**Palestine Journal of Mathematics**

**Vol. 10(2)(2021)** , 533–536 

# Non comaximal graph of commutative rings

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Communicated by Ayman Badawi

MSC 2010 Classifications: 05C10, 05C17, 05C25

Keywords and phrases: Non comaximal; Clique number; Chromatic number; Maximal ideal; Perfect graph.

**Abstract** For a ring $R$ non comaximal graph $NC(R)$ were defined for rings whose vertex set is the collection of all non-trivial (left) ideals of R and any two distinct vertices $I$ and $J$ are adjacent if and only if $I + J \neq R$. In this paper, we investigate connectedness and perfectness of non comaximal graph of commutative rings and determine their diameter, girth etc.

## 1 Preliminaries

The study of the interplay of algebraic properties and graph theoretical properties of algebraic structures is an optimal research area. At first Cayley defined graphs of groups and studied the interplay with graph theoretical and algebraic properties of groups. It was emphasised again with the paper of Beck [3] which dealt mainly with the coloring of graphs. There are more than four hundred papers on zero divisor graphs. Anderson, Badawi [1] contributed a lot to this subject and defined total graph and generalized total graphs to investigate the algebraic structures. Later on Sharma et al. [6] defined another graph on commutative rings $R$, known as comaximal graph of $R$. Here the ring $R$ is the set of vertices and two vertices $a$ and $b$ are adjacent if and only if $Ra + Rb = R$. This concept has been generalized in many ways. The set of vertices was replaced by the set of all proper ideals in $R$ Maimani [4]. Non comaximal graphs are defined where the set of proper ideals of $R$ is the set of vertices and $I$ is adjacent to $J$ if and only if $I + J \neq R$ [2].

We summarize the notations and concepts related to the graph which will be used in our paper. Throughout this paper, we assume that all rings are commutative with non-zero unity and $G(R)$ is the non comaximal graph of a ring $R$. For two distinct vertices $u, v$ of $G$. A $u - v$ *path* of length $k$ is a sequence of vertices $u = v_0, v_1, \cdots, v_k = v$ such that $v_i$ is adjacent to $v_{i+1}$, $1 \leq i \leq k-1$ and it is a cycle if the initial vertex is same as the last vertex. If the shortest $u - v$ path is of length $k$, then d(u,v) = k. The *diameter* of $G$, diam(G), is the supremum of the set $\{d(u, v) : u \ and \ v \ are \ distinct \ vertices \ of \ G\}$. The *girth* of $G$, is the length of the shortest cycle in $G$ and ($girth(G) = \infty$, if $G$ contains no cycles). We say that $G$ is *connected*, if there is a path between any two distinct vertices of $G$. A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. A graph is said to be *planar* if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. A subgraph $H$ of $G$ is said to be an *induced subgraph* of $G$ if it has exactly the edges that appear in $G$. Also, a subgraph of $G$ is called a *spanning subgraph*, if $V(H) = V(G)$. *A* bipartite graph *G is a graph whose vertex set $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that the edge set of such a graph consists of precisely those edges which join vertices in $V_1$ to vertices of $V_2$. In particular, if $E$ consists of all possible such edges, then $G$ is called the* complete bipartite graph *and denoted by $K_{m,n}$ when $|V_1| = m$ and $|V_2| = n$. A $K_{1,n}$ graph is a* star graph. *A* null graph *is a graph containing no edges. By a* clique *in a graph $G$, we mean a complete subgraph of $G$ and the number of vertices in the maximal clique of $G$, is called the* clique number *of $G$ and is denoted by $\omega(G)$. For a graph $G$, let $\chi(G)$, denotes the vertex* chromatic number *of $G$, i.e., the minimum number of colors which can be assigned to the vertices of $G$ such that every two adjacent vertices have different colors. A graph $G$ is* perfect, *if for every* induced subgraph *$H$ of $G$, $\chi(H) = \omega(H)$. A graph is called* weakly perfect, *if its vertex chromatic number equals its clique number. Note that, for every graph $G$, $\omega(G) \leq \chi(G)$. $\overline{G} = (V, E)$ is the complement graph of the graph $G = (V, E)$ if the set of vertices in $\overline{G}$ is same as the set of vertices in $G$ and* *any two vertices $u, v \in V$ are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$.*

*$Max(R)$ is the set of all maximal ideals of $R$, $C_n$ is a cycle of length $n$ and $\overline{K_n}$ is the complement of $K_n$ and $\overline{K_n}$ is a null graph.*

*A commutative ring $R$ is said to be a local ring if it has a unique maximal ideal and it is semilocal ring if it has finitely many maximal ideals. The intersection of all maximal ideals is denoted by $J(R)$.*

***Non comaximal graphs of semilocal rings***

*Generalizing the definition of [2] we consider a graph $G(R)$ where $R$ is the set of vertices and $a, b \in R$ are adjacent if and only if $aR + bR \neq R$. Here $R$ is a semilocal ring therefore $R$ contains finitely many ideals. Since $Ra$ is also an ideal it is contained in a maximal ideal in $R$. This implies that $a$ is adjacent to $b$ if $aR$, $bR$ are contained in the same maximal ideal $M$ in $R$ or $a, b \in M$ [5].*

**Proposition 1.1.** *An element $a \in R$ is an isolated vertex of $R$ if and only if $a$ is a unit.*

*Proof.* If $a$ is a unit, then $aR = R$ and $a$ is not adjacent to any other element of $R$. If $a$ is not a unit then $aR$ is an ideal in $R$ contained in some maximal ideal $M$ in $R$. Now $aR + bR \neq R$ for every $b \in M$ and a can not be isolated. Therefore $a \in R$ is isolated if and only if $a$ is a unit and the set of all units in $R$, $U(R)$ is an independent set in $G(R)$. □

**Proposition 1.2.** *In a semilocal ring $R$*

*(i) If $a \in J(R) \neq 0$, then $deg_G a \neq 0$*

*(ii) $Max\{|M_i|\}$ is the clique number of $G(R)$.*

*Proof.* (i) Since $R$ is a semilocal ring it has finitely many maximal ideals $M_1, M_2, \cdots, M_k$ (say). If $a \in \cap M_i = J(R)$, then $a$ is adjacent to every other element of every $M_i$. Hence $deg_G a \neq 0$.

(ii) If $G_i(R)$ is the subgraph of $G(R)$ generated by the elements of the maximal ideal $M_i$, then it is complete and any element in $G_i(R)$ is not adjacent to any element which is not in $M_i$. Thus we have atleast $k$ complete subgraphs of $G(R)$. The $G_i(R)$ generated by the largest set of vertices is the clique of $G(R)$ and its order i.e., $Max\{|M_i|\}$ is the clique number $\omega(G(R))$ of $G(R)$. □

**Remark 1.3.** If $n = p_1 p_2$, then $Z_n$ has two ideals $\langle p_1 \rangle$ and $\langle p_2 \rangle$ such that $\langle p_1 \rangle \cap \langle p_2 \rangle = 0$ i.e., $J(R) = 0$. Now all the elements are either units or they contained in $\langle p_1 \rangle$ or $\langle p_2 \rangle$. If $n = p_1^{\alpha_1}.p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then there are $k$ maximal ideals in $Z_n$ and $J(R) = \cap \langle p_i \rangle \neq 0$. Then every element of $J(R)$ is adjacent to every element of $\langle p_i \rangle$ for every $i$.

**Proposition 1.4.** *Non comaximal graph $G(R)$ is not connected and $G_1(R)$ is connected if $R$ has more than two maximal ideals and $J(R) \neq 0$. Here $G_1(R)$ is the subgraph of $G$ generated by non units of $R$.*

*Proof.* Let $R$ be a semilocal ring with more than two maximal ideals. For $a, b \in R \setminus U$ and $c \in J(R)$, if $a$ and $b$ are contained in the same maximal ideal $M_i$ of $R$, then $aR + bR \neq R$ and they are adjacent, otherwise if $a \in M_i$ and $b \in M_j$, then $aR + cR \neq R$ $(a, c \in M_i)$ and $bR + cR \neq R$ as $b, c \in M_j$. Thus we have a path $a \to c \to b$. We conclude that the subgraph $G_1(R)$ generated by the elements of $R \setminus U(R)$ is connected but the subgraph $G_2(R)$ generated by $U(R)$ is a null graph. We may say $G(R)$ is the union of a connected graph and the complement of a complete graph. If $|U(R)| = n$, then $G_2(R)$ is $\overline{K_n}$ and $G(R) = G_1(R) \cup \overline{K_n}$. Since $G(R)$ is not connected its subgraph $G_1(R)$ is connected. □

*We may determine the diameter and girth of $G_1(R)$.*

**Proposition 1.5.** *The chromatic number $\chi(G(R)) = max|M_i|$, therefore $G(R)$ is weakly perfect.*

*Proof.* To color a $G(R)$ graph we need $max\{|M_j|\} = n_j$ colors. If $J(R) \neq 0$ such that $|J(R)| = t$, then out of these $n_j$ colors $t$ colors are assigned to the elements of $J(R)$. Now $M_j$ generates a complete subgraph $G_j(R)$ such that the elements of $M_j \setminus J(R)$ require $n_j - t$ colors. The elements of $M_i \setminus J(R)$ are not adjacent to the elements of $M_l \setminus J(R)$ for any $l$. This implies that $n_j - t$ colors are sufficient to color the elements of $M_j \setminus J(R)$ for all $j$. We have seen that $\omega(G(R)) = max\{|M_i|\}$. Therefore $\chi(G(R)) = \omega(G(R))$ and $G(R)$ is weakly perfect. □

**Proposition 1.6.** *For the ring $Z_n$, $n = p_1.p_2$, the diameter of $G(Z_n)$ is infinite.*

*Proof.* If $n = p_1.p_2$, then there are two maximal ideals $M_1$, $M_2$ such that $M_1 \cap M_2 = 0$ i.e., $J(R) = 0$. Now $a, b \in Z_n$ are adjacent if they belong to same $M_i$ and $d(a, b) = 1$. If they belong to different $M_i's$, then there is no path connecting then as in this case $aR + bR = R$. Therefore $d(a, b) = \infty$. This implies that $diam(G(Z_n)) = \infty$. □

**Proposition 1.7.** *The $diam(G_1(R)) = 1, 2$ or $\infty$ and girth of $G_1(R) \leq 4$.*

*Proof.* Let $a, b \in R \setminus U(R)$ and let $R$ be a semilocal ring such that $\{M_i\}$, $i = 1, 2, \cdots, n$ are maximal ideals of $R$. Now $a, b$ are contained in a maximal ideals $M_i$, $M_j$ in $R$. If $a$ and $b$ are contained in same maximal ideal $M_i$, then $d(a, b) = 1$ as they are adjacent otherwise we find an element $c \in J(R)$ to get a path $a \to c \to b$ and $d(a, b) = 2$. In both cases $d(a, b) \leq 2$, therefore $diam(G_1(R)) = 2$. If $J(R) = 0$ and $a, b$ are in different maximal ideals, then $d(a, b) = \infty$.

If $J(R) \neq 0$ which has atleast two elements $a, b \in J(R)$. For any vertices $x, y$ of $G_1(R)$ such that $x, y$ are in different maximal ideals, the cycle $x \to a \to y \to b \to x$ is the shortest cycle of length 4. Hence $girth$ of $G_1(R) \leq 4$.

□

**Proposition 1.8.** *For a ring $R$, $G(R \setminus U)$ is complete if and only if $R$ is isomorphic to $Z(p^k)$ or it has a unique maximal ideal.*

*Proof.* Suppose $R$ is isomorphic to $Z(p^k)$ or it has a unique maximal ideal. For a commutative ring $R$ every ideal is contained in a maximal ideal. Now every $a \in R$ is either a unit or $aR$ is contained in the unique maximal ideal. Clearly for any $a, b \in (R \setminus U)$, $aR + bR \neq R$ and $a$ is adjacent to $b$.

Conversely, if $G(R \setminus U)$ is complete, then the sum of $aR + bR \neq R$ for any $a, b \in R \setminus U$ implying that all the ideals $aR$ are contained in a unique maximal ideal. Therefore $R$ is isomorphic to $Z(p^k)$ or it has a unique maximal ideal.

□

**Proposition 1.9.** *For the ring $Z(p^k)$, $k \geq 5$ the non comaximal graph $G(Z(p^k))$ is not planar.*

*Proof.* It is sufficient to show that $G(Z(p^k))$ has a complete subgraph $K_5$ or a bipartite graph $K_{3,3}$ as subgraphs in $Z(p^k)$. Now all the elements of $Z(p^k)$ which are not units, are contained in the maximal ideal $\langle p \rangle$. Therefore for any two elements $a, b$ in $R \setminus U$, $aR + bR \neq R$ hence they are adjacent. Since $\langle p \rangle$ has more than 5 elements we may consider any 5 elements $a_1, a_2, a_3, a_4$ and $a_5$ such that they form a clique. Thus $G(R \setminus U)$ has $K_5$ as a subgraph and $G(R \setminus U)$ is not planar. □

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Received: March 10, 2020.
Accepted: May 20, 2020.