s}[\xi_\tau(g_{k_0}, g_{k_0+1})\frac{1+\xi_\tau(g_l, g_{l+1})}{1+M(g_l, g_{k_0})} + N(g_l, g_{k_0})]) + s\xi_\tau(Tg_{k_0}, g_{k_0})\\ &\leq \frac{1}{2}\xi_\tau(g_{k_0}, g_{k_0+1}) + \frac{1}{2}N(g_l, g_{k_0}) + s\xi_\tau(Tg_{k_0}, g_{k_0})\\ &\leq \frac{1}{2}\frac{\delta'}{4} + \frac{1}{2}N(g_l, g_{k_0}) + s\frac{\delta'}{4}.\end{aligned}$$

Recall that $N(g_l, g_{k_0}) = \max(\xi_\tau(g_l, g_{k_0}), \frac{\xi_\tau(g_l, g_{l+1}) + \xi_\tau(g_{k_0}, g_{k_0+1})}{2})$.
If $N(g_l, g_{k_0}) = \frac{\xi_\tau(g_l, g_{l+1}) + \xi_\tau(g_{k_0}, g_{k_0+1})}{2}$, then

$$N(g_l, g_{k_0}) \leq \xi_\tau(g_{k_0}, g_{k_0+1}) \leq \frac{\delta'}{4}$$

and consequently,

$$\begin{aligned}\xi_\tau(Tg_l, g_{k_0}) &\leq \frac{1}{2}\frac{\delta'}{4} + \frac{1}{2}N(g_l, g_{k_0}) + s\frac{\delta'}{4}\\ &< \frac{1}{2}\frac{\delta'}{4} + \frac{1}{2}\frac{\delta'}{4} + s\frac{\delta'}{4}\\ &\leq \frac{\delta'}{4} + s\frac{\epsilon'}{4} < \mu.\end{aligned}$$

Now, if $N(g_l, g_{k_0}) = \xi_\tau(g_l, g_{k_0})$, then we have

$$\begin{aligned}\xi_\tau(Tg_l, g_{k_0}) &\leq \frac{1}{2}\frac{\delta'}{4} + \frac{1}{2}N(g_l, g_{k_0}) + s\frac{\delta'}{4}\\ &= \frac{1}{2}\frac{\delta'}{4} + \frac{1}{2}\xi_\tau(g_l, g_{k_0}) + s\frac{\delta'}{4}\\ &\leq \frac{1}{2}\frac{\delta'}{4} + s\epsilon' + s\frac{\delta'}{4} < \mu,\end{aligned}$$

which implies that $Tg_l = g_{l+1} \in D[g_{k_0}, \mu]$.

Now, let $m, n \geq k_0$, then

$$\begin{aligned}\xi_\tau(g_n, g_m) &\leq \tau(g_n, g_{k_0})\xi_\tau(g_n, g_{k_0}) + \tau(g_{k_0}, g_m)\xi_\tau(g_{k_0}, g_m)\\ &\leq s\xi_\tau(g_n, g_{k_0}) + s\xi_\tau(g_{k_0}, g_m)\\ &< s\mu + s\mu = 2s^2(2\epsilon' + \frac{\delta'}{2}) \leq 5s^2\epsilon' = \epsilon.\end{aligned}$$

Thus, $\lim_{n,m} \xi_\tau(g_n, g_m) = 0$. This implies that $(g_n)_n$ is $\xi_\tau-$Cauchy.

Step 5 : T has a fixed point. Since $(g_n)_n$ is $\xi_\tau-$Cauchy and Y is complete, it follows that there exists $u \in Y$ such that

$$\lim_n \xi_\tau(g_n, u) = 0. \tag{2.1}$$

The ξ_τ-orbital continuity of T implies that $Tu = u$.

□

Next, we present the definition of generalized Meir-Keeler contractions of type (I).

Definition 2.2. Let (Y, ξ_τ) be a controlled metric type space and $T : Y \longrightarrow Y$ be an α-admissible mapping. Such T is said to be a generalized Meir-Keeler contraction of type (I) if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq \beta(\xi_\tau(g,h))M(g,h) < \epsilon + \delta \text{ implies } \alpha(g,h)\xi_\tau(Tg,Th) < \epsilon \tag{2.2}$$

where $M(g, h)$ was defined by (1.2) and $\beta : [0,\infty) \longrightarrow [0, \frac{1}{s})$ is a continuous function with a constant $s \geq 1$.

Remark 2.3. If $T : Y \longrightarrow Y$ is a generalized Meir-Keeler contraction of type (I), then

$$\alpha(g,h)\xi_\tau(Tg,Th) < \beta(\xi_\tau(g,h))M(g,h) \tag{2.3}$$

for all $g, h \in Y$ when $M(g, h) > 0$.

Theorem 2.4. *Let (Y, ξ_τ) be a complete controlled metric type space and T be a triangular $\alpha-$admissible mapping. Suppose that the following conditions hold:*

a) T is an $\xi_\tau-$orbitally continuous generalized Meir-Keeler contraction of type (I);

b) there exists $g_0 \in Y$ such that $\alpha(g_0, Tg_0) \geq 1$, $\alpha(Tg_0, g_0) \geq 1$, and the sequences $(\tau(g_n, g))_n$ and $(\tau(g, g_n))_n$ are bounded, where g_n is defined by $g_n = T^n g_0$;

c) if $\{g_n\}$ is a sequence in Y such that $g_n \longrightarrow z$ as $n \longrightarrow \infty$ and $\alpha(g_n, g_m) \geq 1$ for all natural numbers n, m, then $\alpha(z, z) \geq 1$;

d) there exists $s \geq 1$ such that $s = \sup\{\tau(g_n, g_m),\ \ n, m \in \mathbb{N}\}$.

Then T has a fixed point in Y.

Proof. Let $g_0 \in Y$ be such that condition (b) holds and define $\{g_n\}$ in Y so that $g_1 = Tg_0$, $g_{n+1} = Tg_0$ for all natural numbers n. Without loss of generality, we may suppose that $g_{n+1} \neq g_n$ for all $n \geq 0$. Since T is α-admissible, then $\alpha(g_n, g_{n+1}) \geq 1$ for all natural numbers n.
As T is a generalized Meir-Keeler contraction of type (I), then by replacing g by g_n and h by g_{n+1} in (2.10), we observe that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq \beta(\xi_\tau(g_n, g_{n+1}))M(g_n, g_{n+1}) < \epsilon + \delta \Longrightarrow \alpha(g_n, g_{n+1})\xi_\tau(Tg_n, Tg_{n+1}) < \epsilon \tag{2.4}$$

where

$$M(g_n, g_{n+1}) = \max[\xi_\tau(g_n, g_{n+1}), \xi_\tau(g_{n+2}, g_{n+1})]. \tag{2.5}$$

If for some p, $M(g_p, g_{p+1}) = \xi_\tau(g_{p+2}, g_{p+1})$, then equation (2.4) becomes

$$\epsilon \leq \beta(\xi_\tau(g_p, g_{p+1}))\xi_\tau(g_{p+2}, g_{p+1}) < \epsilon + \delta \Longrightarrow \alpha(g_p, g_{p+1})\xi_\tau(Tg_p, Tg_{p+1}) < \epsilon$$

and using that $\alpha(g_p, g_{p+1}) \geq 1$, we have

$$\xi_\tau(Tg_p, Tg_{p+1})\xi_\tau(g_{p+1}, g_{p+2}) < \epsilon \leq \beta(\xi_\tau(g_p, g_{p+1}))\xi_\tau(g_{p+2}, g_{p+1}).$$

Then $\xi_\tau(g_{p+1}, g_{p+2}) < \xi_\tau(g_{p+2}, g_{p+1})$, which gives a contradiction. Thus, $M(g_n, g_{n+1}) = \xi_\tau(g_n, g_{n+1})$ for all $n \geq 0$.
Since $M(g_n, g_{n+1}) > 0$, due to Remark 1.9, we get

$$\begin{aligned} \xi_\tau(g_{n+1}, g_{n+2}) &\leq \alpha(g_n, g_{n+1})\xi_\tau(Tg_n, Tg_{n+1}) \\ &< \epsilon \\ &\leq \beta(\xi_\tau(g_n, g_{n+1}))\xi_\tau(g_n, g_{n+1}) \\ &< \frac{1}{s}\xi_\tau(g_n, g_{n+1}) \leq \xi_\tau(g_n, g_{n+1}). \end{aligned} \tag{2.6}$$

That is, $\{\xi_\tau(g_n, g_{n+1})\}$ is a strictly decreasing positive sequence in $\mathbb{R}^+$ and it converges to some $r \geq 0$. Let us prove that $r = 0$. Assume that $r > 0$. We assert that $0 < r \leq \xi_\tau(g_n, g_{n+1})$ for each $n \geq 0$.
First, suppose that $s > 1$. Applying equation (2.6), we have $\xi_\tau(g_{n+1}, g_{n+2}) < \frac{1}{s}\xi_\tau(g_n, g_{n+1})$. By taking the limit as n tends to infinity, we get $r \leq \frac{1}{s}r < r$, which is a contradiction, and so $r = 0$.

That is,

$$\lim_{n\to\infty} \xi_\tau(g_n, g_{n+1}) = 0.$$

Next, we show that $\{g_n\}$ is an ξ_τ- Cauchy sequence. For this purpose, we will prove that for every $\epsilon > 0$, there exists a natural number N such that for every $l > N$, we have

$$\xi_\tau(g_l, g_{l+k}) < \epsilon. \tag{2.7}$$

Since $\{\xi_\tau(g_n, g_{n+1})\} \longrightarrow 0, n \longrightarrow \infty$, for every $\delta > 0$ there exists a natural number N such that $\xi_\tau(g_n, g_{n+1}) < \delta$ for all $n \geq N$. Choose $\delta < \epsilon$. We will prove (2.7) by using induction on k.
• for $k = 1$, we have $\xi_\tau(g_l, g_{l+1}) < \epsilon \Rightarrow \xi_\tau(g_l, g_{l+1}) < \epsilon$, so (2.7) clearly holds for all $l \geq N$ (due to the choice of δ).
• Assume that the inequality (2.7) holds for some $k = m$, that is, $\xi_\tau(g_l, g_{l+m}) < \epsilon\ \forall l \geq N$.
For $k = m + 1$, we have to show that

$$\xi_\tau(g_l, g_{l+m+1}) < \epsilon \ \ \forall \ \ l \geq N. \tag{2.8}$$

Applying the triangle inequality of the controlled metric type space, one writes

$$\begin{aligned}
\xi_\tau(g_{l-1}, g_{l+m}) &\leq \tau(g_{l-1}, g_{l+m})[\xi_\tau(g_{l-1}, g_l) + \xi_\tau(g_l, g_{l+m})] \\
&\leq s[\xi_\tau(g_{l-1}, g_l) + \xi_\tau(g_l, g_{l+m})] \\
&\leq s[\xi_\tau(g_{l-1}, g_l) + \xi_\tau(g_l, g_{l+m})] \\
&\leq s[\delta + \epsilon]; \ \ \forall l \geq N.
\end{aligned}$$

If $\beta(\xi_\tau(g_{l-1}, g_{l+m}))\xi_\tau(g_{l-1}, g_{l+m}) \geq \epsilon$, then we deduce

$$\begin{aligned}
\epsilon &\leq \beta(\xi_\tau(g_{l-1}, g_{l+m}))\xi_\tau(g_{l-1}, g_{l+m}) \\
&\leq \beta(\xi_\tau(g_{l-1}, g_{l+m}))M(g_{l-1}, g_{l+m}) \\
&= \beta(\xi_\tau(g_{l-1}, g_{l+m}))\max[\xi_\tau(g_{l-1}, g_{l+m}), \xi_\tau(g_l, g_{l-1}), \xi_\tau(g_{l+m+1}, g_{l+m})] \\
&< \beta(\xi_\tau(g_{l-1}, g_{l+m}))\max[s(\delta + \epsilon), \delta, \delta] \\
&< \delta + \epsilon.
\end{aligned}$$

Using (2.4) with $g = g_{l-1}$, $h = g_{l+m}$, we find

$$\epsilon \leq \beta(\xi_\tau(g_{l-1}, g_{l+m}))M(g_{l-1}, g_{l+m}) < \epsilon + \delta,$$

then

$$\alpha(g_{l-1}, g_{l+m})\xi_\tau(Tg_{l-1}, Tg_{l+m}) < \epsilon$$

which gives $\xi_\tau(g_l, g_{l+m+1}) < \epsilon$. Hence, (2.4) holds for $k = m + 1$.

If $\beta(\xi_\tau(g_{l-1}, g_{l+m}))\xi_\tau(g_{l-1}, g_{l+m}) < \epsilon$, then

$$\begin{aligned}
\beta(\xi_\tau(g_{l-1}, g_{l+m}))M(g_{l-1}, g_{l+m}) &= \beta(\xi_\tau(g_{l-1}, g_{l+m}))\max[\xi_\tau(g_{l-1}, g_{l+m}), \\
&\qquad \xi_\tau(g_l, g_{l-1}), \xi_\tau(g_{l+m+1}, g_{l+m})] \\
&< \beta(\xi_\tau(g_{l-1}, g_{l+m}))\max[\xi_\tau(g_{l-1}, g_{l+m}), \delta, \delta] \\
&< \epsilon.
\end{aligned}$$

From Remark 1.9, we get

$$\alpha(g_{l-1},g_{l+m})\xi_\tau(Tg_{l-1},Tg_{l+m}) < \beta(\xi_\tau(g_{l-1},g_{l+m}))M(g_{l-1},g_{l+m}) < \epsilon$$

then

$$\alpha(g_{l-1},g_{l+m})\xi_\tau(g_l,g_{l+m+1}) < \epsilon.$$

So

$$\xi_\tau(g_l,g_{l+m+1}) < \alpha(g_{l-1},g_{l+m})\xi_\tau(g_l,g_{l+m+1}) < \epsilon$$

that is, (2.7) holds for $k = m+1$.
Note that $M(g_{l-1},g_{l+m}) > 0$, otherwise $\xi_\tau(g_l,g_{l-1}) = 0$ and hence $g_l = g_{l-1}$, which is contradiction. Thus, $\xi_\tau(g_l,g_{l+k}) < \epsilon \ \forall l \geq N$ and $k \geq 1$, it means

$$\xi_\tau(g_n,g_m) < \epsilon \ \ \forall \ \ m \geq n \geq N. \tag{2.9}$$

We deduce that $\{g_n\}$ is an $\xi_\tau-$Cauchy sequence. Since Y is a complete controlled metric type space, there exists $u \in Y$ such that $\lim\limits_{n\to\infty} \xi_\tau(g_n,u) = 0$.
Now, we will show that $Tu = u$. For any integer n, the sequences $(\tau(g_n,g))_n$ and $(\tau(g,g_n))_n$ are bounded by some $C \geq 1$.

$$\begin{aligned}
\xi_\tau(u,Tu) &\leq \tau(u,g_{n+1})\xi_\tau(u,g_{n+1}) + \tau(g_{n+1},Tu)\xi_\tau(g_{n+1},Tu)\\
&\leq \tau(u,g_{n+1})\xi_\tau(u,g_{n+1}) + \tau(Tg_n,Tu)\xi_\tau(Tg_n,Tu)\\
&\leq C\xi_\tau(u,g_{n+1}) + C\xi_\tau(Tg_n,Tu).
\end{aligned}$$

Since $\lim_n \xi_\tau(g_{n+1},u) = 0$ and $\lim_n \xi_\tau(Tg_n,Tu) = 0$ (due to the fact that T is ξ_τ-orbitally continuous), thereby $\xi_\tau(u,Tu) = 0$. Thus, $Tu = u$. □

Next, we present the definition of generalized Meir-Keeler contractions of type (II).

Definition 2.5. Let (Y,ξ_τ) be a controlled metric type space. An α-admissible mapping $T : Y \longrightarrow Y$ is said to be a generalized Meir-Keeler contraction of type (II) if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq \beta(\xi_\tau(g,h))N(g,h) < \epsilon+\delta \text{ implies } \alpha(g,h)\xi_\tau(Tg,Th) < \epsilon \tag{2.10}$$

where $N(g,h)$ was defined by (1.3) and $\beta : [0,\infty) \longrightarrow (0,\frac{1}{s})$ is a continuous function with a constant $s \geq 1$.

Theorem 2.6. *Let (Y,ξ_τ) be a complete controlled metric type space and T be a triangular $\alpha-$admissible mapping on Y. Suppose that the following conditions hold:*

- *a) T is an ξ_τ-orbitally continuous generalized Meir-Keeler contraction of type (II);*
- *b) there exists $g_0 \in Y$ such that $\alpha(g_0,Tg_0) \geq 1$, $\alpha(Tg_0,g_0) \geq 1$, the sequences $(\tau(g_n,g))_n$ and $(\tau(g,g_n))_n$ are bounded, where g_n is defined by $g_n = T^n g_0$;*
- *c) if $\{g_n\}$ is a sequence in Y such that $g_n \longrightarrow z$ as $n \longrightarrow \infty$ and $\alpha(g_n,g_m) \geq 1$ for all $n,m \in \mathbb{N}$, then $\alpha(z,z) \geq 1$;*
- *d) assume there exists $s \geq 1$ such that $s = \sup\{\tau(g_n,g_m),\ \ n,m \in \mathbb{N}\}$.*

Then T has a unique fixed point in Y.

Proof. By remark 1.9 , we have $N(g,h) \leq M(g,h)$. Hence, similar to the proof of Theorem 2.4, the result of our theorem will follow as desired. □

Theorem 2.7. *Let (Y,ξ_τ) be a complete controlled metric type space and T be a triangular $\alpha-$admissible mapping on Y. Suppose that the following conditions hold:*

- *a) if $\{g_n\}$ is a sequence in Y which converges to z with respect to τ_{ξ_τ} and satisfies $\alpha(g_{n+1},g_n) \geq 1$ and $\alpha(g_n,g_{n+1}) \geq 1$ for all n, then there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $\alpha(g_z,g_{n_k}) \geq 1$ and $\alpha(g_{n_k},g_z) \geq 1$ for all k;*

b) $T : Y \longrightarrow Y$ is a generalized Meir-Keeler contraction of type (II);

c) there exists $g_0 \in Y$ such that $\alpha(g_0, T_{g_0}) \geq 1$, $\alpha(T_{g_0}, g_0) \geq 1$;

d) assume there exists $s \geq 1$ such that $s = \sup\{\tau(g_n, g_m),\ \ n, m \in \mathbb{N}\}$.

Then T has a fixed point in Y.

Proof. By the proof of Theorem 2.4, one can easily deduce that $\{g_n\}$ defined by $g_1 = T_{g_0}$ and $g_{n+1} = T_{g_n}$ $(n \in \mathbb{N})$ converges to some $z \in Y$ with $\xi_\tau(z, z) = 0$. By condition a), there exists a subsequence $\{g_{n_k}\}$ of g_n such that $\alpha(z, g_{n_k}) \geq 1$ and $\alpha(g_{n_k}, z) \geq 1$ for all k.
Note that, if $N(z, g_{n_k}) = 0$, then $Tz = z$ and we are done.
Now, by remark 1.9 for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \xi_\tau(Tz, g_{n+1}) = \xi_\tau(Tz, Tg_n) &\leq \alpha(z, g_{n_k})\xi_\tau(Tz, T_{g_{n_k}}) \\ &< \beta(\xi_\tau(z, g_{n_k}))N(z, g_{n_k}). \end{aligned}$$

Taking the limit $k \longrightarrow \infty$, we obtain $\lim_{k\to\infty} N(z, g_{n_k}) = \max\{0, \frac{1}{2}\xi_\tau(Tz, z)\} = \frac{1}{2}\xi_\tau(Tz, z)$.
Thus, $\lim_{n\to\infty} \xi_\tau(Tz, g_{n_{k+1}}) \leq \frac{1}{2s}\xi_\tau(Tz, z)$. Hence,

$$\xi_\tau(Tz, z) \leq s\xi_\tau(Tz, g_{n_{k+1}}) + s\xi_\tau(g_{n_{k+1}}, z).$$

Taking the limit $k \longrightarrow \infty$, we obtain

$$\xi_\tau(Tz, z) \leq \frac{1}{2}\xi_\tau(Tz, z),$$

which implies $\xi_\tau(Tz, z) = 0$, and therefore, $Tz = z$ as desired. □

3 Conclusion

Notice that, we proved the existence of fixed point for three types of Meir-Keeler contractive mappings under the condition of ξ_τ-orbitally continuity. It is an open question that for a generalized Meir-Keeler contractive mappings of these type, can we omit ξ_τ-orbitally continuity hypothesis and still get a fixed point or maybe we can change it with a weaker hypothesis.

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