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A NEW NON-LINEAR IDENTITY CLASS FOR TERMS OF A GENERALISED LINEAR DEGREE FOUR RECURRENCE SEQUENCE

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Abstract We state, and prove by a succinct matrix method deployed previously, a result for terms of a sequence generated by a fully general (linear) fourth order recurrence equation which—being characterised by four additional independent parameters—describes a new class of highly non-linear identities.

1 The Result

Consider, with initial values $v_0 = a$, $v_1 = b$, $v_2 = c$ and $v_3 = d$, the linear recursion

$$v_{n+4} = pv_{n+3} + qv_{n+2} + rv_{n+1} + sv_n, \qquad n \geq 0, \tag{1.1}$$

of degree four. In this short paper we state, and prove by a succinct matrix method deployed previously (for recurrences of respective order two [1] and three [2]), an identity class for the sequence $\{v_n\}_{n=0}^{\infty} = \{v_n\}_0^{\infty} = \{v_n(a,b,c,d;p,q,r,s)\}_0^{\infty} = \{a,b,c,d,v_4,v_5,\ldots\}$ which is characterised by four independent parameters (in addition to the variables p,q,r,s,a,b,c,d) and is highly non-linear in nature; when the order is two ($r = s = 0$, $v_{2,3} = v_{2,3}(a,b,p,q)$ determined), the terms of any class instance can align themselves with those of the celebrated Horadam sequence $\{v_n(a,b;p,-q)\}_0^{\infty}$ [1]. We believe the technique underpinning the formulation presented here—though simple enough—is novel, and this article demonstrates its robustness.

Governing Identity. Let $s_1 = (\gamma p + \epsilon)/s$, $s_2 = \alpha = s_5$, $s_3 = \beta = s_9$, $s_4 = \gamma = s_{13}$, $s_6 = [\beta s + \gamma pq + \epsilon q - \alpha ps]/s$, $s_7 = r(\gamma p + \epsilon)/s + \gamma - \beta p = s_{10}$, $s_8 = \epsilon = s_{14}$, $s_{11} = \alpha r - \beta q + \epsilon$, $s_{12} = \alpha s - \gamma q = s_{15}$, and $s_{16} = \beta s - \gamma r$ ($s \neq 0$ assumed), the variables $s_1, \ldots, s_{16}$ being characterised by arbitrary constants α, β, γ and ϵ along with the recurrence parameters p,q,r,s of (1.1). Then, for $i,j \geq 0$ the following result holds, describing a class of identities:

$$\begin{aligned} & v_{i+3}(s_1 v_{j+3} + s_2 v_{j+2} + s_3 v_{j+1} + s_4 v_j) + v_{i+2}(s_5 v_{j+3} + s_6 v_{j+2} + s_7 v_{j+1} + s_8 v_j) \\ & \quad + v_{i+1}(s_9 v_{j+3} + s_{10} v_{j+2} + s_{11} v_{j+1} + s_{12} v_j) + v_i(s_{13} v_{j+3} + s_{14} v_{j+2} + s_{15} v_{j+1} + s_{16} v_j) \\ & \qquad = (ds_1 + cs_5 + bs_9 + as_{13})v_{i+j+3} + (ds_2 + cs_6 + bs_{10} + as_{14})v_{i+j+2} \\ & \qquad\quad + (ds_3 + cs_7 + bs_{11} + as_{15})v_{i+j+1} + (ds_4 + cs_8 + bs_{12} + as_{16})v_{i+j}. \end{aligned}$$

We present the proof of this generalised Governing Identity accordingly.

2 The Proof

Proof. Let $\mathbf{J}(p,q,r,s)$ be the matrix

$$\mathbf{J}(p,q,r,s) = \begin{pmatrix} p & q & r & s \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{P.1}$$

which captures the recurrence equation (1.1) and possesses the important property

$$\begin{pmatrix} v_{n+3} \\ v_{n+2} \\ v_{n+1} \\ v_n \end{pmatrix} = \mathbf{J}^n(p,q,r,s) \begin{pmatrix} v_3 \\ v_2 \\ v_1 \\ v_0 \end{pmatrix}, \qquad n \geq 1. \tag{P.2}$$

The proof hinges partly on a so called 'quasi-commutativity' condition

$$\mathbf{S}(s_1,\ldots,s_{16})\mathbf{J}(p,q,r,s) = [\mathbf{J}(p,q,r,s)]^T\mathbf{S}(s_1,\ldots,s_{16}) \tag{P.3}$$

(where T denotes transposition) that must exist between $\mathbf{J}(p,q,r,s)$ and a matrix

$$\mathbf{S} = \mathbf{S}(s_1,s_2,\ldots,s_{16}) = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_5 & s_6 & s_7 & s_8 \\ s_9 & s_{10} & s_{11} & s_{12} \\ s_{13} & s_{14} & s_{15} & s_{16} \end{pmatrix}, \tag{P.4}$$

say, which generates the system of equations

$$\begin{aligned} s_2 &= s_5, \\ qs_1 + s_3 &= ps_2 + s_6, \\ rs_1 + s_4 &= ps_3 + s_7, \\ ss_1 &= ps_4 + s_8, \\ ps_5 + s_6 &= qs_1 + s_9, \\ qs_5 + s_7 &= qs_2 + s_{10}, \\ rs_5 + s_8 &= qs_3 + s_{11}, \\ ss_5 &= qs_4 + s_{12}, \\ ps_9 + s_{10} &= rs_1 + s_{13}, \\ qs_9 + s_{11} &= rs_2 + s_{14}, \\ rs_9 + s_{12} &= rs_3 + s_{15}, \\ ss_9 &= rs_4 + s_{16}, \\ ps_{13} + s_{14} &= ss_1, \\ qs_{13} + s_{15} &= ss_2, \\ rs_{13} + s_{16} &= ss_3, \\ s_{13} &= s_4, \end{aligned} \tag{P.5}$$

whose (infinite) solution set for $s_1,\ldots,s_{16}$ (containing six arbitrary (that is, 'free') constants $\alpha,\beta,\gamma,\delta,\epsilon,\theta$) forms the entries of $\mathbf{S}$:

$$\begin{aligned} \mathbf{S} &= \mathbf{S}(p,q,r,s,\alpha,\beta,\gamma,\delta,\epsilon,\theta) \\ &= \begin{pmatrix} (\gamma p+\epsilon)/s & \alpha & \beta & \gamma \\ \alpha & [(\gamma p+\epsilon)q+(\beta-\alpha p)s]/s & \delta & \epsilon \\ \beta & \delta & \alpha r-\beta q+\epsilon & \theta \\ \gamma & \epsilon & \theta & \beta s-\gamma r \end{pmatrix}. \end{aligned} \tag{P.6}$$

However, with $\mathbf{S}$ evidently symmetric it follows that the product matrices $\mathbf{SJ}$ and $\mathbf{J}^T\mathbf{S}$, as well as being equal (by (P.3)), will also each be symmetric (trivial reader exercise, or else see the Remark of [2, p. 12]). This is found to force relations

$$\begin{aligned} \delta &= \delta(p,r,s,\beta,\gamma,\epsilon) &&= r(\gamma p+\epsilon)/s+\gamma-\beta p, \\ \theta &= \theta(q,s,\alpha,\gamma) &&= \alpha s-\gamma q, \end{aligned} \tag{P.7}$$

and reduces the number of independent parameters involved from six to four (the interested reader is referred to the Appendix for details); we now write the matrix $\mathbf{S}$ as $\mathbf{S}^{[\alpha,\beta,\gamma,\epsilon]}(p,q,r,s)$ whose δ and θ entries in (P.6) take the form seen in (P.7) and so become redundant ones in themselves (as reflected in the listing of $s_1,\ldots,s_{16}$ in the statement of the Governing Identity).

To construct the result itself, we proceed in a routine algebraic fashion as follows. First, we define a function

$$\begin{aligned}
&\mathbf{L}_n^{[\alpha,\beta,\gamma,\epsilon]}(p,q,r,s;v_0,v_1,v_2,v_3)\\
&\quad= (v_3,v_2,v_1,v_0)\mathbf{S}^{[\alpha,\beta,\gamma,\epsilon]}(p,q,r,s)\begin{pmatrix} v_{n+3}\\ v_{n+2}\\ v_{n+1}\\ v_n\end{pmatrix}\\
&\quad= (v_3s_1+v_2s_5+v_1s_9+v_0s_{13})v_{n+3}+(v_3s_2+v_2s_6+v_1s_{10}+v_0s_{14})v_{n+2}\\
&\qquad+(v_3s_3+v_2s_7+v_1s_{11}+v_0s_{15})v_{n+1}+(v_3s_4+v_2s_8+v_1s_{12}+v_0s_{16})v_n, \qquad \text{(P.8)}
\end{aligned}$$

after a little algebra; in other words,

$$\begin{aligned}
&\mathbf{L}_{i+j}^{[\alpha,\beta,\gamma,\epsilon]}(p,q,r,s;v_0,v_1,v_2,v_3)\\
&\quad= (ds_1+cs_5+bs_9+as_{13})v_{i+j+3}+(ds_2+cs_6+bs_{10}+as_{14})v_{i+j+2}\\
&\qquad+(ds_3+cs_7+bs_{11}+as_{15})v_{i+j+1}+(ds_4+cs_8+bs_{12}+as_{16})v_{i+j}. \qquad \text{(P.9)}
\end{aligned}$$

On the other hand, we may write

$$\begin{aligned}
&\mathbf{L}_{i+j}^{[\alpha,\beta,\gamma,\epsilon]}(p,q,r,s;v_0,v_1,v_2,v_3)\\
&\quad= (v_3,v_2,v_1,v_0)\mathbf{S}^{[\alpha,\beta,\gamma,\epsilon]}(p,q,r,s)\mathbf{J}^{i+j}(p,q,r,s)\begin{pmatrix} v_3\\ v_2\\ v_1\\ v_0\end{pmatrix} \qquad \text{(P.10)}
\end{aligned}$$

using (P.2) (with $n=i+j$),

$$\begin{aligned}
&= (v_3,v_2,v_1,v_0)\mathbf{S}^{[\alpha,\beta,\gamma,\epsilon]}(p,q,r,s)\mathbf{J}^{i}(p,q,r,s)\mathbf{J}^{j}(p,q,r,s)\begin{pmatrix} v_3\\ v_2\\ v_1\\ v_0\end{pmatrix}\\
&= (v_3,v_2,v_1,v_0)[\mathbf{J}^{i}(p,q,r,s)]^T\mathbf{S}^{[\alpha,\beta,\gamma,\epsilon]}(p,q,r,s)\mathbf{J}^{j}(p,q,r,s)\begin{pmatrix} v_3\\ v_2\\ v_1\\ v_0\end{pmatrix}, \qquad \text{(P.11)}
\end{aligned}$$

where (P.3) has been applied (we have used the derivative property that if the matrix $\mathbf{S}$ quasi-commutes with $\mathbf{J}$, then it does so with any power of $\mathbf{J}$—see Remark 2.1 overleaf). This in turn

$$\begin{aligned}
&= \left[\mathbf{J}^{i}(p,q,r,s)\begin{pmatrix} v_3\\ v_2\\ v_1\\ v_0\end{pmatrix}\right]^T\mathbf{S}^{[\alpha,\beta,\gamma,\epsilon]}(p,q,r,s)\mathbf{J}^{j}(p,q,r,s)\begin{pmatrix} v_3\\ v_2\\ v_1\\ v_0\end{pmatrix}\\
&= \begin{pmatrix} v_{i+3}\\ v_{i+2}\\ v_{i+1}\\ v_i\end{pmatrix}^T\mathbf{S}^{[\alpha,\beta,\gamma,\epsilon]}(p,q,r,s)\begin{pmatrix} v_{j+3}\\ v_{j+2}\\ v_{j+1}\\ v_j\end{pmatrix} \qquad \text{(P.12)}
\end{aligned}$$

by (P.2) again,

$$\begin{aligned} = \quad & v_{i+3}(s_1 v_{j+3} + s_2 v_{j+2} + s_3 v_{j+1} + s_4 v_j) + v_{i+2}(s_5 v_{j+3} + s_6 v_{j+2} + s_7 v_{j+1} + s_8 v_j) \\ & + v_{i+1}(s_9 v_{j+3} + s_{10} v_{j+2} + s_{11} v_{j+1} + s_{12} v_j) \\ & + v_i(s_{13} v_{j+3} + s_{14} v_{j+2} + s_{15} v_{j+1} + s_{16} v_j) \end{aligned} \tag{P.13}$$

when expanded; the Governing Identity is immediate on reconciling (P.9) and (P.13). □

Remark 2.1. The proof (for arbitrary matrices $\mathbf{J}, \mathbf{S}$ satisfying the quasi-commutativity condition (P.3)) that $\mathbf{SJ} = \mathbf{J}^T\mathbf{S} \Rightarrow \mathbf{SJ}^n = (\mathbf{J}^n)^T\mathbf{S}$ (for $n \geq 1$) is a simple one by induction (this has been omitted in [1] and [2], and is included here merely for completeness): *Proof.* Assuming the result holds for some $n = k \geq 1$ (it is valid automatically at $n = 1$), then we consider $\mathbf{SJ}^{k+1} = \mathbf{S}(\mathbf{J}^k\mathbf{J}) = (\mathbf{SJ}^k)\mathbf{J} = ((\mathbf{J}^k)^T\mathbf{S})\mathbf{J}$ (by assumption) $= (\mathbf{J}^k)^T(\mathbf{SJ}) = (\mathbf{J}^k)^T(\mathbf{J}^T\mathbf{S}) = ((\mathbf{J}^k)^T\mathbf{J}^T)\mathbf{S} = (\mathbf{JJ}^k)^T\mathbf{S} = (\mathbf{J}^{k+1})^T\mathbf{S}$. Q.E.D.

Remark 2.2. The aforementioned redundancy of δ and θ by use of (P.7) is not a unique move. For instance, computational tests have validated the Governing Identity using alternative re-writes $\alpha = (\gamma q + \theta)/s$, $\gamma = (\alpha s - \theta)/q$, $\beta = [r(\gamma p + \epsilon)/s + \gamma - \delta]/p$ and $\epsilon = s(\beta p + \delta - \gamma)/r - \gamma p$, both separately and in combination (with further choices available, too); while this necessarily alters some of $s_1, \ldots s_{16}$ as stated, the structure and complexity of the identity remain unchanged however the reduction of the parameter set $\{\alpha, \beta, \gamma, \delta, \epsilon, \theta\}$ (from six to four) is effected.

3 Summary

A known methodology to establish an identity class for terms of a linear recurrence sequence—applied recently to the degree 2 and 3 cases [1, 2]—has been used here in the degree 4 case for the first time. The result has been verified algebraically, by computer, for a host of numeric i, j values (using terms of the sequence $\{v_n(a, b, c, d; p, q, r, s)\}_0^\infty$ generated, in the absence of a general term closed form, from the recurrence equation (1.1)) with all parameters $a, b, c, d, p, q, r, s, \alpha, \beta, \gamma, \epsilon$ in symbolic form.

The approach taken here has been demonstrated to be robust, but algebraic complexity limits its use in specific instances of even higher order—for this reason similar treatment of an *arbitrary* degree recurrence sequence would seem to be an intractable one, and is left as an open problem.

Appendix

With $\mathbf{S}, \mathbf{J}$ as in (P.6),(P.1), the product matrices $\mathbf{SJ}$ and $\mathbf{J}^T\mathbf{S}$ are, respectively,

$$\begin{pmatrix} p(\gamma p + \epsilon)/s + \alpha & q(\gamma p + \epsilon)/s + \beta & r(\gamma p + \epsilon)/s + \gamma & \gamma p + \epsilon \\ q(\gamma p + \epsilon)/s + \beta & \alpha q + \delta & \alpha r + \epsilon & \alpha s \\ \beta p + \delta & \alpha r + \epsilon & \beta r + \theta & \beta s \\ \gamma p + \epsilon & \gamma q + \theta & \beta s & \gamma s \end{pmatrix} \tag{A.1}$$

and

$$\begin{pmatrix} p(\gamma p + \epsilon)/s + \alpha & q(\gamma p + \epsilon)/s + \beta & \beta p + \delta & \gamma p + \epsilon \\ q(\gamma p + \epsilon)/s + \beta & \alpha q + \delta & \alpha r + \epsilon & \gamma q + \theta \\ r(\gamma p + \epsilon)/s + \gamma & \alpha r + \epsilon & \beta r + \theta & \beta s \\ \gamma p + \epsilon & \alpha s & \beta s & \gamma s \end{pmatrix}, \tag{A.2}$$

with

$$\mathbf{SJ} - \mathbf{J}^T\mathbf{S} = \begin{pmatrix} 0 & 0 & -f_2 & 0 \\ 0 & 0 & 0 & f_1 \\ f_2 & 0 & 0 & 0 \\ 0 & -f_1 & 0 & 0 \end{pmatrix}, \tag{A.3}$$

where $f_1 = \alpha s - \gamma q - \theta$, $f_2 = \beta p + \delta - r(\gamma p + \epsilon)/s - \gamma$. Solving $f_1 = 0$ for θ and $f_2 = 0$ for δ yields (P.7), forcing (as required) $\mathbf{SJ}$ and $\mathbf{J}^T\mathbf{S}$ to be both equal and symmetric.

References

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