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ON OSCILLATORY SECOND ORDER DIFFERENTIAL EQUATIONS WITH VARIABLE DELAYS

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Abstract In this work, we establish the sufficient conditions for oscillation of the second order neutral delay differential equations of the form:

$$(r(t)((x(t)+p(t)x(\tau(t)))')^{\gamma})'+q(t)x^{\alpha}(\sigma(t))+v(t)x^{\beta}(\eta(t))=0$$

under the assumption that

$$\int_0^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}dt \quad < \quad \infty$$

for various ranges of $p(t)$, where α, β and γ are the quotients of odd positive integers.

1 Introduction

Consider a class of nonlinear neutral delay differential equations of the form:

$$(r(t)((x(t)+p(t)x(\tau(t)))')^{\gamma})'+q(t)x^{\alpha}(\sigma(t))+v(t)x^{\beta}(\eta(t))=0 \tag{1.1}$$

where γ,α,β are quotients of odd positive integers, $r,\ q,\ v,\ \tau,\ \sigma,\eta \in C(\mathbb{R}_+,\mathbb{R}_+)$, $p \in C(\mathbb{R}_+,\mathbb{R})$, $\tau(t)\le t$, $\sigma(t)\le t$, $\eta(t)\le t$ with $\lim_{t\to\infty}\tau(t)=\infty=\lim\limits_{t\to\infty}\sigma(t)=\infty=\lim_{t\to\infty}\eta(t)$. The objective of this work is to examine oscillatory behavior of all solutions to (1.1) under the assumption

(H_0) $\quad\int_0^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}dt<\infty,$

for various range of $p(t)$ with $|p(t)|<\infty$.

In [28] and [29], Tripathy and Sethi have established the sufficient conditions for oscillation, nonoscillation and asymptotic behaviour of solutions of

$$(r(t)(x(t)+p(t)x(\tau(t)))')'+q(t)G(x(\sigma(t)))+v(t)H(x(\eta(t)))=0 \tag{1.2}$$

under the assumptions

$$\int_0^{\infty}\frac{1}{r(t)}dt=\infty,\quad \int_0^{\infty}\frac{1}{r(t)}dt<\infty,$$

where $G,H\in C(\mathbb{R},\mathbb{R})$ such that G and H could be linear, sublinear or superlinear. When $G(x)=x^{\gamma}=H(x)$ and with the nonlinear neutral term, (1.2) reduces to

$$(r(t)((x(t)+p(t)x(\tau(t)))')^{\gamma})'+q(t)x^{\gamma}(\sigma(t))+v(t)x^{\gamma}(\eta(t))=0, \tag{1.3}$$

where γ is a ratio of odd positive integers. The authors have studied the oscillation properties of (1.3) in [24] and [30] by using the Riccati transformation technique under the assumptions

$$\int_0^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}dt=\infty,\quad \int_0^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}dt<\infty$$

and it was a quest to see the application of Riccati technique to nonlinear neutral delay differential equations in both forms $0 < \gamma < 1$ and $\gamma \geq 1$. In this work, we continue our study for (1.1) with any α, β and γ. In [7] Baculkova and Dzurina, respectively have studied the oscillatory behaviour of solutions of

$$(r(t)(x(t)+p(t)x(\tau(t)))')' + q(t)x(\sigma(t)) + v(t)x(\eta(t)) = 0 \tag{1.4}$$

by using the comparison results and same conclusion hold for the equations

$$(r(t)((x(t)+p(t)x(\tau(t)))')^{\gamma})' + q(t)x^{\beta}(\sigma(t)) = 0 \tag{1.5}$$

and

$$(r(t)|z'(t)|^{\alpha-1}z'(t))' + q(t)|x(\sigma(t))|^{\alpha-1}x(\sigma(t)) = 0 \tag{1.6}$$

in the works [6] and [8] respectively. When nothing is known about the differential inequalities concerned in the works [6] and [8], it is interesting to go through the works [28], [29], [24] and [30] for $v(t) = 0$ or $v(t) \neq 0$. Indeed, the equations (1.3) - (1.6) are the special cases of (1.1). Keeping in view of the above fact, we study (1.1) in general without following any comparison results. In this direction, we refer the monographs [14], [16] and some of the works [1], [5], [32]- [11], [17]-[23], [26], [27], [31] and the references cited therein.

Definition 1.1. By a solution of (1.1), we mean a continuously differentiable function $x(t)$ which is defined for $t \geq T^* = \min\{\tau(t_0),\ \sigma(t_0),\ \eta(t_0)\}$ such that $x(t)$ satisfies (1.1) for all $t \geq t_0$. In the sequel, it will always be assumed that the solutions of (1.1) exist on some half line $[t_1, \infty),\ \ t_1 \geq t_0$. A solution of (1.1) is said to be oscillatory, if it has arbitrarily large zeros; otherwise, it is called non-oscillatory. Equation (1.1) is called oscillatory, if all its solutions are oscillatory.

2 Oscillation Criteria when $(0 \leq p(t) < 1)$

This section deals with the sufficient conditions for oscillation of all solutions of (1.1) under the hypothesis (H_0). Throughout our discussion, we use the notation

$$z(t) = x(t) + p(t)x(\tau(t)). \tag{2.1}$$

Lemma 2.1. *[25] Assume that (H_0) holds and $r(t) \in C'[(T_0,\infty),\mathbb{R})$ such that $r'(t) > 0$. Let $x(t)$ be an eventually positive solution of (1.1) such that $(r(t)(x'(t))^{\gamma})' \leq 0$, for $t \geq t_0$. Then $x'(t) > 0$ and $x''(t) < 0$ for $t \geq t_1 > t_0$, where $\gamma \geq 1$ is a quotient of odd positive integers.*

Lemma 2.2. *Assume that (H_0) holds. Let $u(t)$ be an eventually positive continuous function on $[t_0,\infty),\ t_0 \geq 0$ such that $r(t)u'(t)$ is continuous and differentiable function with $(r(t)u'(t))^{\gamma})' \leq 0, \not\equiv 0$ for large $t \in [t_0,\infty)$, where $r(t)$ is positive and continuous function defined on $[t_0,\infty)$. Then the following statements hold:*
(i) If $u'(t) > 0$, then there exists a constant $C > 0$ such that $u(t) > CR(t)$ for large t.
(ii) If $u'(t) < 0$, then $u(t) \geq -(r(t)(u'(t))^{\gamma})^{\frac{1}{\gamma}}R(t)$, where $R(t) = \int_t^{\infty}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} ds$.

Theorem 2.3. *Let $0 \leq p(t) < 1$ and $\gamma < \alpha < \beta$. Assume that (H_0), and $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$, $r'(t) > 0$ hold for any large t. Furthermore, assume that*
(H_1) $\int_{t_0}^{\infty}\Big[q(s)(1-p(s))^{\alpha}R^{\alpha}(\sigma(s)) + v(s)(1-p(s))^{\alpha}R^{\alpha}(\eta(s))\Big]ds = \infty,$
$(H_{1\alpha})$ $a_1(t) = \int_t^{\infty}[(1-p(s))^{\alpha}\{q(s)+v(s)\}]\,ds,\ t \in [t_0,\infty)$ *such that* $\limsup_{t\to\infty} a_1(t) < \infty,$
and
$(H_{2\alpha})$ $\int_{t_0}^{\infty}(\frac{1}{r(\sigma(s))})^{\frac{1}{\gamma}}A_1(s,K_{1\alpha})ds = \infty,$
where $K_{1\alpha} > 0$ is an arbitrary constant and

$$A_1(t,K_{1\alpha}) = \left[a_1(t) + K_{1\alpha}\int_t^{\infty}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}(a_1(s))^{1+\frac{1}{\gamma}}ds\right]^{\frac{1}{\gamma}}.$$

Then every solution of (1.1) oscillates.

Proof. Suppose on the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_0$. Hence, there exist $t_1 > t_0$ such that $x(t) > 0,\ x(\tau(t)) > 0,\ x(\sigma(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$. Using (2.1) in (1.1), we find

$$(r(t)(z'(t))^\gamma)' = -q(t)x^\alpha(\sigma(t)) - v(t)x^\beta(\eta(t)) \leq 0,\ \not\equiv 0\ for\ t \geq t_1. \tag{2.2}$$

So, there exist $t_2 > t_1$ such that $r(t)(z'(t))^\gamma$ is nonincreasing on $[t_2, \infty)$. Consequently, either $z'(t) > 0$ or $z'(t) < 0$ for $t \geq t_2$. By Lemma 2.1, it follows that $z'(t) > 0$ for $t \geq t_2$. Therefore, there exist $t_3 > t_2$ such that

$$\begin{aligned} z(t) - p(t)z(\tau(t)) &= x(t) + p(t)x(\tau(t)) - p(t)x(\tau(t)) \\ &- p(t)p(\tau(t))p(\tau(\tau(t))) \\ &= x(t) - p(t)p(\tau(t))p(\tau(\tau(t))) \\ &\leq x(t) \end{aligned}$$

implies that $x(t) \geq (1 - p(t))z(t)$ on $[t_3, \infty)$. Ultimately, (1.1) becomes

$$(r(t)(z'(t))^\gamma)' + q(t)(1-p(t))^\alpha z^\alpha(\sigma(t)) + v(t)(1-p(t))^\alpha z^\alpha(\eta(t)) \leq 0 (\because \gamma < \alpha < \beta). \tag{2.3}$$

If $z'(t) > 0$ for $t \geq t_3$, then $z(t) \geq CR(t)$ due to Lemma 2.2(i). Therefore, (2.3) implies that

$$q(t)(1-p(t))^\alpha C^\alpha R^\alpha(\sigma(t)) + v(t)(1-p(t))^\alpha C^\alpha R^\alpha(\eta(t)) \leq -(r(t)(z'(t))^\gamma)' \tag{2.4}$$

for $t \geq t_3$. Integrating (2.4) from t_3 to t, we get

$$\begin{aligned} \int_{t_3}^{t} \Big[q(s)(1-p(s))^\alpha C^\alpha R^\alpha(\sigma(s)) + v(s)(1-p(s))^\alpha C^\alpha R^\alpha(\eta(s))\Big] ds &\leq -[(r(s)(z'(s))^\gamma)']_{t_3}^{t} ds \\ &\leq r(t_3)z'(t_3)^\gamma < \infty, \end{aligned}$$

a contradiction to (H_1). Ultimately, $z'(t) < 0$ for $t \geq t_2$.

Using $\eta(t) \geq \sigma(t)$ in (2.3) we obtain

$$\frac{(r(t)(z'(t))^\gamma)'}{z^\alpha(\sigma(t))} + (1-p(t))^\alpha[q(t) + v(t)] \leq 0 \tag{2.5}$$

for $t \geq t_3 > t_2$. Define Riccati transformation

$$w(t) = r(t)\frac{(z'(t))^\gamma}{z^\alpha(\sigma(t))}, t \in [t_3, \infty) \tag{2.6}$$

and by Lemma 2.1, $w(t) > 0$ for $t \geq t_3$. Since

$$w'(t) = \frac{(r(t)(z'(t)^\gamma)'}{z^\alpha(\sigma(t))} - \frac{(r(t)(z'(t)^\gamma)(z^\alpha(\sigma(t)))'}{z^\alpha(\sigma(t))z^\alpha(\sigma(t))},$$

then because of (2.5)

$$w'(t) \leq -(1-p(t))^\alpha[q(t) + v(t)] - \frac{(r(t)(z'(t)^\gamma)(z^\alpha(\sigma(t)))'}{z^\alpha(\sigma(t))z^\alpha(\sigma(t))} \leq 0 \tag{2.7}$$

for $t \geq t_3$. Noting that $(z^\alpha(\sigma(t)))' = \alpha(z(\sigma(t)))^{\alpha-1}z'(\sigma(t))\sigma'(t)$ and using the fact that $z(t)$ is nondecreasing on $[t_3, \infty)$, it is easy to verify that

$$\frac{(z^\alpha(\sigma(t)))'}{z^\alpha(\sigma(t))} \geq \frac{\alpha z'(\sigma(t))}{z(\sigma(t))}$$

for $t \geq t_3$. Therefore, (2.7) yields

$$w'(t) \leq -(1-p(t))^\alpha[q(t) + v(t)] - \alpha w(t)\frac{z'(\sigma(t))}{z(\sigma(t))}. \tag{2.8}$$

Indeed, $(r^{\frac{1}{\gamma}}(\sigma(t))(z'(\sigma(t)) \geq (r^{\frac{1}{\gamma}}(t))(z'(t))$ implies that

$$\frac{(r^{\frac{1}{\gamma}}(\sigma(t))(z'(\sigma(t))}{z^{\frac{\alpha}{\gamma}}(\sigma(t))} \geq \frac{(r^{\frac{1}{\gamma}}(t)(z'(t))}{z^{\frac{\alpha}{\gamma}}(\sigma(t))},$$

that is, $w^{\frac{1}{\gamma}}(\sigma(t)) \geq w^{\frac{1}{\gamma}}(t)$ and hence

$$z'(\sigma(t)) \geq (r(\sigma(t)))^{-\frac{1}{\gamma}} z^{\frac{\alpha}{\gamma}}(\sigma(t)w(t)^{\frac{1}{\gamma}}. \tag{2.9}$$

Substituting (2.9) in (2.8), we get

$$w'(t) \leq -(1-p(t))^{\alpha}[q(t)+v(t)] - \alpha(r(\sigma(t)))^{-\frac{1}{\gamma}} w(t)^{1+\frac{1}{\gamma}} z^{\frac{\alpha}{\gamma}-1}(\sigma(t)). \tag{2.10}$$

Since $z(t)$ is nondecreasing on $[t_3, \infty)$, then there exist $t_4 > t_3$ and $C > 0$ such that $(z(t))^{\frac{\alpha}{\gamma}-1} \geq C$ for $t \geq t_4$. Consequently, (2.10) yields that

$$w'(t) \leq -(1-p(t))^{\alpha}[q(t)+v(t)] - \alpha C r(\sigma(t))^{-\frac{1}{\gamma}} w(t)^{1+\frac{1}{\gamma}} \tag{2.11}$$

for $t \geq t_4$. Integrating (2.11) from t to $u (t < u)$ for $t, u \in (t_4, \infty)$ we obtain

$$-w(t) \leq w(u) - w(t) \leq -\int_t^u \left[(1-p(s))^{\alpha}\{q(s)+v(s)\} + \alpha C(r(\sigma(s)))^{-\frac{1}{\gamma}} w(s)^{1+\frac{1}{\gamma}}\right] ds,$$

that is,

$$w(t) \geq a_1(t) + K_{1\alpha}\int_t^{\infty} r^{-\frac{1}{\gamma}}(\sigma(s)) w(s)^{1+\frac{1}{\gamma}} ds, \ t \geq t_5 > t_4,$$

where $K_{1\alpha} = C\alpha$. Clearly, $w(t) > a_1(t)$ implies that

$$w(t) \geq a_1(t) + K_{1\alpha}\int_t^{\infty} r^{-\frac{1}{\gamma}}(\sigma(s))(a_1(s))^{1+\frac{1}{\gamma}} ds = A_1^{\gamma}(t, K_{1\alpha}).$$

For $\delta > 1$, we notice that

$$\left(z^{1-\delta}(\sigma(t))\right)' \leq (1-\delta) z(\sigma(t))^{-\delta} z'(\sigma(t)). \tag{2.12}$$

Further, from (2.9) it follows that

$$(z'(\sigma(t)) z^{-\frac{\alpha}{\gamma}}(\sigma(t)) \geq (r(\sigma(t))^{-\frac{1}{\gamma}} \left(A_1^{\gamma}(t, K_{1\alpha})\right)^{\frac{1}{\gamma}}$$

which in turn

$$z(\sigma(t))^{-\delta} z'(\sigma(t)) \geq (r(\sigma(t))^{-\frac{1}{\gamma}} \left(A_1(t, K_{1\alpha})\right),$$

where $\delta = \left(\frac{\alpha}{\gamma}\right) > 1)$. Therefore, (2.12) becomes

$$\frac{(z^{1-\delta}(\sigma(t)))'}{1-\delta} \geq (r(\sigma(t))^{-\frac{1}{\gamma}} \left(A_1(t, K_{1\alpha})\right) \tag{2.13}$$

for $t \geq t_5$. Integrating (2.13) from t_5 to t, we get

$$\int_{t_5}^{t} (r(\sigma(s))^{-\frac{1}{\gamma}} \left(A_1(s, K_{1\alpha})\right) ds < \infty,$$

a contradiction to $(H_{2\alpha})$. This completes the proof of the theorem. □

Theorem 2.4. *Let* $0 \leq p(t) < 1$ *and* $\gamma < \beta < \alpha$. *Assume that* $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$ *and* $r'(t) > 0$ *for any large* t. *If* $(H_0), (H_1), (H_{1\beta})$ *and* $(H_{2\beta})$ *hold, then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from the proof of Theorem 2.3 and hence the details are omitted. □

Theorem 2.5. *Let $0 \le p(t) \le a < 1$ and $\gamma = \alpha = \beta$. Assume that $r'(t) > 0$, $\eta(t) \ge \sigma(t)$ and $\sigma'(t) \ge 1$ for any large t. If (H_0), (H_1), $(H_{1\alpha})$ and*

$(H_{3\alpha})$ $\limsup_{t\to\infty} \Big(\int_T^{\sigma(t)} r(s)^{-\frac{1}{\gamma}} ds \Big) A_1(t, K_{1\alpha}) > 1,\ T > t_0 > 0,$

where A_1 is defined in Theorem 2.3 hold, then every solution of (1.1) oscillates.

Proof. On the contrary, we proceed as in the proof of Theorem 2.3 to obtain that $w(t) \ge A^{\gamma}(t, K_{1\alpha})$ for $t \in [t_5, \infty)$. Since $(r(t)(z'(t))^{\gamma})' \le 0$ due to 2.3, then for $t_5 < s \le t$

$$r(s)(z'(s))^{\gamma} \ge r(t)(z'(t))^{\gamma}$$

implies that $z'(s) \ge \frac{r(t)^{\frac{1}{\gamma}} z'(t)}{r(s)^{\frac{1}{\gamma}}}$, that is,

$$\int_{t_5}^{t} z'(s) ds \ge r(t)^{\frac{1}{\gamma}} z'(t) \int_{t_5}^{t} \frac{1}{r(s)^{\frac{1}{\gamma}}} ds.$$

Therefore,

$$z(t) \ge r(t)^{\frac{1}{\gamma}} z'(t) \int_{t_5}^{t} r(s)^{-\frac{1}{\gamma}} ds$$

implies that

$$\frac{r(t)^{\frac{1}{\gamma}} z'(t)}{z(t)} \le \Big(\int_{t_5}^{t} r(s)^{-\frac{1}{\gamma}} ds \Big)^{-1} \tag{2.14}$$

for $t \ge t_5$. As a result,

$$A_1(t, K_{1\alpha}) \le w^{\frac{1}{\gamma}}(t) = \frac{r(t)^{\frac{1}{\gamma}} z'(t)}{z(\sigma(t))} \le \frac{r(\sigma(t))^{\frac{1}{\gamma}} z'(\sigma(t))}{z(\sigma(t))} \le \Big(\int_{t_5}^{\sigma(t)} r(s)^{-\frac{1}{\gamma}} ds \Big)^{-1},$$

implies that

$$\Big(\int_{t_5}^{\sigma(t)} r(s)^{-\frac{1}{\gamma}} ds \Big) A_1(t, K_{1\alpha}) \le 1,$$

which is a contradiction to $(H_{3\alpha})$. Hence, the theorem is proved. □

Theorem 2.6. *Let $0 \le p(t) < 1$ and $\alpha < \beta < \gamma$. Assume that (H_0), (H_1) and $(H_{1\alpha})$, and $r'(t) > 0$, $\eta(t) \ge \sigma(t)$, $\sigma'(t) \ge 1$ hold for any large t. If*

$(H_{4\alpha})$ $\limsup_{t\to\infty} (a_1(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \Big(\int_T^{\sigma(t)} (r(s))^{-\frac{1}{\gamma}} ds \Big) \Big[a_1(t) + K_{2\alpha} \int_t^{\infty} r^{-\frac{1}{\gamma}}(\sigma(s))(a_1(s))^{1+\frac{1}{\alpha}} ds \Big]^{\frac{1}{\gamma}} = \infty,$

where $T > t_0 > 0$, $K_{2\alpha} > 0$ is a constant, then every solution of (1.1) oscillates.

Proof. Proceeding as in the proof of Theorem 2.3, we obtain (2.2), (2.3) and (2.5) and hence $w(t) > a_1(t)$ for $t \in [t_5, \infty)$. It follows from (2.6) that

$$r(\sigma(t))^{\frac{1}{\gamma}} z'(\sigma(t)) \ge z^{\frac{\alpha}{\gamma}}(\sigma(t)) a_1^{\frac{1}{\gamma}}(t)$$

for $t \ge t_5$. Since $(r(t)(z'(t))^{\gamma})' \le 0$, then there exist constant $C > 0$ and $t_6 > t_5$ such that $r(\sigma(t))^{\frac{1}{\gamma}} z'(\sigma(t)) \le C$ for $t \ge t_6$, that is, $C \ge r(\sigma(t))^{\frac{1}{\gamma}} z'(\sigma(t)) \ge z^{\frac{\alpha}{\gamma}}(\sigma(t)) a_1^{\frac{1}{\gamma}}(t)$ implies that $C^{\frac{\gamma}{\alpha}} \ge z(\sigma(t)) a_1(t)^{\frac{1}{\alpha}}$ and hence

$$z(\sigma(t)) \le C^{\frac{\gamma}{\alpha}} (a_1(t))^{-\frac{1}{\alpha}} \ for\ t \in [t_6, \infty). \tag{2.15}$$

As $\alpha < \beta < \gamma$, then (2.15) becomes

$$z(\sigma(t))^{\frac{\alpha-\gamma}{\gamma}} \geq \left(C^{\frac{\gamma}{\alpha}}\right)^{\frac{\alpha-\gamma}{\gamma}} \left((a_1(t))^{-\frac{1}{\alpha}}\right)^{\frac{\alpha-\gamma}{\gamma}} \geq C^{\frac{\alpha-\gamma}{\alpha}} (a_1(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}}. \tag{2.16}$$

Using (2.16) in (2.10) and then integrating as in Theorem 2.3, we obtain

$$w(t) \geq a_1(t) + K_{2\alpha} \int_t^\infty r^{-\frac{1}{\gamma}}(\sigma(s))(a_1(s))^{1+\frac{1}{\alpha}} ds,$$

where $K_{2\alpha} = \alpha C^{\frac{(\alpha-\gamma)}{\gamma}}$ which in turn

$$(z(\sigma(t)))^{\frac{(\gamma-\alpha)}{\gamma}} \frac{r^{\frac{1}{\gamma}}(\sigma(t))z'(\sigma(t))}{z(\sigma(t))} \geq \left[a_1(t) + K_{2\alpha} \int_t^\infty r^{-\frac{1}{\gamma}}(\sigma(s))(a_1(s))^{1+\frac{1}{\alpha}} ds\right]^{\frac{1}{\gamma}} \tag{2.17}$$

due to (2.6). Using (2.14) in (2.17) and then using (2.16), we get

$$C^{\frac{\gamma-\alpha}{\alpha}} (a_1(t))^{\frac{(\alpha-\gamma)}{\alpha\gamma}} \left(\int_{t_5}^{\sigma(t)} (r(s))^{-\frac{1}{\gamma}} ds\right)^{-1} \geq \left[a_1(t) + K_{2\alpha} \int_t^\infty r^{-\frac{1}{\gamma}}(\sigma(s))(a_1(s))^{1+\frac{1}{\alpha}} ds\right]^{\frac{1}{\gamma}}$$

for $t \geq t_5$, that is,

$$(a_1(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \left(\int_{t_5}^{\sigma(t)} (r(s))^{-\frac{1}{\gamma}} ds\right) \left[a_1(t) + K_{2\alpha} \int_t^\infty r^{-\frac{1}{\gamma}}(\sigma(s))(a_1(s))^{1+\frac{1}{\alpha}} ds\right]^{\frac{1}{\gamma}} \leq C^{\frac{\alpha-\gamma}{\alpha}}$$

which contradicts $(H_{4\alpha})$. This completes the proof of the theorem. □

Theorem 2.7. *Let* $0 \leq p(t) < 1$ *and* $\beta < \alpha < \gamma$. *Assume that* $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$ *and* $r'(t) > 0$ *for any large* t. *If* (H_0), (H_1), $(H_{1\beta})$ *and* $(H_{4\beta})$ *hold, then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from the proof of Theorem 2.6 and hence the details are omitted. □

Theorem 2.8. *Let* $0 \leq p(t) < 1$ *and* $\alpha < \gamma < \beta$. *Assume that* $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$ *and* $r'(t) > 0$ *for any large* t. *If* (H_0), (H_1), $(H_{1\alpha})$ *and* $(H_{3\alpha})$ *hold, then every solution of (1.1) oscillates.*

Proof. Proceeding as in the proof of Theorem 2.3, we obtain $w(t) \geq A_1^\gamma(t, K_{1\alpha})$ for $t \geq t_5$ and then using (2.15) we obtain a contradiction to $(H_{3\alpha})$. This completes the proof of the theorem. □

Theorem 2.9. *Let* $0 \leq p(t) < 1$ *and* $\beta < \gamma < \alpha$. *Assume that* $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$ *and* $r'(t) > 0$ *for any large* t. *If* (H_0), (H_1), $(H_{1\beta})$ *and* $(H_{3\beta})$ *hold, then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from the proof of Theorem 2.8 and hence the details are omitted. □

3 Oscillation Criteria when $(1 \leq p(t) < \infty)$

In this section we establish sufficient conditions for oscillation of all solutions of (1.1) under the hypothesis (H_0) when $(1 \leq p(t) < \infty)$.

Theorem 3.1. *Let* $1 \leq p(t) \leq a < \infty$ *and* $\gamma < \alpha < \beta$. *Let* (H_0) *hold. Assume that* $\tau(\sigma(t)) = \sigma(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$, $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$ *and* $r'(t) > 0$ *for any large* t. *Furthermore, assume that*

(H_5) *there exists a* $\lambda > 0$ *such that* $u^\gamma(x) + u^\gamma(y) \geq \lambda u^\gamma(x+y); x, y > 0,\ x, y \in \mathbb{R}$,

(H_6) there exists a $\mu > 0$ such that $v^\gamma(x) + v^\gamma(y) \geq \mu v^\gamma(x+y); x, y > 0,\ x, y \in \mathbb{R}$,
(H_7) $Q(t) = \min\{q(t), q(\tau(t))\}, V(t) = \min\{v(t), v(\tau(t))\}$ for $t \geq t_0$,
(H_8) $\int_{t_0}^{\infty} \Big[Q(s)R^\gamma(\sigma(s)) + V(s)R^\gamma(\eta(s))\Big]ds = \infty$,
(H_9) $a_2(t) = \int_t^\infty [\lambda Q(s) + \mu V(s)]ds,\ t \in [t_0, \infty), \limsup_{t\to\infty} a_2(t) < \infty$,
(H_{10}) $\tau^{-1}, \sigma^{-1}, \eta^{-1} : \mathbb{R} \to \mathbb{R}$ such that $\tau^{-1}, \sigma^{-1}, \eta^{-1}$ are continuous functions and $\tau^{-1}(t) \geq t, \sigma^{-1}(t) \geq t$ and $\eta^{-1}(t) \geq t$,
$(H_{11\alpha})$ $\int_{t_0}^{\infty} (\frac{1}{r(\sigma(s))})^{\frac{1}{\gamma}} A_2(s, K_{3\alpha})ds = \infty$
hold, where $K_{3\alpha} > 0$ is an arbitrary constant and
$A_2(t, K_{3\alpha}) = \left[\frac{a_2(\tau^{-1}(t))}{1+a^\alpha} + K_{3\alpha} \int_{\tau^{-1}(t)}^{\infty} \left(\frac{1}{r(\sigma(s))}\right)^{\frac{1}{\gamma}} ((a_2(\tau^{-1}(s))^{1+\frac{1}{\gamma}} ds\right]^{\frac{1}{\gamma}}$.
Then every solution of (1.1) oscillates.

Proof. We proceed as in the proof of Theorem 2.3 to obtain (2.3) for $t \in [t_1, \infty)$. In what follows, we consider two possible cases $z'(t) > 0$ or $z'(t) < 0$ for $t \geq t_3 > t_2$. From (1.1), it is easy to verify that

$$(r(t)(z'(t))^\gamma)' + a^\alpha(r(\tau(t))(z'(\tau(t))^\gamma)' + q(t)x^\alpha(\sigma(t)) + a^\alpha q(\tau(t))x^\alpha(\sigma(\tau(t)) + v(t)x^\beta(\eta(t)) + a^\alpha v(\tau(t))x^\beta(\eta(\tau(t)) = 0. \tag{3.1}$$

Using (H_5) and (H_6) in (3.1), we get

$$(r(t)(z'(t))^\gamma)' + a^\alpha(r(\tau(t))(z'(\tau(t))^\gamma)' + \lambda Q(t)z^\alpha(\sigma(t)) + \mu V(t)z^\alpha(\eta(t)) \leq 0 (\because \gamma < \alpha < \beta).$$

If $z'(t) > 0$ for $t \geq t_3$, then $z(t) \geq CR(t)$ due to Lemma 2.2(i). Therefore, the above inquality implies that,

$$(r(t)(z'(t))^\gamma)' + a^\alpha(r(\tau(t))(z'(\tau(t))^\gamma)' + \lambda Q(t)C^\alpha R^\alpha(\sigma(t)) + \mu V(t)C^\alpha R^\alpha(\eta(t)) \leq 0. \tag{3.2}$$

Integrating (3.2) from t_3 to t, we get

$$\begin{aligned}\int_{t_3}^{t} \Big[\lambda Q(s)C^\gamma R^\gamma(\sigma(s)) + \mu V(s)C^\gamma R^\gamma(\eta(s))\Big]ds \\ \leq -[(r(s)(z'(s))^\gamma)' + (a^\gamma r(\tau(s))(z'(\tau(s)))^\gamma)']_{t_3}^{t}\, ds \\ \leq r(t_3)z'(t_3)^\gamma + a^\gamma r(\tau(t_3))(z'(\tau(t_3)))^\gamma < \infty,\end{aligned}$$

a contradiction to (H_8). Ultimately, $z'(t) < 0$ for $t \geq t_2$. Using the fact $\eta(t) \geq \sigma(t))$ in the preciding inequality, we obtain

$$\frac{(r(t)(z'(t))^\gamma)'}{z^\alpha(\sigma(t))} + \frac{a^\alpha(r(\tau(t))(z'(\tau(t))^\gamma)'}{z^\alpha(\sigma(t))} + \lambda Q(t) + \mu V(t) \leq 0 \tag{3.3}$$

for $t \geq t_2 > t_1$. Considering the Riccati substitution (2.6) and then proceeding as in Theorem 2.3, we get

$$\begin{aligned} w'(t) + a^\alpha w'(\tau(t)) &\leq \frac{(r(t)(z'(t))^\gamma)'}{z^\alpha(\sigma(t))} - w(t)\frac{\alpha z'(\sigma(t))}{z(\sigma(t))} \\ &\quad + a^\alpha \frac{(r(\tau(t))(z'(\tau(t)))^\gamma)'}{z^\alpha(\sigma(\tau(t)))} - w(\tau(t))\frac{\alpha z'(\sigma(\tau(t)))}{z(\sigma(\tau(t))} \\ &\leq \frac{(r(t)(z'(t))^\gamma)'}{z^\alpha(\sigma(t))} - w(t)^{1+\frac{1}{\gamma}} \alpha r(\sigma(t))^{-\frac{1}{\gamma}} z^{\frac{\alpha}{\gamma}-1}(\sigma(t)) \\ &\quad + a^\alpha \frac{(r(\tau(t))(z'(\tau(t)))^\gamma)'}{z^\alpha(\sigma(\tau(t))} - w(\tau(t))^{1+\frac{1}{\gamma}} \alpha r(\sigma(\tau(t)))^{-\frac{1}{\gamma}} z^{\frac{\alpha}{\gamma}-1}(\sigma(\tau(t))) \end{aligned} \tag{3.4}$$

for $t \geq t_3 > t_2$. Since $z(t)$ is nondecreasing, then there exist $t_4 > t_3$ and $C > 0$ such that $(z(t))^{\frac{\alpha}{\gamma}-1} \geq C$ and hence (3.4) becomes

$$w'(t) + a^\alpha w'(\tau(t)) \leq \frac{(r(t)(z'(t))^\gamma)'}{z^\alpha(\sigma(t))} - w(t)^{1+\frac{1}{\gamma}} \alpha C r(\sigma(t))^{-\frac{1}{\gamma}}$$
$$+ a^\alpha \frac{(r(\tau(t))(z'(\tau(t)))^\gamma)'}{z^\alpha(\sigma(\tau(t))} - w(\tau(t))^{1+\frac{1}{\gamma}} \alpha C r(\sigma(\tau(t))^{-\frac{1}{\gamma}}.$$

Consequently,

$$w'(t) + a^\alpha w'(\tau(t)) \leq -\{\lambda Q(t) + \mu V(t)\}$$
$$- \alpha C \left[w(t)^{1+\frac{1}{\gamma}} r(\sigma(t))^{-\frac{1}{\gamma}} + a^\alpha w(\tau(t))^{1+\frac{1}{\gamma}} r(\sigma(\tau(t))^{-\frac{1}{\gamma}} \right] \tag{3.5}$$

for $t \geq t_4$, that is,

$$w'(t) + a^\alpha w'(\tau(t)) \leq -\{\lambda Q(t) + \mu V(t)\}$$
$$- \alpha C r(\sigma(t))^{-\frac{1}{\gamma}} (1 + a^\alpha) w(t)^{1+\frac{1}{\gamma}} \tag{3.6}$$

due to nonincreasing $w(t)$ and $r'(t) \geq 0$. Integrating (3.6) from t to $v (t < v)$ for $t, v \in [t_4, \infty)$, we obtain

$$-w(t) - a^\alpha w(\tau(t)) < w(v) - w(t) + a^\alpha w(\tau(v)) - a^\alpha w(\tau(t))$$
$$\leq - \int_t^v \{\lambda Q(s) + \mu V(s)\} ds$$
$$- \alpha C(1 + a^\alpha) \int_t^v \left[w(s)^{1+\frac{1}{\gamma}} r(\sigma(s))^{-\frac{1}{\gamma}} \right] ds,$$

that is,

$$w(t) + a^\alpha w(\tau(t)) \geq \int_t^\infty \{\lambda Q(s) + \mu V(s)\} ds$$
$$+ \alpha C(1 + a^\alpha) \int_t^\infty \left[w(s)^{1+\frac{1}{\gamma}} r(\sigma(s))^{-\frac{1}{\gamma}} \right] ds$$
$$= a_2(t) + \alpha C(1 + a^\alpha) \int_t^\infty \left[w(s)^{1+\frac{1}{\gamma}} r(\sigma(s))^{-\frac{1}{\gamma}} \right] ds.$$

Ultimately,

$$(1 + a^\alpha) w(\tau(t)) \geq a_2(t) + \alpha C(1 + a^\alpha) \int_t^\infty \left[w(s)^{1+\frac{1}{\gamma}} r(\sigma(s))^{-\frac{1}{\gamma}} \right] ds. \tag{3.7}$$

Due to (H_{10}), (3.7) yields that

$$w(t) \geq \frac{(a_2(\tau^{-1}(t))}{(1 + a^\alpha)} + \alpha C \int_{\tau^{-1}(t)}^\infty \left[w(s)^{1+\frac{1}{\gamma}} r(\sigma(s))^{-\frac{1}{\gamma}} \right] ds$$

that is, $w(t) \geq \frac{(a_2(\tau^{-1}(t))}{(1+a^\alpha)}$ implies that

$$w(t) \geq \frac{(a_2(\tau^{-1}(t))}{(1 + a^\alpha)} + \alpha C \int_{\tau^{-1}(t)}^\infty \left[r(\sigma(s))^{-\frac{1}{\gamma}} \left(\frac{1}{1 + a^\alpha} \right)^{1+\frac{1}{\gamma}} (a_2(\tau^{-1}(\sigma(s))^{1+\frac{1}{\gamma}} \right] ds = A_2^\gamma(t, K_{3\alpha}),$$

where $K_{3\alpha} = \alpha C \left(\frac{1}{1+a^\alpha} \right)^{1+\frac{1}{\gamma}}$. The rest of the proof follows Theorem 2.3. Hence the theorem is proved. □

Theorem 3.2. *Let* $1 \leq p(t) \leq a < \infty$ *and* $\gamma < \beta < \alpha$. *Assume that* $\tau(\sigma(t)) = \sigma(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$, $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$ *and* $r'(t) > 0$ *for any large* t. *If* (H_0), $(H_5) - (H_{10})$ *and* $(H_{11\beta})$ *hold, then every solution of* (1.1) *oscillates.*

Proof. The proof of the theorem follows from the proof of Theorem 3.1 and hence the details are omitted. □

Theorem 3.3. *Let* $1 \le p(t) \le a < \infty$ *and* $\gamma = \beta = \alpha$. *Assume that* $\tau(\sigma(t)) = \sigma(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$, $\eta(t) \ge \sigma(t)$, $\sigma'(t) \ge 1$ *and* $r'(t) > 0$ *for any large* t. *If* (H_0), $(H_5) - (H_{10})$ *and*

$(H_{12\alpha})$ $\limsup_{t\to\infty} \Big(\int_T^{\sigma(t)} r(s)^{-\frac{1}{\gamma}} ds \Big) A_2(t, K_{3\alpha}) > 1,\ T > t_0 > 0,$

where A_2 *is defined in Theorem 3.1 hold, then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from the proof of Theorem 3.1 and Theorem 2.5 and hence the details are omitted. □

Theorem 3.4. *Let* $1 \le p(t) \le a < \infty$ *and* $\alpha < \beta < \gamma$. *Assume that* $\tau(\sigma(t)) = \sigma(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$, $\eta(t) \ge \sigma(t)$, $\sigma'(t) \ge 1$ *and* $r'(t) > 0$ *for any large* t. *If* (H_0), $(H_5) - (H_{10})$ *and*

$(H_{13\alpha})$ $\limsup_{t\to\infty} (a_2(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \Big(\int_T^{\sigma(t)} (r(s))^{-\frac{1}{\gamma}} ds \Big) \Big[a_2(t) + K_{4\alpha} \int_t^\infty r^{-\frac{1}{\gamma}}(\sigma(s))(a_2(s))^{1+\frac{1}{\alpha}} ds \Big]^{\frac{1}{\gamma}} = \infty,$

$T > t_0 > 0,\ K_{4\alpha} > 0$ *is a constant, where* A_2 *is defined in Theorem 3.1, then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from the proofs of Theorem 3.1 and Theorem 2.6 and hence the details are omitted. □

Theorem 3.5. *Let* $1 \le p(t) \le a < \infty$ *and* $\beta < \alpha < \gamma$. *Assume that* $\tau(\sigma(t)) = \sigma(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$, $\eta(t) \ge \sigma(t)$, $\sigma'(t) \ge 1$ *and* $r'(t) > 0$ *for any large* t. *If* (H_0), $(H_5) - (H_{10})$ *and* $(H_{13\beta})$ *hold, then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from the proof of Theorem 3.4 and hence the details are omitted. □

Theorem 3.6. *Let* $1 \le p(t) \le a < \infty$ *and* $\alpha < \gamma < \beta$. *Assume that* $\tau(\sigma(t)) = \sigma(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$, $\eta(t) \ge \sigma(t)$, $\sigma'(t) \ge 1$ *and* $r'(t) > 0$ *for any large* t. *If* (H_0), $(H_5) - (H_{10})$ *and* $(H_{12\alpha})$ *hold, then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from the proofs of Theorem 3.1 and Theorem 3.3 and hence the details are omitted. □

Theorem 3.7. *Let* $1 \le p(t) \le a < \infty$ *and* $\beta < \gamma < \alpha$. *Assume that* $\tau(\sigma(t)) = \sigma(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$, $\eta(t) \ge \sigma(t)$, $\sigma'(t) \ge 1$ *and* $r'(t) > 0$ *for any large* t. *If* (H_0), $(H_5) - (H_{10})$ *and* $(H_{12\beta})$ *hold, then every solution of (1.1) oscillates.*

Proof. The proof of the theorem is similar to the proof of Theorem 3.6. □

4 Oscillation Criteria when $(-1 < p(t) \le 0)$

In this section we establish sufficient conditions for oscillation of all solutions of (1.1) under the hypothesis (H_0) when $(-1 < p(t) \le 0)$.

Theorem 4.1. *Let* $-1 < p \le p(t) \le 0$ *and* $\gamma < \alpha < \beta$. *Let* (H_0) *hold. Assume that* $\eta(t) \ge \sigma(t)$, $\sigma'(t) \ge 1$ *and* $r'(t) > 0$ *for any large* t. *Furthermore, assume that*

(H_{13}) $\int_{t_0}^\infty \Big[q(s)R^\gamma(\sigma(s)) + v(s)R^\gamma(\eta(s)) \Big] ds = \infty,$

(H_{14}) $a_3(t) = \int_t^\infty [q(s) + v(s)] ds,\ t \in [t_0, \infty), \limsup_{t\to\infty} a_3(t) < \infty$

and

$(H_{15\alpha})$ $\int_{t_0}^\infty \big(\frac{1}{r(\sigma(s))}\big)^{\frac{1}{\gamma}} A_3(s, K_{1\alpha}) ds = \infty,$

where $K_{1\alpha} > 0$ *is an arbitrary constant and*

$$A_3(t, K_{1\alpha}) = \Big[a_3(t) + K_{1\alpha} \int_t^\infty \Big(\frac{1}{r(s)}\Big)^{\frac{1}{\gamma}} (a_3(s))^{1+\frac{1}{\gamma}} ds \Big]^{\frac{1}{\gamma}}$$

hold. Then every unbounded solution of (1.1) oscillates.

Proof. Let $x(t)$ be an unbounded nonoscillatry solution of (1.1). Proceeding as in the proof of Theorem 2.3, we get (2.2) for $t \in [t_1, \infty)$. Since $z(t)$ is monotonic, then either $z(t) > 0$ or $z(t) < 0$ for $t \geq t_2 > t_1$. Suppose that $z(t) < 0$ for $t \geq t_2$. As $x(t)$ is unbounded, we can find a sequence $\{\rho_n\}$ such that $\rho_n \to \infty$ and $x(\rho_n) \to \infty$ as $n \to \infty$ and $x(\rho_n) = \max\{x(s) : t_2 \leq s \leq \rho_n\}$. Indeed,

$$\begin{aligned} z(\rho_n) &= x(\rho_n) + p(\rho_n)x(\tau(\rho_n)) \\ &\geq x(\rho_n) + px(\tau(\rho_n)) \\ &\geq x(\rho_n) + px(\rho_n) \\ &= (1+p)x(\rho_n) (\because 1+p > 0) \end{aligned}$$

implies that $z(t) > 0$, which is absurd. Ultimately, $z(t) > 0$ for $t \geq t_2$. Since $z(t) \leq x(t)$, (1.1) reduces to

$$(r(t)(z'(t))^\gamma)' + q(t)z^\alpha(\sigma(t)) + v(t)z^\alpha(\eta(t)) \leq 0 \ (\because \gamma < \alpha < \beta) \tag{4.1}$$

for $t \geq t_3 > t_2$. Using Lemma 2.2(i) and then integrating from $t_3 (> t_2)$ to ∞, we get a contradiction to (H_{13}) and hence (4.1) can be written as

$$\frac{(r(t)(z'(t))^\gamma)'}{z^\alpha(\sigma(t))} + q(t) + v(t) \leq 0 \ (\because \eta(t) \geq \sigma(t)).$$

For $t \geq t_3$, we define the Riccati transformation as in (2.6) and $w(t) > 0$. The rest of the proof follows from the proof of Theorem 2.3. Hence the theorem is proved. □

Theorem 4.2. *Let $-1 < p \leq p(t) \leq 0$ and $\gamma < \beta < \alpha$. Assume that $r'(t) > 0$, $\eta(t) \geq \sigma(t)$ and $\sigma'(t) \geq 1$ for any large t. If (H_0), (H_{13}), (H_{14}), $(H_{15\beta})$ hold, then every unbounded solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from the proof of the Theorem 4.1. Hence the details are omitted. □

Theorem 4.3. *Let $-1 < p \leq p(t) \leq 0$ and $\gamma = \alpha = \beta$. Assume that $r'(t) > 0$, $\eta(t) \geq \sigma(t)$ and $\sigma'(t) \geq 1$ for any large t. If (H_0), (H_{13}), (H_{14}) and*

$(H_{16\alpha})$ $\limsup_{t\to\infty} \left(\int_T^{\sigma(t)} r(s)^{-\frac{1}{\gamma}} ds \right) A_3(t, K_{1\alpha}) > 1, \ T > t_0 > 0,$

hold, where A_3 is defined in Theorem 4.1, then every unbounded solution of (1.1) oscillates.

Proof. The proof of the theorem follows from the proofs of Theorem 2.5 and Theorem 4.1. Hence the details are omitted. □

Theorem 4.4. *Let $-1 < p \leq p(t) \leq 0$ and $\alpha < \beta < \gamma$. Assume that $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$ and $r'(t) > 0$ for any large t. If (H_0), (H_{13}), (H_{14}) and*

$(H_{17\alpha})$ $\limsup_{t\to\infty} (a_3(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \left(\int_T^{\sigma(t)} (r(s))^{-\frac{1}{\gamma}} ds \right) \left[a_3(t) + K_{2\alpha} \int_t^\infty r^{-\frac{1}{\gamma}}(\sigma(s))(a_3(s))^{1+\frac{1}{\alpha}} ds \right]^{\frac{1}{\gamma}} = \infty,$

$T > t_0 > 0$, $K_{2\alpha} > 0$ is a constant hold, then every unbounded solution of (1.1) oscillates.

Proof. The proof of the theorem follows from the proofs of Theorem 2.6 and Theorem 4.1. Hence the details are omitted. □

Theorem 4.5. *Let $-1 < p \leq p(t) \leq 0$ and $\beta < \alpha < \gamma$. Assume that $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$ and $r'(t) > 0$ for any large t. If (H_0), (H_{13}), (H_{14}) and $(H_{17\beta})$ hold, then every unbounded solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from the proof of Theorem 4.4 and hence the details are omitted. □

Theorem 4.6. *Let $-1 < p \leq p(t) \leq 0$ and $\alpha < \gamma < \beta$. Assume that $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$ and $r'(t) > 0$ for any large t. If (H_0), (H_{13}), (H_{14}) and $(H_{16\alpha})$ hold, then every unbounded solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from the proofs of Theorem 4.1 and Theorem 4.3. Hence the details are omitted. □

Theorem 4.7. *Let $-1 < p \leq p(t) \leq 0$ and $\beta < \gamma < \alpha$. Assume that $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$ and $r'(t) > 0$ for any large t. If (H_0), (H_{13}), (H_{14}) and $(H_{16\beta})$ hold, then every unbounded solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from the proof of Theorem 4.6. Hence the details are omitted. □

5 Discussion and Examples

Often it is challenging to study the second order differential equations of the form (1.1), of course it is interesting for any α, β and γ. In this work, we have made an attempt to establish sufficient conditions for oscillation of all solutions of (1.1).

Theorem 5.1. *Let $-\infty < a \leq p(t) \leq d < -1$ and $\gamma < \alpha < \beta$. Assume that $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$, $r'(t) > 0$, $\tau(\sigma(t)) = \sigma(\tau(t))$ and $\tau(\eta(t)) = \eta(\tau(t))$ for any large t. If (H_0), (H_{13}), (H_{14}), $(H_{15\alpha})$,*

(H_{18}) $\int_T^\infty \left[\frac{1}{r(s)} \int_{t_3}^\theta [q(\theta) + v(\theta)]d\theta\right]^{\frac{1}{\gamma}} ds = \infty$,

(H_{19}) $\int_{t_0}^t \left[\frac{1}{r(\theta)} \int_{t_0}^\theta [R^\alpha(\sigma(s))q(s) + R^\alpha(\sigma(s))v(s)ds]\right]^{\frac{1}{\gamma}} d\theta = \infty$

and

(H_{20}) $\int_{t_0}^\infty [q(\tau(s)) + v(\tau(s))]\, ds = \infty$, *hold, then every bounded solution of (1.1) either oscillates or converges to zero.*

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (2.1). Proceeding as in the proof of Theorem 4.1, we have four possible cases for $t \in [t_2, \infty)$.

$$(i)\;\; z(t) > 0,\; z'(t) > 0,\;\; (ii)\;\; z(t) < 0,\; z'(t) > 0,$$
$$(iii)\;\; z(t) > 0,\; z'(t) < 0,\;\; (iv)\;\; z(t) < 0,\; z'(t) < 0.$$

Case(i) In this case, $z(t) \leq x(t)$ and (2.1) reduces to

$$(r(t)(z'(t))^\gamma)' + q(t)z^\alpha(\sigma(t)) + v(t)z^\alpha(\eta(t)) \leq 0.(\because \gamma < \alpha < \beta)$$

for $t \geq t_3 > t_2$. Proceeding as in the proof of Theorem 4.1, we get a contradiction to (H_{13}).
Case(ii) $\lim_{t\to\infty} z(t)$ exists. Let $\lim_{t\to\infty} z(t) = \zeta, \zeta \in (-\infty, 0]$. We claim that $\zeta = 0$. If not, then there exist $l < 0$ and $t_3 > t_2$ such that $z(\sigma(t)) \leq z(t) < l,\; z(\eta(t)) \leq z(t) < l$ for $t \geq t_3$. From (2.1), it follows that $z(t) > ax(\tau(t))$ and hence $x(\tau(\sigma(t))) > \frac{1}{a}z(\sigma(t))$, that is, $x(\sigma(\tau(t))) > \left(\frac{l}{a}\right)$ for $t \geq t_3$. Also, $x(\eta(\tau(t))) > \left(\frac{l}{a}\right)$ for $t \geq t_3$. Since (2.1) can be written as

$$(r(\tau(t))(z'(\tau(t))^\gamma)' + q(\tau(t))x^\alpha(\sigma(\tau(t)) + v(\tau(t))x^\beta(\eta(\tau(t)) = 0,$$

then for $t \geq t_3$, it follows that

$$(r(\tau(t))(z'(\tau(t))^\gamma)' + \left(\frac{l}{a}\right)^\alpha q(\tau(t)) + \left(\frac{l}{a}\right)^\alpha v(\tau(t)) \leq 0.$$

Consequently,

$$\left(\frac{l}{a}\right)^\alpha \left[\int_{t_3}^t q(\tau(s)) + \int_{t_3}^t v(\tau(s))\right] ds \leq -\left[(r(\tau(t))(z'(\tau(t))^\gamma)'\right]_{t_3}^t$$
$$< -r(\tau(t))(z'(\tau(t)))^\gamma < \infty \text{ as } t \to \infty$$

contradicts (H_{19}). So, our claim holds and hence

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty} z(t) \\ &\le \liminf_{t\to\infty}(x(t)+dx(\tau(t))) \\ &\le \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}(dx(\tau(t))) \\ &= \limsup_{t\to\infty} x(t) + d\limsup_{t\to\infty} x(\tau(t)) \\ &= (1+d)\limsup_{t\to\infty} x(t)\ (\because (1+d)<0) \end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$, that is, $\lim_{t\to\infty} x(t) = 0$.

Case(iii) In this case, $z(t) \le x(t)$. Since $(r(t)(z'(t))^{\gamma})$ is nonincreasing there exists a constant $C < 0$ such that $(r(t)(z'(t))^{\gamma}) \le C$, $t \ge t_2$. From Lemma 2.2(ii), it follows that $z(t) \ge (-C)^{\frac{1}{\gamma}} R(t)$, $t \ge t_2$. Hence, (4.1) can be written as

$$(r(t)(z'(t))^{\gamma})' + q(t)(-C)^{\frac{1}{\gamma}} R^{\alpha}(\sigma(t)) + v(t)(-C)^{\frac{1}{\gamma}} R^{\alpha}(\eta(t)) \le 0.$$

Let $y(t) = (r(t)(z'(t))^{\gamma})$ Then the last inequality becomes

$$y'(t) \quad \le -(-C)^{\frac{1}{\gamma}}[R^{\alpha}(\sigma(t))q(t) + R^{\alpha}(\sigma(t))v(t)].$$

Integrating from $t_3(> t_2)$ to t, we get

$$\int_{t_3}^{t} y'(s)ds \quad \le -(-C)^{\frac{1}{\gamma}} \int_{t_3}^{t}[R^{\alpha}(\sigma(s))q(s) + R^{\alpha}(\sigma(s))v(s)]ds$$

implies that,

$$y(t) - y(t_3) \quad \le -(-C)^{\frac{1}{\gamma}} \int_{t_3}^{t}[R^{\alpha}(\sigma(s))q(s) + R^{\alpha}(\sigma(s))v(s)]ds$$

that is,

$$y(t) \le y(t_3) - (-C)^{\frac{1}{\gamma}} \int_{t_3}^{t}[R^{\alpha}(\sigma(s))q(s) + R^{\alpha}(\sigma(s))v(s)]ds.$$

Further integrating of the last inequality, we obtain

$$\int_{t_3}^{t} z'(s)ds \le -(-C)^{\frac{1}{\gamma}} \int_{t_3}^{t} \left[\frac{1}{r(\theta)} \int_{t_3}^{\theta} [R^{\alpha}(\sigma(s))q(s) + R^{\alpha}(\sigma(s))v(s)ds]\right]^{\frac{1}{\gamma}} d\theta,$$

$$z(t) \quad \le z(t_3) - (-C)^{\frac{1}{\gamma}} \int_{t_3}^{t} \left[\frac{1}{r(\theta)} \int_{t_3}^{\theta}[R^{\alpha}(\sigma(s))q(s) + R^{\alpha}(\sigma(s))v(s)ds]\right]^{\frac{1}{\gamma}} d\theta \to -\infty \ as\ t \to \infty,$$

a contradiction to the fact that $z(t) > 0$.

Case(iv) We have $x(t)+p(t)x(\tau(t)) < 0$ implies that $z(t) > p(t)x(\tau(t)) \ge ax(\tau(t))$. Hence, $z(t) > ax(\tau(t))$. Therefore,

$$z(\tau^{-1}(\sigma(t))) > ax(\tau(\tau^{-1}(\sigma(t)))) > ax(\sigma(t)).$$

Since $z(t)$ is nonincreasing, there exists a constant $C > 0$ such that $-C \ge z(\tau^{-1}(\sigma(t))) > ax(\sigma(t))$, that is, $\frac{-C}{a} \le x(\sigma(t))$. Now

$$(r(t)(z'(t))^{\gamma})' + q(t)x^{\alpha}(\sigma(t)) + v(t)x^{\alpha}(\eta(t)) \le 0. (\because \gamma < \alpha < \beta)$$

implies that

$$(r(t)(z'(t))^{\gamma})' \le -(\frac{-C}{a})^{\alpha} q(t) - (\frac{-C}{a})^{\alpha} v(t).\ t \ge t_2$$

Defining the function $y(t) = (r(t)(z'(t))^{\gamma}$, the last inequality becomes

$$y'(t) \leq -(\frac{-C}{a})^{\alpha} q(t) - (\frac{-C}{a})^{\alpha} v(t).$$

Integrating from $t_3(> t_2)$ to t, we get

$$\int_{t_3}^{t} y'(s)ds \leq -(\frac{-C}{a})^{\alpha} \int_{t_3}^{t} [q(s) + v(s)]ds,$$

that is,

$$\begin{aligned} y(t) &\leq y(t_3) - (\tfrac{-C}{a})^{\alpha} \textstyle\int_{t_3}^{t} [q(s) + v(s)]ds, \\ &\leq -(\tfrac{-C}{a})^{\alpha} \textstyle\int_{t_3}^{t} [q(s) + v(s)]ds. \end{aligned}$$

Further integrating of last inequaity, we obtain

$$\int_{t_3}^{t} z'(s)ds \leq -(\frac{-C}{a})^{\frac{\alpha}{\gamma}} \int_{t_3}^{t} \big[\frac{1}{r(s)} \int_{t_3}^{\theta} q(\theta) + v(\theta)d\theta\big]^{\frac{1}{\gamma}} ds,$$

that is,

$$z(t) \leq z(t_3) - (\frac{-C}{a})^{\frac{\alpha}{\gamma}} \int_{t_3}^{t} \big[\frac{1}{r(s)} \int_{t_3}^{\theta} q(\theta) + v(\theta)d\theta\big]^{\frac{1}{\gamma}} ds \ \rightarrow -\infty \ as\ t \rightarrow \infty$$

which is a contradiction to the fact that $z(t)$ is bounded and $\lim_{t\to\infty}$ z(t) exists. This completes the proof of the theorem. □

Theorem 5.2. *Let $-\infty < a \leq p(t) \leq d < -1$ and $\gamma < \beta < \alpha$. Assume that $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$, $r'(t) > 0$, $\tau(\sigma(t)) = \sigma(\tau(t))$ and $\tau(\eta(t)) = \eta(\tau(t))$ for any large t. If (H_0), (H_{13}), (H_{14}), $(H_{15\beta})$, (H_{18}), (H_{19}) and (H_{20}) hold, then every bounded solution of (1.1) either oscillates or converges to zero.*

Theorem 5.3. *Let $-\infty < a \leq p(t) \leq d < -1$ and $\gamma = \alpha = \beta$. Assume that $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$, $\tau(\sigma(t)) = \sigma(\tau(t))$ and $\tau(\eta(t)) = \eta(\tau(t))$ hold for all large t. If (H_0), (H_{13}), (H_{14}), $(H_{16\alpha})$, (H_{18}), (H_{19}) and (H_{20}) hold, then every bounded solution of (1.1) either oscillates or converges to zero.*

Theorem 5.4. *Let $-\infty < a \leq p(t) \leq d < -1$ and $\alpha < \beta < \gamma$. Assume that $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$, $\tau(\sigma(t)) = \sigma(\tau(t))$ and $\tau(\eta(t)) = \eta(\tau(t))$ hold for all large t. If (H_0), (H_{13}), (H_{14}), (H_{15}), $(H_{17\alpha})$, (H_{18}), (H_{19}) and (H_{20}) hold, then every bounded solution of (1.1) either oscillates or converges to zero.*

Theorem 5.5. *Let $-\infty < a \leq p(t) \leq d < -1$ and $\beta < \alpha < \gamma$. Assume that $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$, $\tau(\sigma(t)) = \sigma(\tau(t))$ and $\tau(\eta(t)) = \eta(\tau(t))$ hold for all large t. If (H_0), (H_{13}), (H_{14}) $(H_{17\beta})$, (H_{18}), (H_{19}) and (H_{20}) hold, then every bounded solution of (1.1) either oscillates or converges to zero.*

Theorem 5.6. *Let $-\infty < a \leq p(t) \leq d < -1$ and $\alpha < \gamma < \beta$. Assume that $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$, $\tau(\sigma(t)) = \sigma(\tau(t))$ and $\tau(\eta(t)) = \eta(\tau(t))$ hold for all large t. If (H_0), (H_{13}), (H_{14}), $(H_{16\alpha})$, (H_{18}), (H_{19}) and (H_{20}) hold, then every bounded solution of (1.1) either oscillates or converges to zero.*

Theorem 5.7. *Let $-\infty < a \leq p(t) \leq d < -1$ and $\beta < \gamma < \alpha$. Assume that $\eta(t) \geq \sigma(t)$, $\sigma'(t) \geq 1$, $\tau(\sigma(t)) = \sigma(\tau(t))$ and $\tau(\eta(t)) = \eta(\tau(t))$ hold for all large t. If (H_0), (H_{13}), (H_{14}), $(H_{16\beta})$, (H_{18}), (H_{19}) and (H_{20}) hold, then every bounded solution of (1.1) either oscillates or converges to zero.*

Note that the above results can be proved in the light of the results of Section 4. We conclude this section with the following example:

Example 5.8. Consider

$$(t(x(t)+x(t-3\pi))')' + e^{-t}x^3(t-2) + te^{-t}x^5(t-1) = 0,\ t > 3\pi, \tag{5.1}$$

where $r(t) = t$, $p(t) = -1$, $q(t) = e^{-t}$, $v(t) = te^{-t}$. Here, $R(t) = \int_{t_0}^{\infty} \left(\frac{1}{t^\gamma}\right)^{\frac{1}{\gamma}} dt = \infty$ and $a_1(t) = \int_t^\infty [q(s)+v(s)]ds = \int_t^\infty [e^{-s} + se^{-s}]ds = e^{-t}(t+2)$,

$$\begin{aligned} A_1(t, K_{1\alpha}) &= \left[a_1(t) + K_{1\alpha}\int_t^\infty \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} (a_1(s))^{1+\frac{1}{\gamma}} ds\right]^{\frac{1}{\gamma}} \\ &= \left[e^{-t}(t+2) + K_{1\alpha}\int_t^\infty \left(\frac{1}{s^\gamma}\right)^{\frac{1}{\gamma}} (e^{-s}(s+2))^{1+\frac{1}{\gamma}} ds\right]^{\frac{1}{\gamma}} \\ &= [e^{-t}(t+2) + K_{1\alpha}\left(\int_t^\infty se^{-2s}ds + 4\int_t^\infty e^{-2s}ds + \int_t^\infty \frac{4}{s}ds\right)] \geq K_{1\alpha}\int_t^\infty \frac{4}{s}ds \end{aligned}$$

and $\int_{t_0}^\infty \left(\frac{1}{r(\sigma(s))}\right)^{\frac{1}{\gamma}} A_1(t, K_{1\alpha}) = \infty$. Hence, all conditions of Theorem 4.1 are satisfied. Therefore, (5.1) is oscillatory.

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