Palestine Journal of Mathematics

Vol. 10(2)(2021) , 502–532

EXISTENCE OF POSITIVE SOLUTIONS OF A CRITICAL SYSTEM IN $\mathbb{R}^N$

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Communicated by Martin Bohner

MSC 2010 Classifications: Primary 35J20, 35J50; Secondary 58E05.

Keywords and phrases: Critical system, Global Compactness result, Brouwer degree theory.

Abstract In this paper we show existence of positive solution to the system

$$(S)\qquad \begin{cases} -\Delta u + a(x)u = \dfrac{1}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N, \\ -\Delta v + b(x)v = \dfrac{1}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N, \\ u,v>0 \text{ in } \mathbb{R}^N, \\ u,v \in D^{1,2}(\mathbb{R}^N), \ \ N\geq 3. \end{cases}$$

We also prove a global compactness result for the associated energy functional similar to that due to Struwe in [14]. The basic tool employed here is some information on a limit system of (S) with $a=b=0$, the concentration compactness due to P. L. Lions [12] and Brouwer degree theory.

1 Introduction

In the celebrated paper [3], Benci and Cerami studied the following semilinear elliptic problem

$$(BC)\qquad \begin{cases} -\Delta u + a(x)u = u^{\frac{N+2}{N-2}} \text{ in } \mathbb{R}^N, \\ u\in D^{1,2}(\mathbb{R}^N), \ \ u\geq 0, \ \ N\geq 3, \end{cases}$$

where

(a_1) $a(x)\geq 0$ and $a(x)\geq a_0>0$, for all $x\in\mathbb{R}^N$ in a neighborhood of a point $\bar{x}$.

(a_2) $a\in L^q(\mathbb{R}^N)$ for all $q\in[p_1,p_2]$ with $1<p_1<\frac{N}{2}<p_2$ with $p_2<\dfrac{N}{4-N}$ if $N=3$.

(a_3) $|a|_{L^{N/2}(\mathbb{R}^N)}<S(2^{2/N}-1)$, where $S=\inf\limits_{u\in D^{1,2}(\mathbb{R}^N),u\neq 0}\dfrac{\int_{\mathbb{R}^N}|\nabla u|^2dx}{\left(\int_{\mathbb{R}^N}|u|^{2^*}dx\right)^{2/2^*}}$.

They used the properties of the solutions of a limit problem given by (BC) with $a=0$, the version to $\mathbb{R}^N$ of Struwe's Global Compactness result [14], Lions's Concentration and Compactness result [12] and arguments of Brouwer degree theory.

We would also like to mention that this kind of problem all these arguments were also used by Cerami and Passasseo in [4] with Neumann boundary conditions in a half-space $\mathbb{R}^N_+$ and by Alves in [1] with p-laplacian operator. As far as the extension to the p-laplacian operator is concerned, some technical difficulties as the lack of linearity and homogeneity must be faced. The version of bi-Laplacian operator was studied by Alves and do Ó in [2]. A multiplicity result involving category theory was studied in [6] by Chabrowski and Yang. More recently, in [17] Xie, Ma and Xu proved a version for [3] considering the Kirchhoff operator. Nascimento and Figueiredo show the same result of [3] considering the fractional Laplacian. A version for

Choquard equation was proved by Gao, E. da Silva, M. Yang, and J. Zhou in [10] and a version for Schrödinger-Poisson system was studied by Cerami and Molle in [5]. In [7], Chen, Wei and Yan showed existence of infinitely many non-radial solutions, whose energy can be made arbitrarily large with a radial. In [13] Penga, Wang and Yan showed the existence of infinitely many non-radial solutions with a partially radial.

A natural, still open question is to know whether Benci and Cerami's results is true in the system of equations case. In this paper, we give a first positive answer to this question. However, the extension to system involves some technical difficulties which are overcome with some refined estimates, as can be seen in Lemma 3.1, Theorem 3.2 and subsection 4.2. More precisely, in these result we give the complete descriptions for the Palais-Smale (PS) sequences of the corresponding energy functionals and by using these descriptions, the existence results of solutions are obtained. Moreover, the main feature of the system is a "double" lack of compactness due to the unboundedness of the domain and the presence of the critical Sobolev exponent. The solutions are sought by means of variational methods, although the functional related to the problem does not satisfy the Palais-Smale compactness condition.

In this paper we show existence of positive solution to the system

$$(S)\qquad \begin{cases} -\Delta u + a(x)u = \dfrac{1}{2^*}K_u(u,v) \ \text{ in } \mathbb{R}^N, \\ -\Delta v + b(x)v = \dfrac{1}{2^*}K_v(u,v) \ \text{ in } \mathbb{R}^N, \\ u, v > 0 \ \text{ in } \mathbb{R}^N, \\ u, v \in D^{1,2}(\mathbb{R}^N), \ \ N \geq 3. \end{cases}$$

Let $\mathbb{R}^2_+ := [0,\infty)\times[0,\infty)$ and set $2^* := 2N/(N-2)$. We state our main hypotheses on the function $K \in C^2(\mathbb{R}^2_+, \mathbb{R})$ as follows.

$(\mathcal{K}_0)$ K is 2^*-homogeneous, that is,

$$K(\lambda s, \lambda t) = \lambda^{2^*} K(s,t) \quad \text{for each } \lambda > 0, \ (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{K}_1)$ there exists $c_1 > 0$ such that

$$|K_s(s,t)| + |K_t(s,t)| \leq c_1 \left(s^{2^*-1} + t^{2^*-1}\right) \quad \text{for each } (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{K}_2)$ $K(s,t) > 0$ for each $s, t > 0$;

$(\mathcal{K}_3)$ $\nabla K(0,1) = \nabla K(1,0) = (0,0)$;

$(\mathcal{K}_4)$ $K_s(s,t), K_t(s,t) \geq 0$ for each $(s,t) \in \mathbb{R}^2_+$.

$(\mathcal{K}_5)$ the 1-homogeneous function $G : \mathbb{R}^2_+ \to \mathbb{R}$ given by $G(s^{2^*}, t^{2^*}) := K(s,t)$ is concave.

To state our main result we need some previous definitions and notations. Let us denote by $\widetilde{S}_K$ the best constant of the immersion $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \times L^{2^*}(\mathbb{R}^N)$, that is,

$$\widetilde{S}_K := \inf_{u,v \in D^{1,2}(\mathbb{R}^N), u,v\neq 0} \frac{\displaystyle\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx}{\left(\displaystyle\int_{\mathbb{R}^N} K(u,v)dx\right)^{2/2^*}}.$$

From now on, we consider the function $\Phi_{\delta,y} \in D^{1,2}(\mathbb{R}^N)$ given by

$$\Phi_{\delta,y}(x) = c\left(\frac{\delta}{\delta^2 + |x-y|^2}\right)^{(N-2)/2}, \quad x, y \in \mathbb{R}^N \ \text{ and } \ \delta > 0, \qquad (1.1)$$

where c is a positive constant. In [15] we can see that every positive solution of

$$(P_\infty) \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u \ \text{ in } \ \mathbb{R}^N, \\ u > 0 \ \text{ in } \ \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \quad N \geq 3. \end{cases}$$

is as (1.1). Moreover, it satisfies

$$\|\Phi_{\delta,y}\|^2 = S \quad \text{and} \quad |\Phi_{\delta,y}|_{2^*} = 1, \tag{1.2}$$

where S was defined in (a_3).

By [9, Lemma 3], there exist $s_o, t_o > 0$ such that $\widetilde{S}_K$ is attained by $(s_o\Phi_{\delta,y}, t_o\Phi_{\delta,y})$. Moreover,

$$M_K\widetilde{S}_K = S, \tag{1.3}$$

where $M_K = \max\limits_{s^2+t^2=1} K(s,t)^{2/2^*} = K(s_o,t_o)^{2/2^*}$.

The hypotheses on the functions $a, b : \mathbb{R}^N \mapsto \mathbb{R}^+$ are given by:

$((a,b)_1)$ The functions a, b are positive in a same set of positive measure.

$((a,b)_2)$ $a, b \in L^q(\mathbb{R}^N)$ for all $q \in [p_1, p_2]$ with $1 < p_1 < \frac{N}{2} < p_2$ and $p_2 < \dfrac{N}{4-N}$ if $N = 3$.

$((a,b)_3)$ $s_o^N|a|_{L^{N/2}(\mathbb{R}^N)} + t_o^N|b|_{L^{N/2}(\mathbb{R}^N)} < \widetilde{S}_K(2^{2/N} - 1)$.

We say that $(u,v) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}$ is a positive weak solution of (S) if $u, v > 0$ in $D^{1,2}(\mathbb{R}^N)$ and for all $\varphi, \psi \in D^{1,2}(\mathbb{R}^N)$ we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla u \nabla \varphi dx + \int_{\mathbb{R}^N} \nabla v \nabla \psi dx + \int_{\mathbb{R}^N} a(x)u\varphi dx + \int_{\mathbb{R}^N} b(x)v\psi dx \\ = \ &\frac{1}{2^*}\int_{\mathbb{R}^N} K_u(u,v)\varphi dx + \frac{1}{2^*}\int_{\mathbb{R}^N} K_v(u,v)\psi dx. \end{aligned}$$

In order to state the main result, we consider the C^1 functional $I : D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \mapsto \mathbb{R}$ associated to system (S) given by

$$I(u,v) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} a(x)u^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} b(x)v^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(u,v)dx,$$

where $\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx$, $\|v\|^2 = \int_{\mathbb{R}^N} |\nabla v|^2 dx$. Note that

$$\begin{aligned} I'(u,v)(\varphi,\psi) \ = \ &\int_{\mathbb{R}^N} \nabla u \nabla \varphi dx + \int_{\mathbb{R}^N} \nabla v \nabla \psi dx + \int_{\mathbb{R}^N} a(x)u\varphi dx + \int_{\mathbb{R}^N} b(x)v\psi dx \\ - \ &\frac{1}{2^*}\int_{\mathbb{R}^N} K_u(u,v)\varphi dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K_v(u,v)\psi dx, \end{aligned}$$

for all $(\varphi,\psi) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$.

Using the above notation we are able to state our main result.

Theorem 1.1. *Assume that* $((a,b)_1) - ((a,b)_3)$ *and* $(\mathcal{K}_0) - (\mathcal{K}_5)$ *hold. Then,* (S) *has a positive solution* $(u_0, v_0) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ *with*

$$\frac{1}{N}\widetilde{S}_K^{N/2} < I(u_0,v_0) < \frac{2}{N}\widetilde{S}_K^{N/2}.$$

The paper is organized as follows. In Section 2 we study the limit system associated to (S). In Section 3 we give the complete descriptions for the Palais-Smale (PS) sequences for the functional I. The proof of the main result is in Section 4.

2 Limit problem

We notice that we can use the homogeneity condition $(\mathcal{K}_0)$ to conclude that

$$K(s,t) = \frac{1}{2^*} s K_s(s,t) + \frac{1}{2^*} t K_t(s,t). \tag{2.1}$$

In this section we study the limit problem given by

$$(S_\infty) \qquad \begin{cases} -\Delta u = \dfrac{1}{2^*} K_u(u,v) \ \text{ in } \ \mathbb{R}^N, \\ -\Delta v = \dfrac{1}{2^*} K_v(u,v) \ \text{ in } \ \mathbb{R}^N, \\ u, v > 0 \ \text{ in } \ \mathbb{R}^N, \\ u, v \in D^{1,2}(\mathbb{R}^N), \ \ N \geq 3, \end{cases}$$

which the functional associated $I_\infty : D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \mapsto \mathbb{R}$ given by

$$I_\infty(u,v) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} K(u,v)dx.$$

Lemma 2.1. *Let (u_n, v_n) be sequence $(PS)_c$ for I_∞. Then*

(i) The sequence (u_n, v_n) is bounded in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$.

(ii) If $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$ and $v_n \rightharpoonup v$ in $D^{1,2}(\mathbb{R}^N)$, then $I'_\infty(u,v) = 0$.

(iii) If $c \in (-\infty, \frac{1}{N}\widetilde{S}_K^{N/2})$, then I_∞ satisfies the $(PS)_c$ condition, i.e, up to a subsequence,

$$(u_n, v_n) \to (u,v) \ \ in \ \ D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N).$$

Proof. Since $I_\infty(u_n, v_n) \to c$ and $I'_\infty(u_n, v_n) \to 0$ and from (2.1), we conclude that there exists $C > 0$ such that

$$C + \|u_n\| + \|v_n\| \geq I_\infty(u_n v_n) - \frac{1}{2^*} I'_\infty(u_n, v_n)(u_n, v_n) = \frac{1}{N}\|u_n\|^2 + \frac{1}{N}\|v_n\|^2 + o_n(1)$$

and the proof of part (i) is over. Now we prove (ii). Since $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$ and $v_n \rightharpoonup v$ in $D^{1,2}(\mathbb{R}^N)$, up to a subsequence, we get

$$u_n \to u \ \ in \ \ L^q_{loc}(\mathbb{R}^N), \ \ v_n \to v \ \ in \ \ L^q_{loc}(\mathbb{R}^N),$$

and

$$u_n(x) \to u(x) \ \ a.e \ \ in \ \ \mathbb{R}^N, \ \ v_n(x) \to v(x) \ \ a.e \ \ in \ \ \mathbb{R}^N.$$

Using a density argument we obtain

$$\int_{\mathbb{R}^N} K_u(u_n, v_n)\varphi dx + \int_{\mathbb{R}^N} K_v(u_n, v_n)\psi dx \to \int_{\mathbb{R}^N} K_u(u,v)\varphi dx + \int_{\mathbb{R}^N} K_v(u,v)\psi dx.$$

for all $\varphi, \psi \in D^{1,2}(\mathbb{R}^N)$, which implies (ii).

In order to prove (iii), consider $w_n = u_n - u$ and $z_n = v_n - v$. Note that applying [11, Lemma 4.6], we get

$$\begin{aligned} o_n(1) &= I'_\infty(u_n, v_n)(u_n, v_n) = \|u_n\|^2 + \|v_n\|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} K_u(u_n, v_n)u_n dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K_v(u_n, v_n)v_n dx \\ &= \|w_n\|^2 + \|u\|^2 + \|z_n\|^2 + \|v\|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} K_u(w_n + u, z_n + v)(w_n + u)dx \\ &- \frac{1}{2^*}\int_{\mathbb{R}^N} K_v(w_n + u, z_n + v)(z_n + v)dx. \end{aligned} \tag{2.2}$$

From [9, Lemma 8], we have

$$\|w_n\|^2+\|u\|^2+\|z_n\|^2+\|v\|^2-\frac{1}{2^*}\int_{\mathbb{R}^N}K_u(w_n,z_n)w_n dx$$
$$-\frac{1}{2^*}\int_{\mathbb{R}^N}K_v(w_n z_n)z_n dx-\frac{1}{2^*}\int_{\mathbb{R}^N}K_u(u,v)u dx-\frac{1}{2^*}\int_{\mathbb{R}^N}K_v(u,v)v dx=o_n(1).$$

Using the item (ii) and (2.1) we obtain

$$\|w_n\|^2+\|z_n\|^2-\int_{\mathbb{R}^N}K(w_n,z_n)dx=o_n(1).$$

Up to a subsequence, we conclude that there exists $\rho\geq 0$ such that

$$0\leq\rho=\lim_{n\to\infty}\left[\|w_n\|^2+\|z_n\|^2\right]=\lim_{n\to\infty}\int_{\mathbb{R}^N}K(w_n,z_n)dx.$$

Suppose, by contradiction, that $\rho>0$. From the inequality

$$\widetilde{S}_K\left(\int_{\mathbb{R}^N}K(w_n,z_n)dx\right)^{2/2^*}\leq\|w_n\|^2+\|z_n\|^2,$$

we get

$$\rho\geq\widetilde{S}_K\rho^{2/2^*}\Rightarrow\rho\geq\widetilde{S}_K^{N/2}. \tag{2.3}$$

Since

$$I_\infty(u,v)=\left(\frac{1}{2}-\frac{1}{2^*}\right)[\|u\|^2+\|v\|^2]=\frac{1}{N}[\|u\|^2+\|v\|^2]\geq 0$$

and

$$c=\frac{1}{N}[\|w_n\|^2+\|z_n\|^2]+I_\infty(u,v)+o_n(1), \tag{2.4}$$

we conclude

$$c = \frac{1}{N}[\|w_n\|^2+\|z_n\|^2]+I_\infty(u,v)+o_n(1)\geq\frac{1}{N}[\|w_n\|^2+\|z_n\|^2]+o_n(1)=\frac{1}{N}\rho\geq\frac{1}{N}\widetilde{S}_K^{N/2},$$

which is a contradiction. Hence $\rho=0$ and

$$\|w_n\|^2=\|u_n-u\|^2\to 0 \ \ and \ \ \|z_n\|^2=\|v_n-v\|^2\to 0.$$

□

3 A compactness result

Now, we establish the following lemma which will be useful to prove a compactness result.

Lemma 3.1. *Let (u_n,v_n) be a $(PS)_c$ sequence for the functional I_∞ with $u_n\rightharpoonup 0$, $v_n\rightharpoonup 0$ and $u_n\not\to 0$, $v_n\not\to 0$. Then, there are sequences $(R_n)\subset\mathbb{R}$, $(x_n)\subset\mathbb{R}^N$ and $(\Upsilon_0,\Upsilon_1)\in D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$ nontrivial solution of (P_∞) and a sequence (τ_n,ζ_n) which is a $(PS)_{\tilde{c}}$ for the I_∞ such that, up to a subsequence of (u_n,v_n),*

$$\tau_n(x)=u_n(x)-R_n^{(N-2)/2}\Upsilon_0(R_n(x-x_n))+o_n(1)$$

and

$$\zeta_n(x)=u_n(x)-R_n^{(N-2)/2}\ \Upsilon_1(R_n(x-x_n))+o_n(1).$$

Proof. Let $(u_n, v_n) \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for the functional I_∞, i.e,

$$I_\infty(u_n, v_n) \to c \text{ and } I'_\infty(u_n, v_n) \to 0. \tag{3.1}$$

From Lemma 2.1, (i), we get that (u_n, v_n) is bounded in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$. Since $u_n \rightharpoonup 0$, $v_n \rightharpoonup 0$ and $u_n \nrightarrow 0$, $v_n \nrightarrow 0$ it follows from Lemma 2.1 (iii) that

$$c \geq \frac{1}{N}\widetilde{S}_K^{N/2}.$$

Note that from (2.1) we obtain

$$c + o_n(1) = I_\infty(u_n, v_n) - \frac{1}{2^*}I'_\infty(u_n, v_n)(u_n, v_n) = \frac{1}{N}\int_{\mathbb{R}^N}[|\nabla u_n|^2 + |\nabla v_n|^2]dx,$$

which implies

$$\int_{\mathbb{R}^N}[|\nabla u_n|^2 + |\nabla v_n|^2]dx = \widetilde{S}_K^{N/2}. \tag{3.2}$$

Let L be a number such that $B_2(0)$ is covered by L balls of radius 1, $(R_n) \subset \mathbb{R}$, $(x_n) \subset \mathbb{R}^N$ such that

$$\sup_{y\in\mathbb{R}^N}\int_{B_{R_n^{-1}}(y)}[|\nabla u_n|^2 + |\nabla v_n|^2]dx = \int_{B_{R_n^{-1}}(x_n)}[|\nabla u_n|^2 + |\nabla v_n|^2]dx = \frac{\widetilde{S}_K^{N/2}}{2L}$$

and the function

$$(w_n(x), z_n(x)) = \left(R_n^{(2-N)/2}u_n\left(\frac{x}{R_n} + x_n\right), R_n^{(2-N)/2}v_n\left(\frac{x}{R_n} + x_n\right)\right).$$

Using a change of variable, we can prove that

$$\int_{B_1(0)}[|\nabla w_n|^2 + \nabla z_n|^2]dx = \frac{\widetilde{S}_K^{N/2}}{2L} = \sup_{y\in\mathbb{R}^N}\int_{B_1(y)}[|\nabla w_n|^2 + |\nabla z_n|^2]dx.$$

Now, for each $(\Phi_1, \Phi_2) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, we define

$$(\tilde{\Phi}_{1,n}, \tilde{\Phi}_{2,n})(x) = (R_n^{(N-2)/2}\ \Phi_1(R_n(x - x_n)), R_n^{(N-2)/2}\ \Phi_2(R_n(x - x_n)))$$

which satisfies

$$\int_{\mathbb{R}^N}[\nabla u_n\nabla\tilde{\Phi}_{1,n} + \nabla v_n\nabla\tilde{\Phi}_{2,n}]dx = \int_{\mathbb{R}^N}[\nabla w_n\nabla\Phi_1 + \nabla z_n\nabla\Phi_2]dx \tag{3.3}$$

and

$$\int_{\mathbb{R}^N}[K_u(u_n, v_n)\tilde{\Phi}_{1,n} + K_v(u_n, v_n)\tilde{\Phi}_{2,n}]dx = \int_{\mathbb{R}^N}[K_w(w_n, z_n)\Phi_1 + K_z(w_n, z_n)\Phi_2]dx, \tag{3.4}$$

where we conclude that

$$I_\infty(w_n, z_n) \to c \text{ and } I'_\infty(w_n, z_n) \to 0. \tag{3.5}$$

From Lemma 2.1, there exists $(\Upsilon_0, \Upsilon_1) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ such that, up to a subsequence, $(w_n, z_n) \rightharpoonup (\Upsilon_0, \Upsilon_1)$ in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ and $I'_\infty(\Upsilon_0, \Upsilon_1) = 0$.

As a consequence of [9, Lemma 6], we get

$$\int_{\mathbb{R}^N}K(w_n, z_n)\phi dx \to \int_{\mathbb{R}^N}K(\Upsilon_0, \Upsilon_1)\phi dx + \sum_{j\in J}\phi(x_j)\nu_j, \quad \forall\phi \in C_0^\infty(\mathbb{R}^N) \tag{3.6}$$

and

$$|\nabla w_n|^2 + |\nabla z_n|^2 \rightharpoonup \mu + \sigma \geq |\nabla \Upsilon_0|^2 + |\nabla \Upsilon_1|^2 + \sum_{j\in J} \phi(x_j)\mu_j + \sum_{j\in J} \phi(x_j)\sigma_j, \ \ \forall \phi \in C_0^\infty(\mathbb{R}^N),$$

for some $\{x_j\}_{j\in J} \subset \mathbb{R}^N$ and for some $\{\nu_j\}_{j\in J}$, $\{\mu_j\}_{j\in J}$, $\{\sigma_j\}_{j\in J} \subset \mathbb{R}^+$.

Since $\widetilde{S}_K \nu_j^{2/2_s^*} \leq \mu_j + \sigma_j$, we can conclude that J is finite. From now on, we denote by $J = \{1, 2, ..., m\}$ and $\Gamma \subset \mathbb{R}^N$ the set given by

$$\Gamma = \{x_j \in \{x_j\}_{j\in J}; |x_j| > 1\}, \ \ (x_j \ \text{given by} \ (3.6)).$$

We are going to show that $(\Upsilon_0, \Upsilon_1) \neq (0,0)$. Suppose, by contradiction, that $(\Upsilon_0, \Upsilon_1) = (0,0)$. Then, by (3.6) we have

$$\int_{\mathbb{R}^N} K(w_n, z_n)\phi dx \to 0, \ \ \forall \phi \in C_0^\infty(\mathbb{R}^N \setminus \{x_1, x_2, ..., x_m\}). \tag{3.7}$$

Since $(\phi_{1,n}, \phi_{2,n}) = (\phi w_n, \phi z_n)$ with $\phi \in C_0^\infty(\mathbb{R}^N \setminus \{x_1, x_2, ..., x_m\})$ is bounded, we obtain

$$I'_\infty(w_n, z_n)(\phi_{1,n}, \phi_{2,n}) = o_n(1),$$

that is,

$$\int_{\mathbb{R}^N} [\nabla w_n \nabla \phi_{1,n} + \nabla z_n \nabla \phi_{2,n}]dx - \frac{1}{2^*}\int_{\mathbb{R}^N} [K_w(w_n, z_n)\phi_{1,n} + K_z(w_n, z_n)\phi_{2,n}]dx = o_n(1).$$

Using the definition of $(\phi_{1,n}, \phi_{2,n})$ and (2.1) , we have

$$\int_{\mathbb{R}^N} [|\nabla w_n|^2 + |\nabla z_n|^2]\phi dx + \int_{\mathbb{R}^N} [w_n \nabla w_n \nabla \phi + z_n \nabla z_n \nabla \phi]dx - \int_{\mathbb{R}^N} K(w_n, z_n)\phi dx = o_n(1).$$

Then,

$$\int_{\mathbb{R}^N} [|\nabla w_n|^2 + |\nabla z_n|^2]\phi dx \leq \int_{\mathbb{R}^N} [|w_n||\nabla w_n||\nabla \phi| + |z_n||\nabla z_n||\nabla \phi|]dx + \int_{\mathbb{R}^N} K(w_n, z_n)\phi dx = o_n(1).$$

Using Hölder inequality we get

$$\begin{aligned}\int_{\mathbb{R}^N} [|\nabla w_n|^2 + |\nabla z_n|^2]\phi dx &\leq |\nabla w_n|_2 \left(\int_{\mathbb{R}^N} |w_n|^2 |\nabla \phi|^2 dx\right)^{1/2} \\ &+ \ \ |\nabla z_n|_2 \left(\int_{\mathbb{R}^N} |z_n|^2 |\nabla \phi|^2 dx\right)^{1/2} + \int_{\mathbb{R}^N} K(w_n, z_n)\phi dx = o_n(1).\end{aligned}$$

Since there exists $R > 0$ such that $supp\phi \subset B_R(0)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} [|\nabla w_n|^2 + |\nabla z_n|^2]\phi dx &\leq C|\nabla w_n|_2 \left(\int_{B_R(0)} |w_n|^2 dx\right)^{1/2} \\ &+ \ \ C|\nabla z_n|_2 \left(\int_{B_R(0)} |z_n|^2 dx\right)^{1/2} + \int_{\mathbb{R}^N} K(w_n, z_n)\phi dx = o_n(1).\end{aligned}$$

Since (w_n, z_n) is bounded in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, from compact embedding and (3.7), we obtain

$$\int_{\mathbb{R}^N} [|\nabla w_n|^2 + |\nabla z_n|^2]\phi dx \to 0, \ \ \forall \phi \in C_0^\infty(\mathbb{R}^N \setminus \{x_1, x_2, ..., x_m\}). \tag{3.8}$$

Let $\rho \in \mathbb{R}$ be a number that satisfies $0 < \rho < \min\{dist(\Gamma, \bar{B}_1(0)), 1)\}$. We will show that

$$\int_{B_{1+\rho}(0) \setminus B_{1+\frac{\rho}{3}}(0)} [|\nabla w_n|^2 + |\nabla z_n|^2]\phi dx \to 0. \tag{3.9}$$

We consider $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \phi(x) \le 1$ and $\phi(x) = 1$ if $x \in B_{1+\rho}(0)$. If $\tilde{\phi} = \phi|_{\mathbb{R}^N \setminus \{x_1,\ldots,x_m\}}$, follows by (3.8) that

$$\int_{\mathbb{R}^N} [|\nabla w_n|^2 + |\nabla z_n|^2]\tilde{\phi}dx \to 0.$$

Since

$$\begin{aligned} 0 &\le \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} [|\nabla w_n|^2 + |\nabla z_n|^2]dx \le \int_{B_{1+\rho}(0)} [|\nabla w_n|^2 + |\nabla z_n|^2]dx \\ &= \int_{B_{1+\rho}(0)} [|\nabla w_n|^2 + |\nabla z_n|^2]\tilde{\phi}dx \le \int_{\mathbb{R}^N} [|\nabla w_n|^2 + |\nabla z_n|^2]\tilde{\phi}dx, \end{aligned}$$

we have that (3.9) is true.

Let $\Psi \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \le \Psi(x) \le 1$ for all $x \in \mathbb{R}^N$ and

$$\Psi(x) = \begin{cases} 1, & x \in B_{1+\frac{\rho}{3}}(0), \\ 0, & x \in B^c_{1+\frac{2\rho}{3}}(0) \end{cases}$$

and consider the sequence $(\Psi_{1,n}, \Psi_{2,n})$ given by $(\Psi_{1,n}, \Psi_{2,n})(x) = (\Psi(x)w_n(x), \Psi(x)z_n(x))$.

Note that

$$\begin{aligned} &\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} [|\nabla \Psi_{1,n}|^2 + |\nabla \Psi_{2,n}|^2]dx \\ \le\ & 4\int_{[B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)]^2} |\Psi|^2|\nabla w_n|^2dx + 4\int_{[B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)]^2} |\Psi|^2|\nabla z_n|^2dx \\ +\ & 4\int_{[B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)]^2} |w_n|^2|\nabla \Psi|^2dx + 4\int_{[B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)]^2} |z_n|^2|\nabla \Psi|^2dx. \end{aligned}$$

From (3.9) we obtain

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} [|\nabla \Psi_{1,n}|^2 + |\nabla \Psi_{2,n}|^2]dx \to 0. \tag{3.10}$$

Since $(\Psi_{1,n}, \Psi_{2,n})$ is bounded in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, we derive that

$$\begin{aligned} &\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} \nabla w_n \nabla \Psi_{1,n}dx + \int_{B_{1+\frac{\rho}{3}}(0)} \nabla w_n \nabla \Psi_{1,n}dx \\ +\ & \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} \nabla z_n \nabla \Psi_{2,n}dx + \int_{B_{1+\frac{\rho}{3}}(0)} \nabla z_n \nabla \Psi_{2,n}dx \\ -\ & \frac{1}{2^*}\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} \Psi_{1,n}K_w(w_n, z_n)dx - \frac{1}{2^*}\int_{B_{1+\frac{\rho}{3}}(0)} \Psi_{1,n}K_w(w_n, z_n)dx \\ -\ & \frac{1}{2^*}\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} \Psi_{2,n}K_z(w_n, z_n)dx - \frac{1}{2^*}\int_{B_{1+\frac{\rho}{3}}(0)} \Psi_{2,n}K_z(w_n, z_n)dx = o_n(1). \end{aligned}$$

From definition of Ψ we have

$$\begin{aligned}
&\int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \nabla w_n \nabla \Psi_{1,n} dx + \int_{B_{1+\frac{\rho}{3}}(0)} |\nabla \Psi_{1,n}|^2 dx \\
+ &\int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \nabla z_n \nabla \Psi_{2,n} dx + \int_{B_{1+\frac{\rho}{3}}(0)} |\nabla \Psi_{2,n}|^2 dx \\
- &\frac{1}{2^*}\int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \Psi_{1,n} K_w(w_n,z_n) dx - \frac{1}{2^*}\int_{B_{1+\frac{\rho}{3}}(0)} \Psi_{1,n} K_w(\Psi_{1,n},\Psi_{2,n}) dx \\
- &\frac{1}{2^*}\int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \Psi_{2,n} K_z(w_n,z_n) dx - \frac{1}{2^*}\int_{B_{1+\frac{\rho}{3}}(0)} \Psi_{2,n} K_z(\Psi_{1,n},\Psi_{2,n}) dx = o_n(1). \quad (3.11)
\end{aligned}$$

Note that from Hölder inequality and (3.10) we get

$$\int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \nabla w_n \nabla \Psi_{1,n} dx + \int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \nabla z_n \nabla \Psi_{2,n} dx \to 0 \text{ when } n \to \infty. \quad (3.12)$$

Moreover, from a direct calculations we have

$$\frac{1}{2^*}\int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \Psi_{1,n} K_w(w_n,z_n) dx + \frac{1}{2^*}\int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \Psi_{2,n} K_z(w_n,z_n) dx = o_n(1) \quad (3.13)$$

From (3.11), (3.12) and (3.13) we obtain

$$\begin{aligned}
&\int_{B_{1+\frac{\rho}{3}}(0)} |\nabla \Psi_{1,n}|^2 dx + \int_{B_{1+\frac{\rho}{3}}(0)} |\nabla \Psi_{2,n}|^2 dx - \frac{1}{2^*}\int_{B_{1+\frac{\rho}{3}}(0)} \Psi_{1,n} K_w(\Psi_{1,n},\Psi_{2,n}) dx \\
- &\frac{1}{2^*}\int_{B_{1+\frac{\rho}{3}}(0)} \Psi_{2,n} K_z(\Psi_{1,n},\Psi_{2,n}) dx = o_n(1). \quad (3.14)
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{\mathbb{R}^N} [|\nabla \Psi_{1,n}|^2 + |\nabla \Psi_{2,n}|^2] dx &= \int_{B_{1+\frac{\rho}{3}}(0)} [|\nabla \Psi_{1,n}|^2 + |\nabla \Psi_{2,n}|^2] dx \\
&= \int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} [|\nabla \Psi_{1,n}|^2 + |\nabla \Psi_{2,n}|^2] dx + \int_{B_{1+\frac{\rho}{3}}(0)} [|\nabla \Psi_{1,n}|^2 + |\nabla \Psi_{2,n}|^2] dx \\
&= o_n(1) + \int_{B_{1+\frac{\rho}{3}}(0)} [|\nabla \Psi_{1,n}|^2 + |\nabla \Psi_{2,n}|^2] dx
\end{aligned}$$

and using (2.1), we get

$$\begin{aligned}
\int_{\mathbb{R}^N} K(\Psi_{1,n},\Psi_{2,n}) dx &= \int_{B_{1+\rho}(0)} K(\Psi_{1,n},\Psi_{2,n}) dx \\
&= \int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} K(\Psi_{1,n},\Psi_{2,n}) dx + \int_{B_{1+\frac{\rho}{3}}(0)} K(\Psi_{1,n},\Psi_{2,n}) dx \\
&= \int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} K(\Psi_{1,n},\Psi_{2,n}) dx + \int_{B_{1+\frac{\rho}{3}}(0)} K(\Psi_{1,n},\Psi_{2,n}) dx,
\end{aligned}$$

we conclude that

$$\int_{\mathbb{R}^N} [|\nabla \Psi_{1,n}|^2 + |\nabla \Psi_{2,n}|^2] dx - \int_{\mathbb{R}^N} K(\Psi_{1,n},\Psi_{2,n}) dx = o_n(1).$$

From definition of $\widetilde{S}_K$, we have

$$\begin{aligned}
&\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2\left[1-\left(\frac{1}{\widetilde{S}_K^{2^*/2}}\right)[\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]^{2^*-2}\right]\\
=\;& [\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]-\frac{1}{\widetilde{S}_K^{2^*/2}}[\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]^{2^*}\\
\le\;& \int_{\mathbb{R}^N}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx-\int_{\mathbb{R}^N}K(\Psi_{1,n},\Psi_{2,n})dx=o_n(1). \qquad (3.15)
\end{aligned}$$

Note that

$$\begin{aligned}
\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2 &= \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx+\int_{B_{1+\frac{\rho}{3}}(0)}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx\\
&= o_n(1)+\int_{B_{1+\frac{\rho}{3}}(0)}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx.
\end{aligned}$$

Since $\Phi_{1,n}=w_n$, $\Phi_{2,n}=z_n$ in $B_{1+\frac{\rho}{3}(0)}$ and that $B_{1+\frac{\rho}{3}(0)}\subset B_2(0)$, we obtain

$$\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2\le o_n(1)+\int_{B_2(0)}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx,$$

which implies

$$\begin{aligned}
\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2 &\le o_n(1)+\int_{\bigcup_{k=1}^L B_1(y_k)}[|\nabla w_n|^2+|\nabla z_n|^2]dx\\
&\le o_n(1)+\sum_{k=1}^L\int_{B_1(y_k)}[|\nabla w_n^2+|\nabla z_n|^2]dx\\
&\le o_n(1)+L\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}[|\nabla w_n|^2+|\nabla z_n|^2]dx\le o_n(1)+\frac{\widetilde{S}_K^{N/2}}{2}.
\end{aligned}$$

Then,

$$\left(\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2\right)^{1/2}\le o_n(1)+\frac{\widetilde{S}_K^{N/4}}{2^{1/2}}$$

implies

$$\left(\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2\right)^{(2^*-2)/2}\le o_n(1)+\left(\frac{\widetilde{S}_K^{N/4}}{2^{1/2}}\right)^{2^*-2}. \qquad (3.16)$$

Using (3.15) and (3.16), we have that

$$\begin{aligned}
&[\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]\left[1+o_n(1)-\frac{1}{\widetilde{S}_K^{2^*/2}}\left(\frac{\widetilde{S}_K^{N/4}}{2^{1/2}}\right)^{2^*-2}\right]\\
=\;& [\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]\left\{1+\frac{1}{\widetilde{S}_K^{2^*/2}}\left[o_n(1)-\left(\frac{\widetilde{S}_K^{N/4}}{2^{1/2}}\right)^{2^*-2}\right]\right\}\\
\le\;& [\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]\left[1-\frac{1}{\widetilde{S}_K^{2^*/2}}[\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]^{2^*-2}\right]=o_n(1).
\end{aligned}$$

But the equality

$$\frac{N}{4}(2^*-2)-\frac{2^*}{2}=\frac{N}{4}\left(\frac{4}{N-2}\right)-\frac{N}{N-2}=0,$$

implies

$$\|\Phi_n\|^2\left[1-\left(\frac{1}{2}\right)^{(2^*-2)/2}\right]\leq o_n(1),$$

where we conclude that $(\Phi_{1,n},\Phi_{1,n})\to(0,0)$ in $D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$.

Since $w_n=\Phi_{1,n}$, $z_n=\Phi_{2,n}$ in $B_1(0)$, we obtain

$$0\leq\int_{B_1(0)}[|\nabla w_n|^2+|\nabla z_n|^2]dx\leq|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2,$$

which implies

$$\int_{B_1(0)}[|\nabla w_n|^2+|\nabla z_n|^2]dx\to 0\ \text{ when }\ n\to\infty.$$

But this last convergence it is a contradiction with

$$\int_{B_1(0)}[|\nabla w_n|^2+|\nabla z_n|^2]dx=\frac{\widetilde{S}_K^{N/2}}{2L},\quad\forall n\in\mathbb{N}.$$

Then, $(\Upsilon_0,\Upsilon_1)\neq(0,0)$. Now we are going to show that there is (τ_n,ζ_n) in $D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$ such that (τ_n,ζ_n) is a $(PS)_{\tilde{c}}$ sequence for I_∞ satisfying

$$\tau_n(x)=u_n(x)-R_n^{(N-2)/2}\Upsilon_0(R_n(x-x_n))+o_n(1),$$

$$\zeta_n(x)=v_n(x)-R_n^{(N-2)/2}\Upsilon_1(R_n(x-x_n))+o_n(1),$$

for some subsequence of (u_n,v_n) that still denote by (u_n,v_n). For this, we consider $\psi\in C_0^\infty(\mathbb{R}^N)$ such that $0\leq\psi(x)\leq 1$ for all $x\in\mathbb{R}^N$ and

$$\psi(x)=\begin{cases}1, & \text{if }\ x\in B_1(0),\\ 0, & \text{if }\ x\in B_2^c(0)\end{cases}$$

and consider (τ_n,ζ_n) a sequence defined by

$$\tau_n(x)=u_n(x)-R_n^{(N-2)/2}\Upsilon_0(R_n(x-x_n))\psi(\bar{R}_n(x-x_n)),\tag{3.17}$$

$$\zeta_n(x)=v_n(x)-R_n^{(N-2)/2}\Upsilon_1(R_n(x-x_n))\psi(\bar{R}_n(x-x_n)),\tag{3.18}$$

where $(\bar{R}_n)$ satisfies $\tilde{R}_n=\dfrac{R_n}{\bar{R}_n}\to\infty$. From (3.17) and (3.18), we obtain

$$R_n^{(2-N)/2}\tau_n(x)=R_n^{(2-N)/2}u_n(x)-\Upsilon_0(R_n(x-x_n))\psi(\bar{R}_n(x-x_n))$$

and

$$R_n^{(2-N)/2}\zeta_n(x)=R_n^{(2-N)/2}v_n(x)-\Upsilon_1(R_n(x-x_n))\psi(\bar{R}_n(x-x_n)).$$

Making change of variable, we conclude

$$R_n^{(2-N)/2}\tau_n\left(\frac{z}{R_n}+x_n\right)=R_n^{(2-N)/2}u_n\left(\frac{z}{R_n}+x_n\right)-\Upsilon_0\psi\left(\frac{z}{\tilde{R}_n}\right)$$

and

$$R_n^{(2-N)/2}\zeta_n\left(\frac{z}{R_n}+x_n\right)=R_n^{(2-N)/2}v_n\left(\frac{z}{R_n}+x_n\right)-\Upsilon_1\psi\left(\frac{z}{\tilde{R}_n}\right).$$

Now we define

$$\widetilde{\tau}_n=R_n^{(2-N)/2}\tau_n\left(\frac{z}{R_n}+x_n\right)$$

and

$$\widetilde{\zeta}_n=R_n^{(2-N)/2}\zeta_n\left(\frac{z}{R_n}+x_n\right).$$

Since

$$w_n(x) = R_n^{(2-N)/2} u_n\left(\frac{x}{R_n} + x_n\right)$$

and

$$z_n(x) = R_n^{(2-N)/2} v_n\left(\frac{x}{R_n} + x_n\right),$$

we get,

$$\widetilde{\tau}_n(z) = w_n(z) - \Upsilon_0(z)\psi\left(\frac{z}{\tilde{R}_n}\right) \tag{3.19}$$

and

$$\widetilde{\zeta}_n(z) = \zeta_n(z) - \Upsilon_1(z)\psi\left(\frac{z}{\tilde{R}_n}\right). \tag{3.20}$$

If

$$\psi_n(z) = \psi\left(\frac{z}{\tilde{R}_n}\right) \tag{3.21}$$

we have that

$$\psi_n(z) = \begin{cases} 1, & \text{if } \ z \in B_{\tilde{R}_n}(0), \\ 0, & \text{if } \ z \in B^c_{2\tilde{R}_n}(0). \end{cases}$$

From (3.19), (3.20) and (3.21), we derive that

$$\widetilde{\tau}_n(z) = w_n(z) - \Upsilon_0(z)\psi_n(z)$$

and

$$\widetilde{\zeta}_n(z) = z_n(z) - \Upsilon_1(z)\psi_n(z).$$

Since $\tilde{R}_n \to \infty$, it is not difficult to show that $\Upsilon_i\psi_n \to \Upsilon_i$ in $D^{1,2}(\mathbb{R}^N)$, $i = 0, 1$. Then

$$\widetilde{\tau}_n(z) = w_n(z) - \Upsilon_0(z) + o_n(1) \tag{3.22}$$

and

$$\widetilde{\zeta}_n(z) = z_n(z) - \Upsilon_1(z) + o_n(1). \tag{3.23}$$

To finish the proof, it is enough to show that (τ_n, ζ_n) is a $(PS)_{\tilde{c}}$ sequence for I_∞. Note that making a change of variable we get

$$I_\infty(\tau_n, \zeta_n) = I_\infty(\widetilde{\tau}_n, \widetilde{\zeta}_n)$$

Using (3.22) and (3.23) and applying [11, Lemma 4.6], [9, Lemma 8] and (3.5), we have

$$I_\infty(\tau_n, \zeta_n) \quad = \quad I_\infty(w_n, z_n) - I_\infty(\Upsilon_0, \Upsilon_1) + o_n(1) = \widetilde{c} + o_n(1),$$

where $\widetilde{c} = c - I_\infty(\Upsilon_0, \Upsilon_1)$.

Now, since

$$0 \leq \|I'_\infty(\tau_n, \zeta_n)\|_{D'} \leq \|I'_\infty(\widetilde{\tau}_n, \widetilde{\zeta}_n)\|_{D'},$$

it is sufficient to prove that $\|I'_\infty(\widetilde{\tau}_n, \widetilde{\zeta}_n)\|_{D'} \to 0$ which is equivalent to show that

$$\|I'_\infty(\widetilde{\tau}_n, \widetilde{\zeta}_n) - I'_\infty(w_n, z_n) + I'_\infty(\Upsilon_0, \Upsilon_1)\|_{D'} \to 0. \tag{3.24}$$

But the last convergence is a direct consequence of [9, Lemma 8]. □

The next result is a version for a gradient system in $\mathbb{R}^N$ of the result due to Struwe that can be found in [14].

Theorem 3.2. *(A global compactness result) Let (u_n, v_n) be a $(PS)_c$ sequence for I with $u_n \rightharpoonup u_0$ in $D^{1,2}(\mathbb{R}^N)$ and $v_n \rightharpoonup v_0$ in $D^{1,2}(\mathbb{R}^N)$. Then, up to a subsequence, (u_n, v_n) satisfies either,*

(a) $(u_n, v_n) \to (u_0, v_0)$ in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ or,

(b) there exists $k \in \mathbb{N}$ and nontrivial solutions $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), ..., (z_0^k, \zeta_0^k)$ for the system (S_∞), such that

$$\|u_n\|^2 + \|v_n\|^2 \to \|u_0\|^2 + \|v_0\|^2 + \sum_{j=1}^{k} [\|z_0^j\|^2 + \|\zeta_0^j\|^2]$$

and

$$I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j).$$

Proof. From the weak convergence and a density argument, we have that (u_0, v_0) is a critical point of I. Suppose that $u_n \nrightarrow u_0$, $v_n \nrightarrow v_0$ in $D^{1,2}(\mathbb{R}^N)$ and let $(w_n^1, z_n^1) \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ be the sequence given by $w_n^1 = u_n - u_0$ and $z_n^1 = v_n - v_0$. Then, $w_n^1 \rightharpoonup 0$, $z_n^1 \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^N)$ and $w_n^1 \nrightarrow 0$, $z_n^1 \nrightarrow 0$ in $D^{1,2}(\mathbb{R}^N)$.

Applying [11, Lemma 4.6] and [9, Lemma 8], we obtain

$$I_\infty(w_n^1, z_n^1) = I(u_n, v_n) - I(u_0, v_0) + o_n(1) \tag{3.25}$$

and

$$I'_\infty(w_n^1, z_n^1) = I'(u_n, v_n) - I'(u_0, v_0) + o_n(1). \tag{3.26}$$

Then, we conclude from (3.25) and (3.26) that (w_n^1, z_n^1) is a $(PS)_{c_1}$ sequence for I_∞. Hence, by Lemma 3.1, there are sequences $(R_{n,1}) \subset \mathbb{R}$, $(x_{n,1}) \subset \mathbb{R}^N$, $(z_0^1, \zeta_0^1) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ nontrivial solution for the system (P_∞) and a $(PS)_{c_2}$ sequence $(w_n^2, z_n^2) \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ for I_∞ such that

$$w_n^2(x) = w_n^1(x) - R_{n,1}^{(N-2)/2} z_0^1(R_{n,1}(x - x_{n,1})) + o_n(1)$$

and

$$z_n^2(x) = z_n^1(x) - R_{n,1}^{(N-2)/2} \zeta_0^1(R_{n,1}(x - x_{n,1})) + o_n(1).$$

If we define

$$\Phi_n^1(x) = R_{n,1}^{(2-N)/2} w_n^1\left(\frac{x}{R_{n,1}} + x_{n,1}\right), \tag{3.27}$$

$$\Psi_n^1(x) = R_{n,1}^{(2-N)/2} z_n^1\left(\frac{x}{R_{n,1}} + x_{n,1}\right) \tag{3.28}$$

and

$$\widetilde{w}_n^2(x) = R_{n,1}^{(2-N)/2} w_n^2\left(\frac{x}{R_{n,1}} + x_{n,1}\right),$$

$$\widetilde{z}_n^2(x) = R_{n,1}^{(2-N)/2} z_n^2\left(\frac{x}{R_{n,1}} + x_{n,1}\right),$$

we get

$$\widetilde{w}_n^2(x) = \Phi_n^1(x) - z_0^1(x) + o_n(1), \tag{3.29}$$

$$\widetilde{z}_n^2(x) = \Psi_n^1(x) - \zeta_0^1(x) + o_n(1) \tag{3.30}$$

and

$$\|\Phi_n^1\| = \|w_n^1\|, \ \ \|\Psi_n^1\| = \|z_n^1\| \ \text{ and } \int_{\mathbb{R}^N} K(\Phi_n^1, \Psi_n^1)dx = \int_{\mathbb{R}^N} K(w_n^1, z_n^1)dx. \tag{3.31}$$

Hence,

$$I_\infty(\Phi_n^1, \Psi_n^1) = I_\infty(w_n^1, z_n^1) \tag{3.32}$$

and

$$I'_\infty(\Phi_n^1, \Psi_n^1) \to 0 \ \text{ in } \ (D^{1,2}(\mathbb{R}^N))'. \tag{3.33}$$

From (3.32), (3.33) and from item (a) by lemma 2.1, we have that (Φ_n^1, Ψ_n^1) is a bounded sequence in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ and, up to a subsequence,

$$\Phi_n^1 \rightharpoonup z_0^1, \ \ \Psi_n^1 \rightharpoonup \zeta_0^1 \ \text{ in } \ D^{1,2}(\mathbb{R}^N). \tag{3.34}$$

Applying [11, Lemma 4.6] and [9, Lemma 8] again, we obtain

$$I_\infty(\widetilde{w}_n^2, \widetilde{z}_n^2) = I_\infty(\Phi_n^1, \Psi_n^1) - I_\infty(z_0^1, \zeta_0^1) + o_n(1) = I(u_n, v_n) - I(u_0, v_0) - I_\infty(z_0^1, \zeta_0^1) + o_n(1) \tag{3.35}$$

and

$$I'_\infty(\widetilde{w}_n^2, \widetilde{z}_n^2) = I'_\infty(\Phi_n^1, \Psi_n^1) - I'_\infty(z_0^1, \zeta_0^1) + o_n(1). \tag{3.36}$$

If $\widetilde{w}_n^2, \widetilde{z}_n^2 \to 0$ in $D^{1,2}(\mathbb{R}^N)$, the proof is over for $k = 1$, because in this case, we have

$$\|u_n\|^2 + \|v_n\|^2 \to \|u_0\|^2 + \|v_0\|^2 + \|z_0^1\|^2 + \zeta_0^1\|^2.$$

Moreover, from continuity of I_∞, we get

$$I(u_n, v_n) \to I(u_0, v_0) + I_\infty(z_0^1, \zeta_0^1).$$

If $\widetilde{w}_n^2 \nrightarrow 0, \widetilde{z}_n^2 \nrightarrow 0$ in $D^{1,2}(\mathbb{R}^N)$, using (3.30) and (3.34) that $\widetilde{w}_n^2, \widetilde{z}_n^2 \rightharpoonup 0 \ D^{1,2}(\mathbb{R}^N)$, by (3.35) and (3.36), we conclude that $(\widetilde{w}_n^2, \widetilde{z}_n^2)$ is a $(PS)_{c_2}$ sequence for I_∞.

By Lemma 3.1, there are sequences $(R_{n,2}) \subset \mathbb{R}$, $(x_{n,2}) \subset \mathbb{R}^N$, $(z_0^2, \zeta_0^2) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ nontrivial solutions of (S_∞) and a $(PS)_{c_3}$ sequence $(w_n^3, z_n^3) \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ for I_∞ such that

$$w_n^3(x) = \widetilde{w}_n^2(x) - R_{n,2}^{(N-2)/2} z_0^2(R_{n,2}(x - x_{n,2})) + o_n(1),$$

$$z_n^3(x) = \widetilde{z}_n^2(x) - R_{n,2}^{(N-2)/2} \zeta_0^2(R_{n,2}(x - x_{n,2})) + o_n(1).$$

If

$$\Phi_n^2(x) = R_{n,2}^{(2-N)/2} \widetilde{w}_n^2\left(\frac{x}{R_{n,2}} + x_{n,2}\right),$$

$$\Psi_n^2(x) = R_{n,2}^{(2-N)/2} \widetilde{z}_n^2\left(\frac{x}{R_{n,2}} + x_{n,2}\right)$$

and

$$\widetilde{w}_n^3(x) = R_{n,2}^{(2-N)/2} w_n^3\left(\frac{x}{R_{n,2}} + x_{n,2}\right),$$

$$\widetilde{z}_n^3(x) = R_{n,2}^{(2-N)/2} z_n^3\left(\frac{x}{R_{n,2}} + x_{n,2}\right),$$

we have that

$$\widetilde{w}_n^3(x) = \Phi_n^2(x) - z_0^2(x) + o_n(1), \tag{3.37}$$

$$\widetilde{z}_n^3(x) = \Psi_n^2(x) - \zeta_0^2(x) + o_n(1). \tag{3.38}$$

Arguing as before, we conclude

$$\|\widetilde{w}_n^3\|^2 + \|\widetilde{z}_n^3\|^2 = \|u_n\|^2 + \|v_n\|^2 - \|u_0\|^2 - \|v_0\|^2 - \|z_0^1\|^2 - \|\zeta_0^1\|^2 - \|z_0^2\|^2 - \|\zeta_0^2\|^2 + o_n(1) \tag{3.39}$$

$$I_\infty(\widetilde{w}_n^3, \widetilde{z}_n^3) = I(u_n, v_n) - I(u_0, v_0) - I_\infty(z_0^1, \zeta_0^1) - I_\infty(z_0^2, \zeta_0^2) + o_n(1). \tag{3.40}$$

and

$$I'_\infty(\widetilde{w}_n^3, \widetilde{z}_n^3) = I'_\infty(\Phi_n^2, \Psi_n^2) - I'_\infty(z_0^2, \zeta_0^2) + o_n(1). \tag{3.41}$$

If $\tilde{w}_n^3, \tilde{z}_n^3 \to 0$ in $D^{1,2}(\mathbb{R}^N)$, the proof is over with $k = 2$, because $\|\tilde{w}_n^3\|^2 \to 0$, $\|\tilde{z}_n^3\|^2 \to 0$ and from (3.39), we have

$$\|u_n\|^2 + \|v_n\|^2 \to \|u_0\|^2 + \|v_0\|^2 + \sum_{j=1}^{2}[\|z_0^j\|^2 + \|\zeta_0^j\|^2].$$

Moreover, from continuity of I_∞, we have that $I_\infty(\tilde{z}_n^3) \to 0$, now using (3.40) we get

$$I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{2} I_\infty(z_0^j, \zeta_0^j).$$

If $\widetilde{w}_n^3, \widetilde{z}_n^3 \nrightarrow 0$ in $D^{1,2}(\mathbb{R}^N)$, we can repeat the same arguments before and we can find $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), ..., (z_0^{k-1}, \zeta_0^{k-1})$ nontrivial solutions for the system (S_∞) satisfying

$$\|\widetilde{w}_n^k\|^2 + \|\widetilde{z}_n^k\|^2 = \|u_n\|^2 + \|v_n\|^2 - \|u_0\|^2 - \|v_0\|^2 - \sum_{j=1}^{k-1}[\|z_0^j\|^2 - \|\zeta_0^j\|^2] + o_n(1), \tag{3.42}$$

and

$$I_\infty(\widetilde{z}_n^k, \widetilde{z}_n^k) = I(u_n, v_n) - I(u_0, v_0) - \sum_{j=1}^{k-1} I_\infty(z_0^j, \zeta_0^j) + o_n(1). \tag{3.43}$$

From definition of $\widetilde{S}_K$, we conclude that

$$\left(\int_{\mathbb{R}^N} K(z_0^j, \zeta_0^j)dx\right)^{2/2^*} \widetilde{S}_K \le \|z_0^j\|^2 + \|\zeta_0^j\|^2, \quad j = 1, 2, ..., k-1. \tag{3.44}$$

Since (z_0^j, ζ_0^j) is nontrivial solution of (S_∞), for all $j = 1, 2, ..., k-1$, we get

$$\|z_0^j\|^2 + \|\zeta_0^j\|^2 = \int_{\mathbb{R}^N} K(z_0^j, \zeta_0^j)dx.$$

Hence,

$$-\|z_0^j\|^2 - \|\zeta_0^j\|^2 \le -\widetilde{S}_K^{N/2}, \quad j = 1, 2, ..., k-1. \tag{3.45}$$

From (3.42) and (3.45), we have

$$\begin{aligned} \|\widetilde{w}_n^k\|^2 + \|\widetilde{z}_n^k\|^2 &= \|u_n\|^2 + \|v_n\|^2 - \|u_0\|^2 - \|v_0\|^2 \\ &- \sum_{j=1}^{k-1}[\|z_0^j\|^2 + \|\zeta_0^j\|^2] + o_n(1) \\ &\le \|u_n\|^2 + \|v_n\|^2 - \|u_0\|^2 - \|v_0\|^2 - (k-1)\widetilde{S}_K^{N/2} + o_n(1). \end{aligned} \tag{3.46}$$

Since (u_n, v_n) is bounded in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, for k sufficient large, we conclude that $\widetilde{w}_n^k$, $\widetilde{z}_n^k \to 0$ in $D^{1,2}(\mathbb{R}^N)$ and the proof is over. □

Corollary 3.3. *Let (u_n, v_n) be a $(PS)_c$ sequence for I with $c \in (0, \frac{1}{N}\widetilde{S}_K^{N/2})$. Then, up to a subsequence, (u_n, v_n) strong converges in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$.*

Proof. We have that (u_n, v_n) is bounded in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$,

$$u_n \rightharpoonup u_0, \;\; v_n \rightharpoonup v_0 \;\text{ in }\; D^{1,2}(\mathbb{R}^N)$$

and by a density argument $I'(u_0, v_0) = 0$. Suppose, by contradiction, that

$$u_n \nrightarrow u_0, \;\; v_n \nrightarrow v_0 \;\text{ in }\; D^{1,2}(\mathbb{R}^N).$$

From Theorem 3.2, there are $k \in \mathbb{N}$ and nontrivial solutions $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), ..., (z_0^k, \zeta_0^k)$ of the system (S_∞) such that,

$$\|u_n\|^2 + \|v_n\|^2 \to \|u_0\|^2 + \|v_0\|^2 + \sum_{j=1}^{k}[|z_0^j\|^2 + |\zeta_0^j\|^2]$$

and

$$I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j).$$

Note that by (2.1) we have

$$\begin{aligned} I(u_0, v_0) &= \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} a(x)u_0^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} b(x)v_0^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(u_0, v_0)dx \\ &= \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\Big(\int_{\mathbb{R}^N} K(u_0, v_0)dx - \|u_0\|^2 - \|v_0\|^2\Big) - \frac{1}{2^*}\int_{\mathbb{R}^N} K(u_0, v_0)dx \\ &= \frac{1}{N}\int_{\mathbb{R}^N} K(u_0, v_0)dx \geq 0. \end{aligned}$$

Then,

$$c = I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) \geq \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) \geq \frac{k}{N}\widetilde{S}_K^{N/2} \geq \frac{1}{N}\widetilde{S}_K^{N/2},$$

which is a contradiction with $c \in (0, \frac{1}{N}\widetilde{S}_K^{N/2})$. □

Corollary 3.4. *The functional $I : D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ satisfies the Palais-Smale condition in $(\frac{1}{N}\widetilde{S}_K^{N/2}, \frac{2}{N}\widetilde{S}_K^{N/2})$.*

Proof. Let (u_n, v_n) be a sequence in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ that satisfies

$$I(u_n, v_n) \to c \;\text{ and }\; I'(u_n, v_n) \to 0.$$

Since (u_n, v_n) is bounded in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, up to a subsequence, we have

$$u_n \rightharpoonup u_0, \;\; v_n \rightharpoonup v_0 \;\text{ in }\; D^{1,2}(\mathbb{R}^N).$$

Moreover, $I(u_0, v_0) \geq 0$. Suppose, by contradiction, that

$$u_n \nrightarrow u_0, \;\; v_n \nrightarrow v_0 \;\text{ in }\; D^{1,2}(\mathbb{R}^N).$$

From Theorem 3.2, there are $k \in \mathbb{N}$ and nontrivial solutions $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), ..., (z_0^k, \zeta_0^k)$ of the system (S_∞) such that

$$\|u_n\|^2 + \|v_n\|^2 \to \|u_0\|^2 + \|v_0\|^2 + \sum_{j=1}^{k}[\||z_0^j\|^2 + \|\zeta_0^j\|^2]$$

and

$$I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) = c.$$

Since $I(u_0, v_0) \geq 0$, then $k = 1$ and z_0^1, ζ_0^1 cannot change of the sign. Hence,

$$c = I(u_0, v_0) + I_\infty(z_0^1, \zeta_0^1) = I(u_0, v_0) + \frac{1}{N}\widetilde{S}_K^{N/2}.$$

From definition of $\widetilde{S}_K$, $I'(u_0, v_0)(u_0, v_0) = 0$ and

$$I(u_0, v_0) = \frac{1}{N}\int_{\mathbb{R}^N} K(u_0, v_0)dx,$$

we have,

$$\frac{2}{N}\widetilde{S}_K^{N/2} \leq I(u_0, v_0) + \frac{1}{N}\widetilde{S}_K^{N/2} = c,$$

which a contradiction with $c \in (\frac{1}{N}\widetilde{S}_K^{N/s}, \frac{2}{N}\widetilde{S}_K^{N/2})$. □

Corollary 3.5. *Let $(u_n, v_n) \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for I with $c \in (\frac{k}{N}\widetilde{S}_K^{N/2}, \frac{(k+1)}{N}\widetilde{S}_K^{N/2})$, where $k \in \mathbb{N}$. Then, the weak limit (u_0, v_0) of (u_n, v_n) is not trivial.*

Proof. Suppose, by contradiction, that $u_0, v_0 \equiv 0$. Since $c > 0$, then $u_n, v_n \nrightarrow 0$ in $D^{1,2}(\mathbb{R}^N)$. From Theorem 3.2, up to subsequence, we get

$$\|u_n\|^2 + \|v_n\|^2 \to \|u_0\|^2 + \|v_0\|^2 + \sum_{j=1}^{k}[\|z_0^j\|^2 + \|\zeta_0^j\|^2] = \sum_{j=1}^{k}[\|z_0^j\|^2 + \|\zeta_0^j\|^2]$$

and

$$I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) = \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) = c \geq \frac{(k+1)}{N}\widetilde{S}_K^{N/2},$$

which a contradiction with $c \in (\frac{k}{N}\widetilde{S}_K^{N/2}, \frac{(k+1)}{N}\widetilde{S}_K^{N/2})$. □

From now on we consider the functional $f : D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$f(u, v) = \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} a(x)u^2 dx + \int_{\mathbb{R}^N} b(x)v^2 dx$$

and the manifold $\mathcal{M} \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ given by

$$\mathcal{M} = \left\{(u, v) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} K(u, v)dx = 1\right\}.$$

The next results are direct consequence of the corollaries above.

Lemma 3.6. *Let $(u_n, v_n) \subset \mathcal{M}$ be a sequence that satisfies*

$$f(u_n, v_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n, v_n) \to 0.$$

Then, the sequence $(w_n, z_n) \subset D^{1,2}(\mathbb{R}^N)$, where $(w_n, z_n) = (c^{(N-2)/4}u_n, c^{(N-2)/4}v_n)$, satisfies the following limits.

$$I(w_n, z_n) \to \frac{1}{N}c^{N/2} \quad \text{and} \quad I'(w_n, z_n) \to 0.$$

Lemma 3.7. *Suppose that there are a sequence $(u_n, v_n) \subset \mathcal{M}$ and $c \in (\widetilde{S}_K, 2^{2/N}\widetilde{S}_K)$ such that*

$$f(u_n, v_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n, v_n) \to 0.$$

Then, up to a subsequence, $u_n \to u$, $v_n \to v$ in $D^{1,2}(\mathbb{R}^N)$, for some $u, v \in D^{1,2}(\mathbb{R}^N)$.

Corollary 3.8. *Suppose that there are a sequence $(u_n, v_n) \subset \mathcal{M}$ and $c \in (\widetilde{S}_K, 2^{2/N}\widetilde{S}_K)$ such that*

$$f(u_n, v_n) \to c \quad \text{and} \quad f'(u_n, v_n) \to 0.$$

Then I has a critical point $(u_0, v_0) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ with $I(u_0, v_0) = \frac{1}{N}c^{N/2}$.

4 Existence of positive solution to (P)

Now we recall some properties on the function $\Phi_{\delta,y}$ given by in (1.1). Note that

$$(\Phi_{\delta,y},\Phi_{\delta,y})\in\Sigma=\Big\{(u,v)\in D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N);u,v\geq 0\Big\}. \tag{4.1}$$

Moreover, making a change of variable we can prove that

$$\Phi_{\delta,y}\in L^q(\mathbb{R}^N)\quad\text{for}\quad q\in\left(\frac{N}{N-2},2^*\right],\quad\forall\delta>0\ \text{ and }\ \forall y\in\mathbb{R}^N. \tag{4.2}$$

The proof of next result can be seen in [1, Lemma 4].

Lemma 4.1. *For each $y\in\mathbb{R}^N$, we have*

(i) $\|\Phi_{\delta,y}\|_{H^{1,\infty}(\mathbb{R}^N)}\to 0$ *when* $\delta\to+\infty$,

(ii) $|\Phi_{\delta,y}|_q\to 0$ *when* $\delta\to 0$, $\forall q\in\left(\frac{N}{N-2},2^*\right)$,

(iii) $|\Phi_{\delta,y}|_q\to +\infty$ *when* $\delta\to+\infty$, $\forall q\in\left(\frac{N}{N-2},2^*\right)$.

The proof of next result can be seen in [1, Lemma 5].

Lemma 4.2. *For each $\varepsilon>0$, we have*

$$\int_{\mathbb{R}^N\setminus B_\varepsilon(0)}|\nabla\Phi_{\delta,0}|^2dx\to 0\quad\text{when}\quad\delta\to 0.$$

4.1 Technical Lemmas

Lemma 4.3. *Suppose that $a,b\in L^q(\mathbb{R}^N)$, $\forall q\in[p_1,p_2]$, where $1<p_1<\frac{N}{2}<p_2$ with $p_2<3$ if $N=3$. Then, for each $\varepsilon>0$, there are $\underline{\delta}=\underline{\delta}(\varepsilon)>0$ and $\bar{\delta}=\bar{\delta}(\varepsilon)>0$ such that*

$$\sup_{y\in\mathbb{R}^N}f(s_o\Phi_{\delta,y},t_o\Phi_{\delta,y})<\widetilde{S}_K+\varepsilon,\quad\delta\in(0,\underline{\delta}]\cup[\bar{\delta},\infty).$$

Proof. Consider $y\in\mathbb{R}^N$, $q\in\left(\frac{N}{2},p_2\right]$ and $t\in(1,+\infty)$ with $\frac{1}{q}+\frac{1}{t}=1$. Making a direct calculations we have

$$\frac{N}{N-2}<2t<2^*. \tag{4.3}$$

Since $\Phi_{\delta,b}\in L^d(\mathbb{R}^N),\forall d\in\left(\frac{N}{N-2},2^*\right)$, we get $|\Phi_{\delta,b}|^2\in L^t(\mathbb{R}^N)$. Then, using Hölder inequality and change of variable, we have

$$\int_{\mathbb{R}^N}a(x)|\Phi_{\delta,b}|^2dx\leq|a|_q|\Phi_{\delta,0}|^2_{2t},\quad\forall y\in\mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N}b(x)|\Phi_{\delta,b}|^2dx\leq|b|_q|\Phi_{\delta,0}|^2_{2t},\quad\forall y\in\mathbb{R}^N.$$

From item (iii) of Lemma 4.1, given $\varepsilon>0$, there exists $\underline{\delta}=\underline{\delta}(\varepsilon)>0$ such that

$$\sup_{y\in\mathbb{R}^N}f(s_o\Phi_{\delta,y},t_o\Phi_{\delta,y})\leq\widetilde{S}_K+\frac{\varepsilon}{2}<\widetilde{S}_K+\varepsilon,\quad\forall\delta\in(0,\underline{\delta}].$$

Suppose that $q \in \left[p_1, \dfrac{N}{2}\right)$ with $t \in (1, +\infty)$ and $\dfrac{1}{q} + \dfrac{1}{t} = 1$. Note that $2t - 2^* > 0$ and for $\delta > 1$,

$$|\Phi_{\delta,y}| \in L^{\infty}(\mathbb{R}^N) \tag{4.4}$$

and $|\Phi_{\delta,y}|^{2^*} \in L^1(\mathbb{R}^N)$. Then, $|\Phi_{\delta,y}|^2 \in L^t(\mathbb{R}^N)$. Using Hölder inequality with q and t, we get

$$\begin{aligned} s_o^2 \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,y}|^2 dx &\leq s_o^2 |a|_q \left(\int_{\mathbb{R}^N} |\Phi_{\delta,0}|^{2t} dz \right)^{1/t} \\ &= s_o^2 |a|_q \left(\int_{\mathbb{R}^N} |\Phi_{\delta,0}|^{2_s^*} |\Phi_{\delta,0}|^{2t-2_s^*} dz \right)^{1/t} \\ &\leq s_o^2 |a|_q |\Phi_{\delta,0}|_{\infty}^{(2t-2^*)/t} \left(\int_{\mathbb{R}^N} |\Phi_{\delta,0}|^{2^*} dz \right)^{1/t} \leq s_o^2 |a|_q |\Phi_{\delta,0}|_{\infty}^{(2t-2^*)/t} \\ &\leq s_o^2 |a|_q c^{(2t-2^*)/t} \delta^{((2-N)/2)((2t-2^*)/t)}, \quad \forall y \in \mathbb{R}^N. \end{aligned}$$

Then, given $\varepsilon > 0$, there is $\bar{\delta} = \bar{\delta}(\varepsilon) > 1$ such that

$$\delta^{((2-N)/2)/2)((2t-2^*)/t)} < \frac{\varepsilon}{2 s_o^2 |a|_q c^{(2t-2^*)/t}}, \quad \forall \delta \in [\bar{\delta}, \infty).$$

Arguing as the same way, we have

$$t_o^2 \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,y}|^2 dx \quad \leq \quad t_o^2 |b|_q c^{(2t-2^*)/t} \delta^{((2-N)/2)((2t-2^*)/t)}, \quad \forall y \in \mathbb{R}^N.$$

Then

$$\begin{aligned} f(s_o \Phi_{\delta,y}, t_o \Phi_{\delta,y}) &= \widetilde{S}_K + s_o^2 \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,y}|^2 dx + t_o^2 \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,y}|^2 dx \\ &\leq S + s_o^2 \sup_{y \in \mathbb{R}^N} \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,y}|^2 dx + t_o^2 \sup_{y \in \mathbb{R}^N} \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,y}|^2 dx \\ &\leq \widetilde{S}_K + \frac{\varepsilon}{2} < \widetilde{S}_K + \varepsilon, \quad \forall y \in \mathbb{R}^N \quad \text{and} \quad \forall \delta \in [\bar{\delta}, \infty). \end{aligned}$$

□

Lemma 4.4. *Suppose that $(a, b)_3$ is true. Then,*

$$\sup_{\substack{y \in \mathbb{R}^N \\ \delta \in (0,+\infty)}} f(s_o \Phi_{\delta,y}, t_o \Phi_{\delta,y}) < 2^{2/N} \widetilde{S}_K.$$

Proof. Using Hölder inequality with $N/2$ and $N/(N-2)$, we get

$$f(s_o \Phi_{\delta,y}, t_o \Phi_{\delta,y}) \leq \widetilde{S}_K + s_o^N |a|_{L^{N/2}(\mathbb{R}^N)} + t_o^N |b|_{L^{N/2}(\mathbb{R}^N)}.$$

From $(a, b)_3$ we conclude

$$\sup_{\substack{y \in \mathbb{R}^N \\ \delta \in (0,\infty)}} f(s_o \Phi_{\delta,y}, t_o \Phi_{\delta,y}) \leq \widetilde{S}_K + \widetilde{S}_K (2^{2/N} - 1) = 2^{2/N} \widetilde{S}_K.$$

□

Consider the function

$$\xi(x) = \begin{cases} 0, & \text{if} \quad |x| < 1 \\ 1, & \text{if} \quad |x| \geq 1 \end{cases}$$

and define $\alpha : D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \to \mathbb{R}^{N+1}$ by

$$\alpha(u,v) = \frac{1}{\widetilde{S}_K}\int_{\mathbb{R}^N}\left(\frac{x}{|x|},\xi(x)\right)[s_o^2|\nabla u|^2 + t_o^2|\nabla v|^2]dx = (\beta(u,v),\gamma(u,v)),$$

where

$$\beta(u,v) = \frac{1}{\widetilde{S}_K}\int_{\mathbb{R}^N}\frac{x}{|x|}[s_o^2|\nabla u|^2 + t_o^2|\nabla v|^2]dx$$

and

$$\gamma(u,v) = \frac{1}{\widetilde{S}_K}\int_{\mathbb{R}^N}\xi(x)[s_o^2|\nabla u|^2 + t_o^2|\nabla v|^2]dx.$$

Lemma 4.5. *If* $|y| \geq \frac{1}{2}$, *then*

$$\beta(\Phi_{\delta,y},\Phi_{\delta,y}) = \frac{y}{|y|} + o_\delta(1) \quad \text{when} \quad \delta \to 0.$$

Proof. Given $\varepsilon > 0$, from Lemma 4.2, there is $\hat{\delta} > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_\varepsilon(y)}|\nabla\Phi_{\delta,y}|^2dx = \int_{\mathbb{R}^N\setminus B_\varepsilon(0)}|\nabla\Phi_{\delta,0}|^2dz < \varepsilon, \quad \forall\delta \in (0,\hat{\delta}).$$

Then,

$$\begin{aligned}\left|\beta(\Phi_{\delta,y},\Phi_{\delta,y}) - \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{B_\varepsilon(y)}\frac{x}{|x|}|\nabla\Phi_{\delta,y}|^2dx\right| &\leq \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{\mathbb{R}^N\setminus B_\varepsilon(y)}|\nabla\Phi_{\delta,y}|^2dx \\ &< \varepsilon, \quad \forall\delta\in(0,\hat{\delta}). \end{aligned} \tag{4.5}$$

Note that

$$\left|\frac{y}{|y|} - \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{B_\varepsilon(y)}\frac{x}{|x|}|\nabla\Phi_{\delta,y}|^2dx\right| < 4\varepsilon+\varepsilon = C\varepsilon, \quad \forall\delta\in(0,\hat{\delta}). \tag{4.6}$$

From (4.5) and (4.6), we have

$$\begin{aligned}\left|\beta(\Phi_{\delta,y},\Phi_{\delta,y}) - \frac{y}{|y|}\right| &= \Bigg|\beta(\Phi_{\delta,y},\Phi_{\delta,y}) - \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{B_\varepsilon(y)}\frac{x}{|x|}|\nabla\Phi_{\delta,y}|^2dx \\ &+ \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{B_\varepsilon(y)}\frac{x}{|x|}|\nabla\Phi_{\delta,y}|^2dx - \frac{y}{|y|}\Bigg| \\ &\leq \left|\beta(\Phi_{\delta,y},\Phi_{\delta,y}) - \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{B_\varepsilon(y)}\frac{x}{|x|}|\nabla\Phi_{\delta,y}|^2dx\right| \\ &+ \left|\frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{B_\varepsilon(y)}\frac{x}{|x|}|\nabla\Phi_{\delta,y}|^2dx - \frac{y}{|y|}\right| \\ &< \varepsilon + C\varepsilon \\ &= K\varepsilon, \quad \forall\delta\in(0,\hat{\delta}).\end{aligned}$$

□

Lemma 4.6. *Suppose that* $a, b \in L^q(\mathbb{R}^N)$, $\forall q \in [p_1,p_2]$, *where* $1 < p_1 < \frac{N}{2} < p_2$ *with* $p_2 < 3$ *if* $N = 3$. *Then, for every* $\delta > 0$, *we have*

$$\lim_{|y|\to\infty} f(s_o\Phi_{\delta,y}, t_o\Phi_{\delta,y}) = \widetilde{S}_K.$$

Proof. Since

$$f(s_o\Phi_{\delta,y},t_o\Phi_{\delta,y}))=\widetilde{S}_K+s_o^2\int_{\mathbb{R}^N}a(x)|\Phi_{\delta,y}|^2dx+t_o^2ds\int_{\mathbb{R}^N}b(x)|\Phi_{\delta,y}|^2dx,$$

we need to prove that

$$\lim_{|y|\to\infty}\int_{\mathbb{R}^N}a(x)|\Phi_{\delta,y}|^2dx=0,\quad\forall\delta>0 \tag{4.7}$$

and

$$\lim_{|y|\to\infty}\int_{\mathbb{R}^N}b(x)|\Phi_{\delta,y}|^2dx=0,\quad\forall\delta>0. \tag{4.8}$$

Note that given $\varepsilon>0$, there is $k_0>0$ such that

$$\left(\int_{\mathbb{R}^N\setminus B_\rho(0)}a(x)^{N/2}dx\right)^{2/N}<\varepsilon,\quad\forall\rho>k_0.$$

and

$$\left(\int_{\mathbb{R}^N\setminus B_\rho(y)}|\Phi_{\delta,y}|^{2^*}dx\right)^{1/2^*}=\left(\int_{\mathbb{R}^N\setminus B_\rho(0)}|\Phi_{\delta,0}|^{2^*}dz\right)^{1/2^*}<\varepsilon,\quad\forall\rho>k_0. \tag{4.9}$$

Consider

$$k_0<2\rho<|y|\quad(\rho\ \text{ fixed}) \tag{4.10}$$

and note that

$$B_\rho(0)\cap B_\rho(y)=\emptyset. \tag{4.11}$$

Using Hölder inequality with $N/2$ and $N/(N-2)$, we get

$$\begin{aligned}
\int_{\mathbb{R}^N}a(x)|\Phi_{\delta,y}|^2dx &\le \left(\int_{\mathbb{R}^N\setminus(B_\rho(0)\cup B_\rho(y))}a^{N/2}dx\right)^{2/N}\left(\int_{\mathbb{R}^N\setminus(B_\rho(0)\cup B_\rho(y))}|\Phi_{\delta,y}|^{2^*}dx\right)^{(N-2)/N}\\
&+\left(\int_{B_\rho(0)}a^{N/2}dx\right)^{2/N}\left(\int_{B_\rho(0)}|\Phi_{\delta,y}|^{2^*}dx\right)^{(N-2)/N}\\
&+\left(\int_{B_\rho(y)}a^{N/2}dx\right)^{2/N}\left(\int_{B_\rho(y)}|\Phi_{\delta,y}|^{2^*}dx\right)^{(N-2)/N}\\
&\le \left(\int_{\mathbb{R}^N\setminus B_\rho(0)}a^{N/2}dx\right)^{2/N}\left(\int_{\mathbb{R}^N\setminus B_\rho(y)}|\Phi_{\delta,y}|^{2^*}dx\right)^{(N-2)/N}\\
&+\left(\int_{\mathbb{R}^N}a^{N/2}dx\right)^{2/N}\left(\int_{\mathbb{R}^N\setminus B_\rho(y)}|\Phi_{\delta,y}|^{2^*}dx\right)^{(N-2)/N}\\
&+\left(\int_{\mathbb{R}^N\setminus B_\rho(0)}a^{N/2}dx\right)^{2/N}\left(\int_{\mathbb{R}^N}|\Phi_{\delta,y}|^{2^*}dx\right)^{(N-2)/N}\\
&=\left(\int_{\mathbb{R}^N\setminus B_\rho(0)}a^{N/2}dx\right)^{2/N}\\
&<\varepsilon\varepsilon^2+|a|_{N/2}\varepsilon^2+\varepsilon.
\end{aligned}$$

Arguing of the same way for the term (4.8), the proof is over. □

Now we define the set

$$\Im = \left\{(u, v) \in \mathcal{M}; \alpha(u, v) = \left(0, \frac{1}{2}\right)\right\}.$$

and note that from Lemma 4.2 and Lemma 4.1, item (i), there is $\delta_1 > 0$ such that $(\Phi_{\delta_1,0}, \Phi_{\delta_1,0}) \in \Im$.

Lemma 4.7. *The number* $c_0 = \inf_{u\in\Im} f(u, v)$ *satisfies the inequality* $c_0 > \widetilde{S}_K$.

Proof. Since $\Im \subset \mathcal{M}$, we have

$$\widetilde{S}_K \leq c_0.$$

Suppose, by contradiction, that $\widetilde{S}_K = c_0$. By Ekeland variational principle [16], there exists $(u_n, v_n) \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} K(u_n, v_n)dx = 1, \quad \alpha(u_n, v_n) \to \left(0, \frac{1}{2}\right) \tag{4.12}$$

and

$$f(u_n, v_n) \to \widetilde{S}_K, \quad f'|_{\mathcal{M}}(u_n, v_n) \to 0. \tag{4.13}$$

Then, (u_n, v_n) is bounded in $D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$ and, up to a subsequence, $u_n \rightharpoonup u_0$, $v_n \rightharpoonup v_0$ in $D^{1,2}(\mathbb{R}^N)$.

If $w_n = S^{(N-2)/4}u_n$, $z_n = S^{(N-2)/4}v_n$ and $w_0 = S^{(N-2)/4}u_0$, $z_0 = S^{(N-2)/4}v_0$, we have that $w_n \rightharpoonup w_0$, $z_n \rightharpoonup z_0$ in $D^{1,2}(\mathbb{R}^N)$. Moreover, from (4.13) and Lemma 3.6, we get

$$I(w_n, z_n) \to \frac{1}{N}\widetilde{S}_K^{N/2} \text{ and } I'(w_n, z_n) \to 0.$$

We are going to show that $(w_0, z_0) \equiv (0, 0)$. Note that

$$u_n \nrightarrow u_0, \quad u_n \nrightarrow u_0 \text{ in } D^{1,2}(\mathbb{R}^N), \tag{4.14}$$

since otherwise, $(u_0, v_0) \in \mathcal{M}$ implies $u_0 \neq 0$, $v_0 \neq 0$. Then,

$$\begin{aligned}\widetilde{S}_K &\leq \frac{\int_{\mathbb{R}^N}|\nabla u_0|^2dx + \int_{\mathbb{R}^N}|\nabla v_0|^2dx}{\left(\int_{\mathbb{R}^N}K(u_0, v_0)dx\right)^{2/2^*}} = \int_{\mathbb{R}^N}|\nabla u_0|^2dx + \int_{\mathbb{R}^N}|\nabla v_0|^2dx \\ &< \int_{\mathbb{R}^N}|\nabla u_0|^2dx + \int_{\mathbb{R}^N}|\nabla v_0|^2dx + \int_{\mathbb{R}^N}a(x)|u_0|^2dx + \int_{\mathbb{R}^N}b(x)|v_0|^2dx = \widetilde{S}_K,\end{aligned}$$

which it is an absurd. Hence, $w_n \nrightarrow w_0$, $z_n \nrightarrow z_0$ in $D^{1,2}(\mathbb{R}^N)$ and, since (w_n, z_n) is a $(PS)_c$ sequence for I, by Theorem 3.2 we obtain that

$$I(w_n.z_n) \to I(w_0, z_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) = \frac{1}{N}\widetilde{S}_K^{N/2}.$$

Since $I'_\infty(z_0^j, \zeta_0^j) = 0$, we have that

$$I(w_0, z_0) = 0, \tag{4.15}$$

$$k = 1, \tag{4.16}$$

$$z_0^1, \zeta_0^1 > 0, \tag{4.17}$$

$$I(w_0, z_0) = \frac{1}{N}\int_{\mathbb{R}^N} K(w_0, z_0)dx$$

and from (4.15), we conclude that $w_0 \equiv 0$ and $z_0 \equiv 0$. Then, (w_n, z_n) is a $(PS)_c$ sequence for I such that $w_n \rightharpoonup 0$, $z_n \rightharpoonup 0$ and $w_n \not\to 0$, $z_n \not\to 0$.

Note that $\int_{\mathbb{R}^N} a(x)|w_n|^2dx = o_n(1)$ and $\int_{\mathbb{R}^N} b(x)|z_n|^2dx = o_n(1)$. Then,

$$\frac{1}{N}\widetilde{S}_K^{N/2} + o_n(1) = I(w_n, z_n) = I_\infty(w_n, z_n) + \int_{\mathbb{R}^N} a(x)|w_n|^2dx + \int_{\mathbb{R}^N} b(x)|z_n|^2dx = I_\infty(v_n) + o_n(1) \quad (4.18)$$

and

$$\|I'_\infty(w_n, z_n)\|_{D'} \leq \|I'(w_n, z_n)\|_{D'} + o_n(1). \quad (4.19)$$

From (4.18) and (4.19) we conclude that (w_n, z_n) is a $(PS)_c$ sequence for I_∞ and by Lemma 3.1, there are sequences $(R_n) \subset \mathbb{R}$, $(x_n) \subset \mathbb{R}^N$, (z_0^1, ζ_0^1) nontrivial solution of (S_∞) and (Φ_n, Ψ_n) a $(PS)_c$ sequence for I_∞ such that

$$w_n(x) = \Phi_n(w) + R_n^{(N-2)/2} z_0^1(R_n(x - x_n)) + o_n(1) \text{ and } z_n(x) = \Psi_n(w) + R_n^{(N-2)/2}\zeta_0^1(R_n(x - x_n)) + o_n$$

Note that if we define

$$\widetilde{\Phi}_n(x) = R_n^{(N-2)/2} z_0^1(R_n(x - x_n)), \quad \widetilde{\Psi}_n(x) = R_n^{(N-2)/2}\zeta_0^1(R_n(x - x_n)),$$

making change of variable, we have

$$I'_\infty(\widetilde{\Phi}_n, \widetilde{\Psi}_n)(\varphi, \psi) = I'_\infty(z_0^1, \zeta_0^1)(\varphi_n, \psi_n) = 0, \quad \forall(\varphi, \psi) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N), \quad \forall n \in \mathbb{N},$$

i.e, $(\widetilde{\Phi}_n, \widetilde{\Psi}_n)$ is a solution of (S_∞) , for all $n \in \mathbb{N}$.

Moreover, from definition of $(\widetilde{\Phi}_n, \widetilde{\Psi}_n)$ and by (4.17), we get

$$\widetilde{\Phi}_n(x) = \widetilde{\Psi}_n(x) = c\left(\frac{\delta_n}{\delta_n^2 + |x - y_n|^2}\right)^{(N-2)/2}.$$

By (4.20), we obtain

$$u_n(x) = \widehat{\Phi}_n(x) + \Phi_{\delta_n,y_n}(x) + o_n(1), \quad v_n(x) = \widehat{\Psi}_n(x) + \Phi_{\delta_n,y_n}(x) + o_n(1)$$

where

$$\widehat{\Phi}_n(x) = \frac{1}{\widetilde{S}_K^{(N-2)/4}}\Phi_n(x), \quad \widehat{\Psi}_n(x) = \frac{1}{\widetilde{S}_K^{(N-2)/4}}\Psi_n(x).$$

Using (4.16), we derive that $\Phi_n \to 0$, $\Psi_n \to 0$ in $D^{1,2}(\mathbb{R}^N)$, which implies that $\widehat{\Phi}_n \to 0$, $\widehat{\Psi}_n \to 0$ in $D^{1,2}(\mathbb{R}^N)$. From (4.12) we have

$$\left(0, \frac{1}{2}\right) + o_n(1) = \alpha(u_n, v_n) = \alpha(\widehat{\Phi}_n(x) + \Phi_{\delta_n,y_n}(x), \widehat{\Psi}_n(x) + \Phi_{\delta_n,y_n}(x)) + o_n(1)) = \alpha(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n})$$

which implies

$$(i) \quad \beta(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) \to 0$$

and

$$(ii) \quad \gamma(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) \to \frac{1}{2}.$$

Passing to a subsequence, one of these cases can occur.

(a) $\delta_n \to +\infty$ when $n \to +\infty$;

(b) $\delta_n \to \tilde{\delta} \neq 0$ when $n \to +\infty$;

(c) $\delta_n \to 0$ and $y_n \to \tilde{y}$ when $n \to +\infty$ with $|\tilde{y}| < \frac{1}{2}$;

(d) $\delta_n \to 0$ when $n \to +\infty$ and $|y_n| \geq \dfrac{1}{2}$ for n sufficient large.

Suppose that (a) is true. Then,

$$\gamma(\Phi_{\delta_n,y_n}) = 1 - \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{B_1(0)} |\nabla \Phi_{\delta_n,y_n}|^2 dx,$$

which implies by Lemma 4.1,

$$|\gamma(\Phi_{\delta_n,b_n}) - 1| = \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{B_1(0)} |\nabla \Phi_{\delta_n,y_n}|^2 dx \leq \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{\mathbb{R}^N} |\nabla \Phi_{\delta_n,y_n}|^2 dx = o_n(1),$$

which contradicts (ii).

Suppose that (b) is true. In this case we can suppose that $|y_n| \to +\infty$, because if $y_n \to \tilde{y}$, we can prove that

$$\Phi_{\delta_n,y_n} \to \Phi_{\tilde{\delta},\tilde{y}} \ \text{ in } \ D^{1,2}(\mathbb{R}^N).$$

Since $\widehat{\Phi}_n, \widehat{\Psi}_n \to 0$ in $D^{1,2}(\mathbb{R}^N)$ and $u_n = \widehat{\Phi}_n + \Phi_{\delta_n,y_n} + o_n(1)$, $v_n = \widehat{\Psi}_n + \Phi_{\delta_n,y_n} + o_n(1)$, we have that (u_n, v_n) converges in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ but this is a contradiction with (4.14).

Then,

$$\begin{aligned}\gamma(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) &= \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{\mathbb{R}^N} \xi(x)|\nabla \Phi_{\delta_n,y_n}|^2 dx = \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{\mathbb{R}^N\setminus B_1(0)} |\nabla \Phi_{\delta_n,y_n}|^2 dx \\ &= 1 - \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{B_1(-y_n)} |\nabla \Phi_{\delta_n,0}|^2 dx. \end{aligned} \tag{4.21}$$

From Lebesgue Theorem we can prove that

$$\int_{B_1(-b_n)} |\nabla \Phi_{\delta_n,0}|^2 dx \to 0$$

and from (4.21), we obtain

$$\gamma(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) \to 1 \quad \text{when} \quad n \to +\infty,$$

which is a contradiction with (ii).

Suppose that (c) is true. We have that

$$\begin{aligned}\gamma(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) &= \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{\mathbb{R}^N} \xi(x)|\nabla \Phi_{\delta_n,y_n}|^2 dx = \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{\mathbb{R}^N\setminus B_1(0)} |\nabla \Phi_{\delta_n,y_n}|^2 dx \\ &= \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{\mathbb{R}^N} |\nabla \Phi_{\delta_n,y_n}|^2 dx - \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{B_1(-y_n)} |\nabla \Phi_{\delta_n,0}|^2 dz \\ &= 1 - \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{B_1(-y_n)} |\nabla \Phi_{\delta_n,0}|^2 dz. \end{aligned} \tag{4.22}$$

Note that using Lebesgue Theorem again, we can prove that

$$\lim_{n\to+\infty} \frac{s_o^2+t_0^2}{\widetilde{S}_K}\int_{B_1(-y_n)} |\nabla \Phi_{\delta_n,0}|^2 dz = 1.$$

Then, by (4.22) we have that

$$\gamma(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) \to 0,$$

which is a contradiction with (ii).

Suppose that (d) is true. Since $|y_n| \geq \dfrac{1}{2}$ for n large, then $y_n \nrightarrow 0$ in $\mathbb{R}^N$. From Lemma 4.5, we get

$$\beta(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) = \frac{y_n}{|y_n|} + o_n(1).$$

Hence,

$$\beta(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) \nrightarrow 0,$$

which is a contradiction with (i). The, we conclude that $\widetilde{S}_K < c_0$ and the proof is over. □

Lemma 4.8. *There is* $\delta_1 \in (0, 1/2)$ *such that*

(a) $f(s_0\Phi_{\delta_1,y}, t_0\Phi_{\delta_1,y}) < \dfrac{\widetilde{S}_K + c_0}{2}, \quad \forall y \in \mathbb{R}^N$;

(b) $\gamma(\Phi_{\delta_1,y}, \Phi_{\delta_1,y}) < \dfrac{1}{2}, \quad \forall y \in \mathbb{R}^N$ *such that* $|y| < \dfrac{1}{2}$;

(c) $\left|\beta(\Phi_{\delta_1,y}, \Phi_{\delta_1,y}) - \dfrac{y}{|y|}\right| < \dfrac{1}{4}, \quad \forall y \in \mathbb{R}^N$ *such that* $|y| \geq \dfrac{1}{2}$.

Proof. From Lemma 4.3, we can choose $\varepsilon = \dfrac{c_0 - S}{2} > 0$, $\delta_2 < \min\{\underline{\delta}, 1/2\}$ and conclude that

$$f(s_0\Phi_{\delta,y}, t_0\Phi_{\delta,y}) \leq \sup_{y\in\mathbb{R}^N} f(s_0\Phi_{\delta,y}, t_0\Phi_{\delta,y}) < \widetilde{S}_K + \frac{c_0 - \widetilde{S}_K}{2} = \frac{\widetilde{S}_K + c_0}{2}, \quad \forall y \in \mathbb{R}^N. \tag{4.23}$$

Now by definition of ξ, we have

$$\gamma(\Phi_{\delta,y}, \Phi_{\delta,y}) = 1 - \frac{s_o^2 + t_0^2}{\widetilde{S}_K} \int_{B_1(-y)} |\nabla\Phi_{\delta,0}|^2 dz.$$

From Lebesgue Theorem

$$\frac{s_o^2 + t_0^2}{\widetilde{S}_K} \int_{B_1(-y)} |\nabla\Phi_{\delta,0}|^2 dz = 1$$

and the proof of this item is over.

Note that from Lemma 4.5, we conclude that

$$\beta(\Phi_{\delta,y}, \Phi_{\delta,y}) = \frac{y}{|y|} + o_\delta(1) \quad \text{when} \quad \delta \to 0, \quad \forall y \in \mathbb{R}^N; \ |y| \geq \frac{1}{2}$$

and the proof is finished. □

Lemma 4.9. *There is* $\delta_2 > 1$ *such that*

(a) $f(s_0\Phi_{\delta_2,y}, t_0\Phi_{\delta_2,y}) < \dfrac{\widetilde{S}_K + c_0}{2}, \quad \forall y \in \mathbb{R}^N$,

(b) $\gamma(\Phi_{\delta_2,y}, \Phi_{\delta_2,y}) > \dfrac{1}{2}, \quad \forall y \in \mathbb{R}^N$.

Proof. From Lemma 4.3, we can choose $\varepsilon = \dfrac{c_0 - S}{2} > 0$, $\delta_3 > \max\{\bar{\delta}, 1\}$ we have

$$f(s_0\Phi_{\delta,y}, t_0\Phi_{\delta,y}) \leq \sup_{y\in\mathbb{R}^N} f(s_0\Phi_{\delta,y}, t_0\Phi_{\delta,y}) < \widetilde{S}_K + \frac{c_0 - \widetilde{S}_K}{2} = \frac{\widetilde{S}_K + c_0}{2}, \quad \forall y \in \mathbb{R}^N. \tag{4.24}$$

Moreover, from definition of ξ and Lemma 4.1, we can conclude that

$$\gamma(\Phi_{\delta,y}, \Phi_{\delta,y}) \to 1 \quad \text{when} \quad \delta \to +\infty$$

and the proof is over. □

Lemma 4.10. *There is* $R > 0$ *such that*

(a) $f(s_0\Phi_{\delta,y}, t_0\Phi_{\delta,y}) < \dfrac{\widetilde{S}_K + c_0}{2}, \quad \forall y;\ |y| \geq R$ *and* $\delta \in [\delta_1, \delta_2]$,

(b) $(\beta(\Phi_{\delta,y}, \Phi_{\delta,y})|y))_{\mathbb{R}^N} > 0 \ \ \forall y;\ |y| \geq R$ *and* $\delta \in [\delta_1, \delta_2]$.

Proof. The first item follows by Lemma 4.3 and the choose of $\varepsilon = \dfrac{c_0 - S}{2} > 0$. The second item follows of the definition of β and $\Phi_{\delta,y}$ and adaptations the same arguments explored in [3] □

Consider the set

$$\mathcal{V} = \{(y,\delta) \in \mathbb{R}^{\mathbb{N}} \times (0,\infty); |y| < R \ \text{ and } \ \delta \in (\delta_1,\delta_2)\},$$

where δ_1, δ_2 and R are given by Lemmas 4.8, 4.9 and 4.10, respectively.
Let $Q : \mathbb{R}^N \times (0,+\infty) \to D^{1,2}(\mathbb{R}^N)$ be the continuous function given by

$$Q(y,\delta) = \Phi_{\delta,y}.$$

Consider now the sets

$$\Theta = \{(Q(y,\delta), Q(y,\delta)); (y,\delta) \in \overline{\mathcal{V}}\},$$

$$\mathcal{H} = \left\{ h \in C(\Sigma \cap \mathcal{M}); h(u,v) = (u,v), \forall (u,v) \in \Sigma \cap \mathcal{M}; f(s_o u, t_o v) < \frac{\widetilde{S}_K + c_0}{2} \right\}$$

and

$$\Gamma = \{\mathcal{A} \subset \Sigma \cap \mathcal{M}; \mathcal{A} = h(\Theta), h \in \mathcal{H}\}.$$

Note that $\Theta \subset \Sigma \cap \mathcal{M}$, $\Theta = Q(\overline{\mathcal{V}}) \times Q(\overline{\mathcal{V}})$ is compact and $\mathcal{H} \neq \emptyset$, because the identity function is in $\mathcal{H}$.

Lemma 4.11. *Let $\mathcal{F} : \overline{\mathcal{V}} \to \mathbb{R}^{N+1}$ be a function given by*

$$\mathcal{F}(y,\delta) = (\alpha \circ (Q,Q))(y,\delta) = \frac{s_o^2 + t_0^2}{\widetilde{S}_K} \int_{\mathbb{R}^N} \left(\frac{x}{|x|}, \xi(x) \right) |\nabla \Phi_{\delta,y}|^2 dx.$$

Then,

$$d(\mathcal{F}, \mathcal{V}, (0,1/2)) = 1. \ (Topological\ degree)$$

Proof. Let

$$\mathcal{Z} : [0,1] \times \overline{\mathcal{V}} \to \mathbb{R}^{N+1}$$

be the homotopy given by

$$\mathcal{Z}(t,(y,\delta)) = t\mathcal{F}(y,\delta) + (1-t) I_{\overline{\mathcal{V}}}(y,\delta),$$

where $I_{\overline{\mathcal{V}}}$ is the identity operator. Using lemma 4.8 and Lemma 4.9, we can show that $(0,1/2) \notin \mathcal{Z}([0,1] \times (\partial \mathcal{V}))$, i.e,

$$t\beta(\Phi_{\delta,y}, \Phi_{\delta,y}) + (1-t)y \neq 0, \quad \forall t \in [0,1] \ \text{ and } \ \forall (y,\delta) \in \partial \mathcal{V} \tag{4.25}$$

or

$$t\gamma(\Phi_{\delta,y}, \Phi_{\delta,y}) + (1-t)\delta \neq \frac{1}{2}, \quad \forall t \in [0,1] \ \text{ and } \ \forall (y,\delta) \in \partial \mathcal{V}. \tag{4.26}$$

Hence $(0,1/2) \notin \mathcal{Z}([0,1] \times \partial \mathcal{V})$ where we conclude that $d(\mathcal{F}, \mathcal{V}, (0,1/2))$, $d(i_{\overline{\mathcal{V}}}, \mathcal{V}, (0,1/2))$ and $d(\mathcal{Z}(t,\cdot), \mathcal{V}, (0,1/2))$ are well defined and

$$d(\mathcal{F}, \mathcal{V}, (0,1/2)) = d(i_{\overline{\mathcal{V}}}, \mathcal{V}, (0,1/2)) = 1.$$

□

Lemma 4.12. *If $\mathcal{A} \in \Gamma$, then $\mathcal{A} \cap \Im \neq \emptyset$.*

Proof. It is sufficient to prove that for all $h \in \mathcal{H}$, there exists $(y_0, \delta_0) \in \overline{\mathcal{V}}$ such that

$$(\alpha \circ \mathcal{H} \circ (Q,Q))(y_0,\delta_0) = \left(0, \frac{1}{2}\right).$$

Given $h \in \mathcal{H}$, let

$$\mathcal{F}_h : \overline{\mathcal{V}} \to \mathbb{R}^{N+1}$$

be the continuous function given by

$$\mathcal{F}_h(y,\delta) = (\alpha \circ h \circ (Q,Q))(y,\delta).$$

We are going to show that $\mathcal{F}_h = \mathcal{F}$ in $\partial\mathcal{V}$. Note that

$$\partial\mathcal{V} = \Pi_1 \cup \Pi_2 \cup \Pi_3, \tag{4.27}$$

where

$$\Pi_1 = \{(y,\delta_1); |y| \le R\},$$

$$\Pi_2 = \{(y,\delta_2); |y| \le R\}$$

and

$$\Pi_3 = \{(y,\delta); |y| = R \ \text{ and } \ \delta \in [\delta_1,\delta_2]\}.$$

If $(y,\delta) \in \Pi_1$, then $(y,\delta) = (y,\delta_1)$ and by item (a) from Lemma 4.8, we have

$$f(s_oQ(y,\delta),t_oQ(y,\delta)) = f(s_oQ(y,\delta_1),t_oQ(y,\delta_1)) = f(s_0\Phi_{\delta_1,y},t_0\Phi_{\delta_1,y}) < \frac{\widetilde{S}_K + c_0}{2}, \quad \forall (y,\delta) \in \Pi_1. \tag{4.28}$$

If $(y,\delta) \in \Pi_2$, then $(y,\delta) = (y,\delta_2)$ and by item(a) from Lemma 4.9, we get

$$f(s_oQ(y,\delta),t_oQ(y,\delta)) = f(s_oQ(y,\delta_2),t_oQ(y,\delta_2)) = f(s_0\Phi_{\delta_2,y},t_0\Phi_{\delta_2,y}) < \frac{\widetilde{S}_K + c_0}{2}, \quad \forall (y,\delta) \in \Pi_2. \tag{4.29}$$

If $(y,\delta) \in \Pi_3$, then $|y| = R$ and $\delta \in [\delta_1,\delta_2]$ and by item (a) from Lemma 4.10, we obtain

$$f(s_oQ(y,\delta),t_oQ(y,\delta)) = f(s_0\Phi_{\delta,y},t_0\Phi_{\delta,y}) < \frac{\widetilde{S}_K + c_0}{2}, \quad \forall (y,\delta) \in \Pi_3. \tag{4.30}$$

From (4.27), (4.28), (4.29) and (4.30) we conclude that

$$f(s_oQ(y,\delta),t_oQ(y,\delta)) < \frac{\widetilde{S}_K + c_0}{2}, \quad \forall (y,\delta) \in \partial\mathcal{V}.$$

Hence,

$$\begin{aligned}
\mathcal{F}_h(y,\delta) &= (\alpha \circ h \circ (Q,Q))(y,\delta) = (\alpha \circ h)(Q(y,\delta),Q(y,\delta)) \\
&= \alpha(h((Q(y,\delta),Q(y,\delta)))) = \alpha((Q(y,\delta),Q(y,\delta))) \\
&= (\alpha \circ (Q,Q))(y,\delta) = \mathcal{F}(y,\delta), \quad \forall (y,\delta) \in \partial\mathcal{V}.
\end{aligned}$$

Since $(0,1/2) \notin \mathcal{F}(\partial\mathcal{V})$, we have

$$d(\mathcal{F},\mathcal{V},(0,1/2)) = d(\mathcal{F}_h,\mathcal{V},(0,1/2)).$$

From Lemma 4.11, we get

$$d(\mathcal{F}_h,\mathcal{V},(0,1/2)) = d(\mathcal{F},\mathcal{V},(0,1/2)) = 1,$$

and there exists $(y_0,\delta_0) \in \mathcal{V}$ such that

$$\mathcal{F}_h(y_0,\delta_0) = (\alpha \circ h \circ (Q,Q))(y_0,\delta_0) = \left(0, \frac{1}{2}\right)$$

and the proof is over. □

4.2 Proof of Theorem 1.1

Consider the number

$$c = \inf_{\mathcal{A}\in\Gamma} \max_{(u,v)\in\mathcal{A}} f(u,v)$$

and for each $q \in \mathbb{R}$,

$$f^q = \{(u,v) \in \Sigma \cap \mathcal{M}; f(u,v) \leq q\}.$$

We are going to show that

$$\widetilde{S}_K < c < 2^{2/N}\widetilde{S}_K. \tag{4.31}$$

Note that

$$c = \inf_{\mathcal{A}\in\Gamma} \max_{(u,v)\in\mathcal{A}} f(u,v) \leq \max_{(u,v)\in\Theta} f(u,v) \leq \sup_{\substack{y\in\mathbb{R}^N \\ \delta\in(0,+\infty)}} f(s_o\Phi_{\delta,y}, t_o\Phi_{\delta,y}) < 2^{2/N}\widetilde{S}_K.$$

On the other hand, from Lemma 4.12, we have that

$$c_0 = \inf_{u\in\Im} f(u,v) \leq c = \inf_{\mathcal{A}\in\Gamma} \max_{u\in\mathcal{A}} f(s_o u, t_o v) \leq \sup_{\substack{y\in\mathbb{R}^N \\ \delta\in(0,+\infty)}} f(s_o\Phi_{\delta,y}, t_o\Phi_{\delta,y}) < 2^{2/N}\widetilde{S}_K. \tag{4.32}$$

From Lemma 4.7, we have that $\widetilde{S}_K < c_0$ and the proof is over.
Using the definition of c, there exists $(u_n, v_n) \subset \Sigma \cap \mathcal{M}$ such that

$$f(u_n, v_n) \to c. \tag{4.33}$$

Suppose, by contradiction, that

$$f'|_{\mathcal{M}}(u_n, v_n) \nrightarrow 0.$$

Then, there exists $(u_{nj}, v_{nj}) \subset (u_n, v_n)$ such that

$$\|f'|_{\mathcal{M}}(u_{nj}, v_{nj})\|_* \geq C > 0, \quad \forall j \in \mathbb{N}.$$

Using a Deformation Lemma [16], there exists a continuous application $\eta : [0,1] \times (\Sigma \cap \mathcal{M}) \to (\Sigma \cap \mathcal{M})$, $\varepsilon_0 > 0$ such that

(1) $\eta(0,u,v) = (u,v)$;

(2) $\eta(t,u,v) = (u,v), \forall (u,v) \in f^{c-\varepsilon_0} \cup \{(\Sigma \cap \mathcal{M}) \setminus f^{c+\varepsilon_0}\}, \forall t \in [0,1]$;

(3) $\eta(1, f^{c+\frac{\varepsilon_0}{2}}) \subset f^{c-\frac{\varepsilon_0}{2}}$.

From definition of c, there exists $\tilde{\mathcal{A}} \in \Gamma$ such that

$$c \leq \max_{(u,v)\in\tilde{\mathcal{A}}} f(u,v) < c + \frac{\varepsilon_0}{2},$$

where

$$\tilde{\mathcal{A}} \subset f^{c+\frac{\varepsilon_0}{2}}. \tag{4.34}$$

Since $\tilde{\mathcal{A}} \in \Gamma$, we have $\tilde{\mathcal{A}} \subset (\Sigma \cap \mathcal{M})$ and there exists $\bar{h} \in \mathcal{H}$ such that

$$\bar{h}(\Theta) = \tilde{\mathcal{A}}. \tag{4.35}$$

From definition of η, we have

$$\eta(1, \tilde{\mathcal{A}}) \subset (\Sigma \cap \mathcal{M}). \tag{4.36}$$

Let $\hat{h} : (\Sigma \cap \mathcal{M}) \to (\Sigma \cap \mathcal{M})$ be the function given by $\hat{h}(u,v) = \eta(1, \bar{h}(u,v))$ and note that $\hat{h} \in C(\Sigma \cap \mathcal{M})$. We are going to show that

$$f^{c+\varepsilon_0} \setminus f^{c-\varepsilon_0} \subset f^{2^{2s/N} S} \setminus f^{(S+c_0)/2}. \tag{4.37}$$

Considering $(u,v) \in f^{c+\varepsilon_0} \setminus f^{c-\varepsilon_0}$, we have

$$c - \varepsilon_0 < f(u,v) \leq c + \varepsilon_0$$

and by (4.31), for ε_0 sufficiently small, we get

$$c - \varepsilon_0 < f(u,v) \leq c + \varepsilon_0 < 2^{2/N} \widetilde{S}_K. \tag{4.38}$$

Now from Lemma 4.7 and (4.32), we obtain

$$\frac{\widetilde{S}_K + c_0}{2} < c_0 - \varepsilon_0 < c - \varepsilon_0 < 2^{2/N} \widetilde{S}_K$$

and

$$\frac{\widetilde{S}_K + c_0}{2} < c_0 - \varepsilon_0 \leq c - \varepsilon_0 < f(u,v), \tag{4.39}$$

which implies

$$(u,v) \in f^{2^{2/N} \widetilde{S}_K} \setminus f^{(\widetilde{S}_K + c_0)/2}.$$

Consider $(u,v) \in (\Sigma \cap \mathcal{M})$ such that

$$f(u,v) < \frac{\widetilde{S}_K + c_0}{2}. \tag{4.40}$$

Then,

$$\bar{h}(u,v) = (u,v)$$

and from (4.40), we have that $(u,v) \notin f^{2^{2/N} \widetilde{S}_K} \setminus f^{(\widetilde{S}_K + c_0)/2}$ and by (4.37), we get

$$(u,v) \notin f^{c+\varepsilon_0} \setminus f^{c-\varepsilon_0}.$$

Then,

$$(u,v) \in f^{c-\varepsilon_0} \cup \{(\Sigma \cap \mathcal{M}) \setminus f^{c+\varepsilon_0}\}$$

and from Deformation Lemma, we obtain

$$\eta(1,u,v) = (u,v).$$

Hence,

$$\hat{h}(u,v) = \eta(1, \bar{h}(u,v)) = \eta(1,u,v) = (u,v)$$

where we conclude that $\hat{h} \in \mathcal{H}$, which implies

$$\hat{h}(\Theta) = \eta(1, \bar{h}(\Theta))$$

and from (4.35), we conclude that

$$\hat{h}(\Theta) = \eta(1, \bar{h}(\Theta)) = \eta(1, \tilde{\mathcal{A}}). \tag{4.41}$$

From (4.36), we have $\eta(1, \tilde{\mathcal{A}}) \in \Gamma$, which implies

$$c = \inf_{\mathcal{A} \in \Gamma} \max_{u \in \mathcal{A}} f(u,v) \leq \max_{u \in \eta(1, \tilde{\mathcal{A}})} f(u,v). \tag{4.42}$$

From Deformation Lemma again and by (4.34), we get

$$\eta(1, \tilde{\mathcal{A}}) \subset \eta(1, f^{c+\frac{\varepsilon_0}{2}}) \subset f^{c-\frac{\varepsilon_0}{2}}.$$

Then,

$$f(u,v) \le c - \frac{\varepsilon_0}{2}, \quad \forall (u,v) \in \eta(1, \tilde{\mathcal{A}}),$$

which implies

$$\max_{u \in \eta(1,\tilde{\mathcal{A}})} f(u,v) \le c - \frac{\varepsilon_0}{2}$$

and using (4.42), we conclude that

$$c \le \max_{u \in \eta(1,\tilde{\mathcal{A}})} f(u,v) \le c - \frac{\varepsilon_0}{2},$$

which is an absurd.

Then,

$$f(u_n, v_n) \to c \ \text{ and } \ f'|_{\mathcal{M}}(u_n, v_n) \to 0$$

and from Lemma 3.7, up to a subsequence, $u_n \to \widetilde{u}_0$, $v_n \to \widetilde{v}_0$ in $D^{1,2}(\mathbb{R}^N)$, which implies that $\widetilde{u}_0, \widetilde{v}_0 \ge 0$,

$$f(\widetilde{u}_0, \widetilde{v}_0) = c \ \text{ and } \ f'|_{\mathcal{M}}(\widetilde{u}_0, (\widetilde{v}_0) = 0$$

and from(4.31)

$$\widetilde{S}_K < f(\widetilde{u}_0, \widetilde{v}_0) < 2^{2/N} \widetilde{S}_K.$$

The positivity of $\widetilde{u}_0$ and $\widetilde{v}_0$ is a consequence of the classical maximum principle.

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Received: June 18, 2020.
Accepted: August 22, 2020.